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EXCEPTIONALLY RAMIFIED MEROMORPHIC FUNCTIONS WITH A NON-ENUMERABLE SET OF ESSENTIAL SINGULARITIES

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§1. Introduction

In the complex function theory, Picard's Great Theorem plays an essential and important role. It is well-known as generalizations of this theorem that in a neighborhood of an isolated essential singularity, a meromorphic function cannot be exceptionally ramified (see W. Gross [2]) and that even it cannot be normal (see O. Lehto and K. I. Virtanen [7]). We are therefore interested in the behaviour of meromorphic functions with non-isolated essential singularities as well as in generalizations of the Gross' result. Several approaches in this direction have been made by G. af Hällström [3], S. Kametani [4], K. Noshiro [13], K. Matsumoto [8], [9], [10], [11], [12], S. Toppila [15], etc..

As for the functions with "more than two Picard exceptional values", K. Matsumoto ([10], [11]) has given sufficient conditions on Cantor sets E whose complements do not admit such functions. One of his basic results is

THEOREM A. *Let E be a Cantor set with successive ratios ξ_n satisfying the condition*

$$\xi_{n+1} = o(\xi_n^2) ,$$

then the domain complementary to E does not admit meromorphic functions with "more than two Picard exceptional values" at each singularities.

Having been inspired by this theorem, we are led to ask whether there is a Cantor set admitting no meromorphic functions with weaker conditions, such as "exceptionally ramified" (or "normal"). An exceptionally ramified

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meromorphic function is defined as follows: A meromorphic function f on the extended complex plane $\hat{C}$ is said to be exceptionally ramified, if there exist w_k, $1 \leqq k \leqq q$, in $\hat{C}$ such that the multiplicities $\ell_{k,j}$ of the roots $z_{k,j}$ of the equation $f(z) = w_k$ satisfy

$$\ell_{k,j} \geqq \nu_k \qquad \text{except finite } j\text{'s} ,$$

for a sequence of integers $\nu_k \geqq 2$ with the property

$$\sum_{k=1}^{q} \left(1 - \frac{1}{\nu_k}\right) > 2 . \tag{1.1}$$

Our main theorem is stated as follows:

THEOREM. *Let E be a Cantor set with successive ratios ξ_n satisfying the condition*

$$\xi_{n+1} = o(\xi_n^5) , \tag{1.2}$$

then the domain complementary to E admits no exceptionally ramified meromorphic functions with E as the set of essential singularities.

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§ 2. Preliminaries

2.1. Introducing the chordal distance $\chi(w, \zeta)$ on $\hat{C}$, we denote by $|S|$ the diameter of a subset S in $\hat{C}$. Let Δ be a τ-ply connected domain bounded by positively oriented analytic curves $\{\Gamma_i\}_{i=1,2,\cdots,\tau}$, Γ_i: $z = z_i(t)$ $(a \leqq t \leqq b)$ and let f be meromorphic on the closure $\bar{\Delta}$ of Δ. For $\zeta_1, \zeta_2 \notin f(\Gamma_i)$, $O(\Gamma_i; \zeta_1, \zeta_2)$ denotes the variation of $(1/2\pi) \arg (f(z) - \zeta_1)/(f(z) - \zeta_2)$ as z describes the curve Γ_i positively once.

We shall deal with an exceptionally ramified meromorphic function f on $\bar{\Delta}$ with q totally ramified values $\{w_k\}_{k=1,2,\cdots,q}$ satisfying the following three conditions:

(1) There exist mutually disjoint simply connected sectionally analytic domains $\{D_j\}_{j=1,\cdots,\alpha}$, $1 \leqq \alpha \leqq \tau$, with

$$|D_j| < \tfrac{1}{2} \min_{k \neq m} \chi(w_k, w_m) \tag{2.1}$$

and the images $\{f(\Gamma_i)\}_{i=1,\cdots,\tau}$ are covered with $\{D_j\}_{j=1,\cdots,\alpha}$, each D_j containing

$f(\Gamma_i)$ for at least one i.

(2) The number $\nu(w, f, \Delta)$ of roots of the equation $f(z) = w$ in Δ is $\geqq 1$, for $w \in \hat{C} - \bigcup_{j=1}^{\alpha} \overline{D}_j$.

(3) f has no ramified values on each boundary $\partial D_j \equiv C_j$.
Here the multiplicity is always taken into account.

For each C_j, the inverse image $f^{-1}(C_j)$ of C_j consists of a finite number of simple closed analytic curves $\{\Gamma_k^{(j)}\}_k$ in Δ. Then $\mathscr{F}$ denote the family of all subdomains of Δ which are bounded by some of $\{\Gamma_k^{(j)}\}_{k,j}$. By introducing a partial order into $\mathscr{F}$ by inclusion, we choose a maximal element Δ' of $\mathscr{F}$. The boundary $\partial\Delta'$ consists of a subfamily $\{\Gamma'_i\}_{i=1,\cdots,\tau'}(\tau' \leqq \tau)$ of $\{\Gamma_k^{(j)}\}_{k,j}$. We may assume that Γ'_i is positively oriented with respect to Δ'. Denoting by $j(i)$ the number j with $C_j \supset f(\Gamma'_i)$, we assume that $C_{j(i)}$, $i = 1, \cdots, \tau'$, form a subset $\{C_j\}_{j=1,\cdots,\alpha'}$ of $\{C_j\}_{j=1,\cdots,\alpha}$. For $\zeta_0 \in \hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j$, $\zeta_{j(i)} \in D_{j(i)}$, we set

$$s_i = O(\Gamma'_i; \zeta_0, \zeta_{j(i)}) .$$

Since Δ' is maximal in $\mathscr{F}$, we see that

$$s_i > 0 \qquad (i = 1, 2, \cdots, \tau')$$

and

$$\nu(\zeta, f, \Delta') \geqq 1 \qquad \text{for } \zeta \in \hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j .$$

Since the Riemannian image $\tilde{S}$ of Δ' under f may be viewed as a covering surface of $\hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j$, the exact value of the Euler characteristic $\rho(\Delta')$ of Δ':

$$\begin{aligned} \rho(\Delta') &= \rho(\tilde{S}) \\ &= \sum_{j=1}^{\alpha'} \rho(D_j)\nu(\zeta_j, f, \Delta') + \rho\Big(\hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j\Big)\nu(\zeta_0, f, \Delta') + v , \end{aligned}$$

for $\zeta_0 \in \hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j$, $\zeta_j \in D_j$, where v denotes the ramification index of $\tilde{S}$, that is,

$$\tau' - 2 = -\sum_{j=1}^{\alpha'} \nu(\zeta_j, f, \Delta') + (\alpha' - 2)\nu(\zeta_0, f, \Delta') + v .$$

Hence

$$2\nu(\zeta_0, f, \Delta') - 2 = \sum_{j=1}^{\alpha'} \{\nu(\zeta_0, f, \Delta') - \nu(\zeta_j, f, \Delta')\} - \tau' + v .$$

The argument principle proves

$$\nu(\zeta_0, f, \Delta') - \nu(\zeta_j, f, \Delta') = \sum_{i=1}^{\tau'} O(\Gamma'_i; \zeta_0, \zeta_j) , \tag{2.2}$$

so that

$$\begin{aligned} 2\nu(\zeta_0, f, \Delta') - 2 &= \sum_{i=1}^{\tau'} \left(\sum_{j=1}^{\alpha'} O(\Gamma'_i; \zeta_0, \zeta_j) - 1 \right) + v \\ &= \sum_{i=1}^{\tau'} (O(\Gamma'_i; \zeta_0, \zeta_{j(i)}) - 1) + v \\ &= \sum_{i=1}^{\tau'} (s_i - 1) + v . \end{aligned}$$

Putting $n = \nu(\zeta_0, f, \Delta')$, we have

LEMMA 1.

$$2n - 2 = \sum_{i=1}^{\tau'} (s_i - 1) + v . \tag{2.3}$$

2.2. Using Lemma 1 and (1.1) we shall show that Δ and Δ' are at least triply connected.

Let m_k denote the number of roots $z_{k,j}$ of the equation $f(z) = w_k$ restricted to Δ' and let $l_{k,j}$ be the multiplicities of $z_{k,j}$, $j = 1, 2, \cdots, m_k$. For a totally ramified value w_k $(1 \leqq k \leqq q)$, we write

$$N_k = \begin{cases} \{i \mid j(i) = j_k\}, & \text{if } w_k \in D_{j_k} \text{ for some } j_k , \\ \emptyset & , \text{ if } w_k \notin \bigcup_{j=1}^{\alpha'} D_j \end{cases}$$

and

$$\sigma_k = \text{the number of } N_k .$$

Obviously, by (2.1)

$$N_k \cap N_m = \emptyset , \qquad \text{if } k \neq m$$

and

$$0 \leqq \sigma_1 + \sigma_2 + \cdots + \sigma_q \leqq \tau' . \tag{2.4}$$

Since $O(\Gamma'_i; \zeta_0, \zeta_{j_k}) = 0$ for i with $j(i) \neq j_k$, i.e. $i \notin N_k$, the equality

$$n = \sum_{j=1}^{m_k} \ell_{k,j} + \sum_{i \in N_k} s_i \tag{2.5}$$

comes from (2.2), whence we have

(2.6) $$n \geqq \nu_k m_k + \sigma_k \qquad (k = 1, 2, \cdots, q)\,,$$

because $l_{k,j} \geqq \nu_k$ and $s_i \geqq 1$ for $i \in N_k$.
Hence

(2.7) $$n \sum_{k=1}^{q} \frac{1}{\nu_k} \geqq \sum_{k=1}^{q} m_k\,.$$

From (2.3) and (2.5), it follows that

(2.8) $$\begin{aligned} 2n - 2 &= \sum_{i=1}^{\tau'} (s_i - 1) + v \\ &\geqq \sum_{k=1}^{q} \sum_{i \in N_k} (s_i - 1) + \sum_{k=1}^{q} \sum_{j=1}^{m_k} (\ell_{k,j} - 1) \\ &= qn - \sum_{k=1}^{q} m_k - \sum_{k=1}^{q} \sigma_k\,, \end{aligned}$$

so that

(2.8)′ $$\sum_{k=1}^{q} m_k + \sum_{k=1}^{q} \sigma_k - 2 \geqq (q-2)n\,.$$

Using (1.1), (2.7) and (2.8)′, we obtain

(2.9) $$\sum_{k=1}^{q} m_k + \sum_{k=1}^{q} \sigma_k - 2 \geqq (q-2)n > n \sum_{k=1}^{q} \frac{1}{\nu_k} \geqq \sum_{k=1}^{q} m_k$$

and hence, by (2.4),

(2.10) $$\tau \geqq \tau' \geqq \sum_{k=1}^{q} \sigma_k \geqq 3\,.$$

Thus we have the following

LEMMA 2. *A simply, or doubly, connected domain $\varDelta$ does not admit any exceptionally ramified meromorphic functions satisfying the conditions* (1), (2) *and* (3).

§ 3. Classification of covering surfaces generated by exceptionally ramified meromorphic functions

3.1. For approach it is essential to determine all covering surfaces generating by an exceptionally ramified meromorphic function f with three totally ramified values on a triply connected domain $\varDelta(q = 3$ and $\tau = 3)$. With this choice of q and τ, the inequalities (2.9) and (2.10) imply

(3.1) $$n = m_1 + m_2 + m_3 + 1$$

and

(3.2) $$\tau = \tau' = \sigma_1 + \sigma_2 + \sigma_3 = 3\,.$$

The inequality in (2.8) should be equality, so that f cannot have any ramified value other than $\{w_k\}_{k=1,2,3}$. By (3.2), each D_j $(1 \leqq j \leqq \alpha')$ contains one of the $\{w_k\}_{k=1,2,3}$. Since both Δ and Δ' are triply connected, each component of $\Delta - \Delta'$ is a ring domain. The image of a component under f is contained in one of the $\{D_j\}_{j=1,\dots,\alpha'}$. Consequently $\alpha = \alpha'$.

Combining (3.1) with (2.6), we have

(3.3) $$m_1 + m_2 + m_3 + 1 \geqq \nu_k m_k + \sigma_k, \qquad k = 1, 2, 3\,.$$

There are four possibilities:

(i) $$m_1 \geqq 1\,, \quad m_2 \geqq 1\,, \quad m_3 \geqq 1\,.$$

(ii) $$m_1 \geqq 1\,, \quad m_2 \geqq 1\,, \quad m_3 = 0\,.$$

(iii) $$m_1 \geqq 1\,, \quad m_2 = m_3 = 0\,.$$

(iv) $$m_1 = m_2 = m_3 = 0\,.$$

Case (i). By (3.2) and (3.3), we have

(3.4) $$0 \geqq (\nu_1 - 3)m_1 + (\nu_2 - 3)m_2 + (\nu_3 - 3)m_3\,.$$

From (1.1) and (3.4), follow

$$\nu_1 = 2\,, \quad \nu_2 \geqq 3 \quad \text{and} \quad \nu_3 \geqq 4\,.$$

From (3.3) and (3.4) follows

(3.5) $$1 \geqq (\nu_2 - 4)m_2 + (\nu_3 - 4)m_3\,.$$

By (1.1), the following two possibilities occur

(i_α) $$\nu_2 = 4\,, \qquad \nu_3 \geqq 5$$

(i_β) $$\nu_2 = 3\,, \qquad \nu_3 \geqq 7\,.$$

Case (i_α). From (3.5), $m_3 = 1$ and $\nu_3 = 5$ follow. Hence by (3.3), there are the following possibilities:

(a) $$m_1 = 2\,, \qquad m_2 = 1\,.$$

(b) $$m_1 = 3\,, \qquad m_2 = 1\,.$$

(c) $$m_1 = 4\,, \qquad m_2 = 2\,.$$

In each case, the numbers n, $\ell_{k,j}$, σ_k, s_i are determined by (2.5), (3.1) and (3.2). Since $\sum_{i=1}^{3} s_i \geqq 3$, the case (a) does not occur.

Case (b). Since $n = 6$, we have

$$\sum_{j=1}^{3} \ell_{1,j} + \sum_{i \in N_1} s_i = \ell_{2,1} + \sum_{i \in N_2} s_i = \ell_{3,1} + \sum_{i \in N_3} s_i = 6 .$$

This implies

$$\begin{cases} n = 6, \ \ell_{1,j} = 2 \text{ for } j = 1, 2, 3, \ \ell_{2,1} = 4, \ \ell_{3,1} = 5 , \\ \sigma_1 = 0, \ \sigma_2 = 2, \ \sigma_3 = 1, \ \{s_i\}_{i \in N_2} = \{1, 1\} , \\ \{s_i\}_{i \in N_3} = \{1\} . \end{cases}$$

This covering surface is said to be of class 1.

Case (c). Similarly as above, we have

$$\begin{cases} n = 8, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 4, \ \ell_{2,1} = \ell_{2,2} = 4 , \\ \ell_{3,1} = 5, \ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 1\} . \end{cases}$$

This covering surface is said to be of class 2.

Case (i_β). The inequality (3.3) with (3.2) gives

(3.6). $$m_1 \geqq 4m_3 .$$

From (3.3), it follows that

(3.7) $$2(m_3 + 1) \geqq m_2 ,$$

so that by (3.3), (3.6) and (3.7), we have

$$m_3 = 1, \ 2 \text{ or } 3.$$

Hence, using the inequalities (3.3) and (3.6) again, we have seven possibilities:

(d) $m_1 = 4 , \quad m_2 = 2 , \quad m_3 = 1 .$

(e) $m_1 = 4 , \quad m_2 = 3 , \quad m_3 = 1 .$

(f) $m_1 = 5 , \quad m_2 = 3 , \quad m_3 = 1 .$

(g) $m_1 = 6 , \quad m_2 = 4 , \quad m_3 = 1 .$

(*h*) $m_1 = 8 , \quad m_2 = 5 , \quad m_3 = 2 .$

(i) $m_1 = 9 , \quad m_2 = 6 , \quad m_3 = 2 .$

(j) $m_1 = 12 , \quad m_2 = 8 , \quad m_3 = 3 .$

In each case, the numbers n, $\ell_{k,j}$, σ_k and s_i are determined as follows:

Case (d).
$$\begin{cases} n = 8,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 4\,, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \ell_{3,1} = 7\,, \\ \sigma_1 = 0,\ \sigma_2 = 2,\ \sigma_3 = 1\,, \\ \{s_i\}_{i \in N_2} = \{1, 1\},\ \{s_i\}_{i \in N_3} = \{1\}\,. \end{cases}$$

This covering surface is said to be of class 3.

Case (e).
$$\begin{cases} n = 9,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 4\,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 3,\ \ell_{3,1} = 7\,, \\ \sigma_1 = 1,\ \sigma_2 = 0,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_1} = \{1\}\ \{s_i\}_{i \in N_3} = \{1, 1\}\,. \end{cases}$$

This covering surface is said to be of class 4.

Case (f).
$$\begin{cases} n = 10,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 5\,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 3,\ \ell_{3,1} = 7\,, \\ \sigma_1 = 0,\ \sigma_2 = 1,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 2\}\,, \end{cases}$$

$$\begin{cases} n = 10,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 5\,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 3,\ \ell_{3,1} = 8\,, \\ \sigma_1 = 0,\ \sigma_2 = 1,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 1\}\,, \end{cases}$$

$$\begin{cases} n = 10,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 5\,, \\ \{\ell_{2,1}, \ell_{2,2}, \ell_{2,3}\} = \{3, 3, 4\},\ \ell_{3,1} = 7\,, \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 1\}\,. \end{cases}$$

These covering surfaces are said to be of classes 5, 6 and 7, respectively.

Case (g).
$$\begin{cases} n = 12,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6\,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4,\ \ell_{3,1} = 7\,, \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 3\}\,, \end{cases}$$

$$\begin{cases} n = 12,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6\,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4,\ \ell_{3,1} = 7\,, \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 2, 2\}\,, \end{cases}$$

$$\begin{cases} n = 12,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6\,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4,\ \ell_{3,1} = 8\,, \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 2\}\,, \end{cases}$$

$$\begin{cases} n = 12, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6 \,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4, \ \ell_{3,1} = 9 \,, \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 1\} \,. \end{cases}$$

These covering surfaces are said to be of classes 8, 9, 10 and 11, respectively.

Case (h).
$$\begin{cases} n = 16, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 8 \,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 5, \ \ell_{3,1} = \ell_{3,2} = 7 \,, \\ \sigma_1 = 0, \ \sigma_2 = 1, \ \sigma_3 = 2 \,, \\ \{s_i\}_{i \in N_2} = \{1\}, \ \{s_i\}_{i \in N_3} = \{1, 1\} \,. \end{cases}$$

This covering surface is said to be of class 12.

Case (i).
$$\begin{cases} n = 18, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 9 \,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 6, \ \ell_{3,1} = \ell_{3,2} = 7 \,, \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 2\} \,, \end{cases}$$
$$\begin{cases} n = 18, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 9 \,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 6, \ \{\ell_{3,1}, \ell_{3,2}\} = \{7, 8\} \,, \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 1\} \,. \end{cases}$$

These covering surfaces are said to be of classes 13 and 14, respectively.

Last case (j).

$$\begin{cases} n = 24, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 12 \,, \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 8, \ \ell_{3,j} = 7 \text{ for } j = 1 \text{ to } 3 \,, \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 1\} \,. \end{cases}$$

This covering surface is said to be of class 15.

3.2. *Case* (ii). The inequality (3.3) yields

$$m_1 + m_2 + 1 \geqq \nu_k m_k + \sigma_k \,, \qquad k = 1, 2 \,. \tag{3.8}$$

From (1.1), the following possibilities occur:

(ii$_\alpha$) $\nu_1 = 2, \ \nu_2 \geqq 3 \,.$

(ii$_\beta$) $\nu_1 \geqq 3, \ \nu_2 \geqq 3 \,.$

Case (ii$_\alpha$). The inequality (3.8) implies the following five possibilities:

(k) $m_1 = 1 \,, \quad m_2 = 1 \,, \quad \sigma_1 = \sigma_2 = 0 \,.$

(l) $m_1 = 2 \,, \quad m_2 = 1 \,, \quad \sigma_1 = \sigma_2 = 0 \,.$

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(m) $m_1 = 2\,,\quad m_2 = 1\,,\quad \sigma_1 = 0\,,\quad \sigma_2 = 1\,.$

(n) $m_1 = 1\,,\quad m_2 = 1\,,\quad \sigma_1 = 1\,,\quad \sigma_2 = 0\,.$

(o) $m_1 = 3\,,\quad m_2 = 2\,,\quad \sigma_1 = \sigma_2 = 0\,.$

Using (2.5), (3.1) and (3.2) in each case, we have:

Case (k). $\begin{cases} n = 3,\ \ell_{1,1} = \ell_{2,1} = 3,\ \sigma_3 = 3\,, \\ \{s_i\}_{i \in N_3} = \{1, 1, 1\}\,. \end{cases}$

Case (l). $\begin{cases} n = 4,\ \ell_{1,1} = \ell_{1,2} = 2,\ \ell_{2,1} = 4,\ \sigma_3 = 3\,, \\ \{s_i\}_{i \in N_3} = \{1, 1, 2\}\,. \end{cases}$

Case (m). $\begin{cases} n = 4,\ \ell_{1,1} = \ell_{1,2} = 2,\ \ell_{2,1} = 3,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 3\}\,, \end{cases}$

$\begin{cases} n = 4,\ \ell_{1,1} = \ell_{1,2} = 2,\ \ell_{2,1} = 3,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{2, 2\}\,. \end{cases}$

Case (n). $\begin{cases} n = 3,\ \ell_{1,1} = 2,\ \ell_{2,1} = 3,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_1} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 2\}\,. \end{cases}$

Case (o). $\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 3\,, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 4\}\,, \end{cases}$

$\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 3\,, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 2, 3\}\,, \end{cases}$

$\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 3, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{2, 2, 2\}\,. \end{cases}$

These covering surfaces are said to be of classes 16 to 23, respectively.

Case (ii$_\beta$). The inequality (3.8) yields $\sigma_1 = \sigma_2 = 0$ and $m_1 = m_2 = 1$, that is, the case (k).

Case (iii). The inequality (3.3) yields $m_1 = 1$. Hence we have

$$\begin{cases} n = 2,\ \ell_{1,1} = 2,\ \sigma_1 = 0,\ \sigma_2 = 1,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{2\},\ \{s_i\}_{i \in N_3} = \{1, 1\}\,. \end{cases}$$

This covering surface is said to be of class 24.

Case (iv). We have easily

$$n = 1\,,\quad \{s_i\}_{i \in N_k} = \{1\} \qquad \text{for } k = 1, 2, 3\,.$$

This surface covers univalently the base domain $\hat{C} - \bigcup_{j=1}^{3} \bar{D}_j$ and is said

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to be of class 25.

Summing up the above discussion, we state the following

LEMMA 3. *Let Δ be a triply connected domain bounded by analytic curves $\{\Gamma_i\}_{i=1,2,3}$ and let f be exceptionally ramified meromorphic on $\bar{\Delta}$ with three totally ramified values $\{w_k\}_{k=1,2,3}$ and satisfy the conditions* (1), (2) *and* (3).

Then, for the above domain Δ' mentioned, we have:

1°) *Δ' is a triply connected subdomain of Δ, and the covering surface generated by f restricted to Δ' belongs to one of the 25 classes (see Table 1).*

2°) *f has no ramified values other than $\{w_k\}_{k=1,2,3}$ in Δ'.*

3°) *Each component of $\Delta - \Delta'$ is doubly connected and its image is contained in one of the $\{D_j\}_{j=1,\dots,\alpha}$ ($\alpha \leqq 3$).*

4°) *Each D_j contains one of the $\{w_k\}_{k=1,2,3}$.*

Table 1

class	ν_1	ν_2	ν_3	m_1 $\ell_{1,j}$	m_2 $\ell_{2,j}$	m_3 $\ell_{3,j}$	n	σ_1 $\{s_i\}_{i\in N_1}$	σ_2 $\{s_i\}_{i\in N_2}$	σ_3 $\{s_i\}_{i\in N_3}$
1	2	4	5	3 $\ell_{1,j}=2$	1 $\ell_{2,1}=4$	1 $\ell_{3,1}=5$	6	0	2 {1, 1}	1 {1}
2	2	4	5	4 $\ell_{1,j}=2$	2 $\ell_{2,j}=4$	1 $\ell_{3,1}=5$	8	0	0	3 {1, 1, 1}
3	2	3	7	4 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	8	0	2 {1, 1}	1 {1}
4	2	3	7	4 $\ell_{1,j}=2$	3 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	9	1 {1}	0	2 {1, 1}
5	2	3	7	5 $\ell_{1,j}=2$	3 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	10	0	1 {1}	2 {1, 2}
6	2	3	7	5 $\ell_{1,j}=2$	3 $\ell_{2,j}=3$	1 $\ell_{3,1}=8$	10	0	1 {1}	2 {1, 1}
7	2	3	7	5 $\ell_{1,j}=2$	3 $\{\ell_{2,1}, \ell_{2,2}, \ell_{2,3}\}=\{3, 3, 4\}$	1 $\ell_{3,1}=7$	10	0	0	3 {1, 1, 1}
8	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	12	0	0	3 {1, 1, 3}
9	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	12	0	0	3 {1, 2, 2}
10	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=8$	12	0	0	3 {1, 1, 2}
11	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=9$	12	0	0	3 {1, 1, 1}

class	ν_1	ν_2	ν_3	m_1 $\ell_{1,j}$	m_2 $\ell_{1,j}$	m_3 $\ell_{3,j}$	n	σ_1 $\{s_i\}_{i\in N_1}$	σ_1 $\{s_i\}_{i\in N_2}$	σ_3 $\{s_i\}_{i\in N_3}$
12	2	3	7	8 $\ell_{1,j}=2$	5 $\ell_{2,j}=3$	2 $\ell_{3,j}=7$	16	0	1 $\{1\}$	2 $\{1, 1\}$
13	2	3	7	9 $\ell_{1,j}=2$	6 $\ell_{2,j}=3$	2 $\ell_{3,j}=7$	18	0	0	3 $\{1, 1, 2\}$
14	2	3	7	9 $\ell_{1,j}=2$	6 $\ell_{2,j}=3$	2 $\{\ell_{3,1}, \ell_{3,2}\}=\{7, 8\}$	18	0	0	3 $\{1, 1, 1\}$
15	2	3	7	12 $\ell_{1,j}=2$	8 $\ell_{2,j}=3$	3 $\ell_{3,j}=7$	24	0	0	3 $\{1, 1, 1\}$
16	2	3		1 $\ell_{1,1}=3$	1 $\ell_{2,1}=3$	0	3	0	0	3 $\{1, 1, 1\}$
17	2	3		2 $\ell_{1,j}=2$	1 $\ell_{2,1}=4$	0	4	0	0	3 $\{1, 1, 2\}$
18	2	3		2 $\ell_{1,j}=2$	1 $\ell_{2,1}=3$	0	4	0	1 $\{1\}$	2 $\{1, 3\}$
19	2	3		2 $\ell_{1,j}=2$	1 $\ell_{2,1}=3$	0	4	0	1 $\{1\}$	2 $\{2, 2\}$
20	2	3		1 $\ell_{1,1}=2$	1 $\ell_{2,1}=3$	0	3	1 $\{1\}$	0	2 $\{1, 2\}$
21	2	3		3 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	0	6	0	0	3 $\{1, 1, 4\}$
22	2	3		3 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	0	6	0	0	3 $\{1, 2, 3\}$
23	2	3		3 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	0	6	0	0	3 $\{2, 2, 2\}$
24	2			1 $\ell_{1,1}=2$	0	0	2	0	1 $\{2\}$	2 $\{1, 1\}$
25				0	0	0	1	1 $\{1\}$	1 $\{1\}$	1 $\{1\}$

§ 4. Key Lemma

4.1. We form a Cantor set in the usual manner. Let $\{\xi_n\}$ be a sequence of positive numbers satisfying $0<\xi_n<2/3$, $n=1, 2, 3, \cdots$. We remove first an open interval of length $(1-\xi_1)$ from the interval $I_{0,1}$: $[-1/2, 1/2]$, so that on both sides there remains a closed interval of length $\xi_1/2\equiv\eta_1$. The remained intervals are denoted by $I_{1,1}$ and $I_{1,2}$. Inductively we remove an open interval of length $(1-2\eta_n)\prod_{p=1}^{n-1}\eta_p$, with $\eta_p=(1/2)\xi_p$ $(p=1, 2, \cdots)$, from each $I_{n-1,k}$, $k=1, 2, \cdots, 2^{n-1}$, so that on both sides there remains a closed interval of length $\prod_{p=1}^{n}\eta_p$. The remained intervals are denoted by

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$I_{n,2k-1}$ and $I_{n,2k}$. By repeating this procedure endlessly, we obtain an infinite sequence of closed intervals $\{I_{n,k}\}_{n=1,2,\ldots,\ k=1,2,\ldots,2^n}$. The set given by

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} I_{n,k}$$

is said to be the Cantor set in the interval $I_{0,1}$ with successive ratios ξ_n.

Set

$$S_{n,k} = \left\{ z \,\middle|\, \prod_{p=1}^{n} \eta_p < |z - z_{n,k}| < \frac{1}{3} \prod_{p=1}^{n-1} \eta_p \right\}$$

and

$$\Gamma_{n,k} = \left\{ z \,\middle|\, |z - z_{n,k}| = \prod_{p=1}^{n-1} \eta_p \sqrt{\frac{\eta_n}{3}} \right\},$$

where $z_{n,k}$ is the midpoint of $I_{n,k}$. Denoting by $\mu_n = \mu(S_{n,k})$ the harmonic modulus of $S_{n,k}$, we have

$$\mu_n = \log \frac{1}{3\eta_n} = \log \frac{2}{3\xi_n} . \tag{4.1}$$

We give Lemma 4 which will be a key of our proof of Theorem.

LEMMA 4. *Let E be the Cantor set with successive ratios ξ_n satisfying the condition*

$$\lim_{n\to\infty} \xi_n = 0 .$$

Let f be an exceptionally ramified meromorphic function in the complement E^c. Then, for a sufficiently large $n(\geqq L_1)$, we have with a positive constant M depending only on E and f,

$$|f(\Gamma_{n,k})| < M \exp(-\mu_n/2) .$$

In order to prove Lemma 4, we use Lemma 5 due to L. Carleson and K. Matsumoto.

LEMMA 5. *Let f be meromorphic in an annulus $\bar{R}$: $1 \leqq |z| \leqq \exp \mu$ $(0 < \mu < \infty)$. If the image $f(\bar{R})$ is contained in the open disc $D(\zeta_0, d)$ with center ζ_0 and radius d $(0 < d < 1/2)$, then by putting $L = \{|z| = \exp \mu/2\}$ we have with some positive constant A depending only on d*

$$|f(L)| < A \exp(-\mu/2) ,$$

whenever μ is sufficiently large $(\mu \geqq \mu_0)$.

Moreover, we can choose A with

$$A = O(d) \quad as \quad d \to 0$$

(cf. L. Sario and K. Noshiro [14], 128–129).

4.2. *Proof of Lemma* 4. Since f is exceptionally ramified in E^c, f is normal. Hence, denoting by $d\sigma_{E^c}(z)$ (resp. $d\sigma_{S_{n,k}}$) the element of hyperbolic length of E^c (resp. $S_{n,k}$), we have

$$\{|f'(z)|/(1 + |f(z)|^2)\}|dz| \leqq C d\sigma_{E^c}(z) \leqq C d\sigma_{S_{n,k}}(z) ,$$

in E^c with some constant C depending only on f and E (cf. O. Lehto and K. I. Virtanen [7]). Denote by $\zeta = \phi\ (z)$ the conformal mapping of $S_{n,k}$ onto G': $1 < |\zeta| < \exp \mu_n$ and put $g(\zeta) = f(\phi^{-1}(\zeta))$. Both of $d\sigma_{S_{n,k}}(z)$ and $\{|f'(z)|/(1 + |f(z)|^2)\}|dz|$ are conformally invariant, so that

$$\{|g'(\zeta)|/(1 + |g(\zeta)|^2)\}|d\zeta| < C d\sigma_{G'}(\zeta) = \left\{C\pi/2\mu_n|\zeta| \sin\left(\frac{\pi}{\mu_n}\log|\zeta|\right)\right\}|d\zeta| .$$

Denoting by $L_{n,k}^{(1)}$ and $L_{n,k}^{(2)}$ the inverse images of L'_{ν_0} $:|\zeta| = \exp \nu_0$ and of L''_{ν_0} $:|\zeta| = \exp(\mu_n - \nu_0)$ under ϕ, respectively, we have

$$\begin{aligned}
&\int_{z \in L_{n,k}^{(1)}} \{|f'(z)|/(1 + |f(z)|^2)\}|dz| \\
&\quad = \int_{\zeta \in L'_{\nu_0}} \{|g'(\zeta)|/(1 + |g(\zeta)|^2)\}|d\zeta| \\
&\quad < \int_{\zeta \in L'_{\nu_0}} \left\{C\pi/2\mu_n|\zeta| \sin\left(\frac{\pi}{\mu_n}\log|\zeta|\right)\right\}|d\zeta| \\
&\quad = \int_0^{2\pi} \left\{C\pi/2\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right)\right\}d\theta \\
&\quad = C\pi^2/\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right) .
\end{aligned}$$

Similarly,

$$\int_{z \in L_{n,k}^{(2)}} \{|f'(z)|/(1 + |f(z)|^2)\}|dz| < C\pi^2/\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right) .$$

We take a fixed ν_0 with $\nu_0 > 32C$ and a sufficiently large n with $\mu_n > \bar{\mu} = \max(\mu_0, \nu_0)$ $(n \geqq L_2)$. From

$$C\pi^2/\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right) < \frac{C\pi}{\nu_0} + \frac{\pi}{32} < \frac{\pi}{16}$$

follow

$$|f(L^{(1)}_{n,k})| < \tfrac{1}{16} \quad \text{and} \quad |f(L^{(2)}_{n,k})| < \tfrac{1}{16}\,.$$

Hence there are discs D_i with $|D_i| < 1/8$ such that $D_i \supset f(L^{(i)}_{n,k})$ $(i = 1, 2)$. Lemma 2 implies therefore that, with the ring domain $T_{n,k}$ bounded by $L^{(1)}_{n,k}$ and $L^{(2)}_{n,k}$,

$$\nu(w, f, T_{n,k}) = 0 \qquad \text{for } w \in \hat{C} - (\bar{D}_1 \cup \bar{D}_2)\ ,$$

because if $\nu(w, f, T_{n,k}) \geqq 1$ for $w \in \hat{C} - (\bar{D}_1 \cup \bar{D}_2)$, f is not exceptionally ramified.

Consequently,

$$\bar{D}_1 \cap \bar{D}_2 \neq \phi \quad \text{and} \quad f(\bar{T}_{n,k}) \subset \bar{D}_1 \cup \bar{D}_2\,.$$

Applying Lemma 5 to f in $T_{n,k}$, we obtain the desired inequality

$$\begin{aligned} |f(\Gamma_{n,k})| &< A \exp\{-\tfrac{1}{2}(\mu_n - 2\nu_0)\} \\ &= Ae^{\nu_0} \exp(-\mu_n/2) = M \exp(-\mu_n/2)\ , \end{aligned}$$

where $M = Ae^{\nu_0}$.

§5. Proof of Theorem

5.1. Assuming that, for a Cantor set E satisfying our condition (1.2), there is an exceptionally ramified meromorphic function f in E^c with an essential singularity at each point of E, we shall arrive at a contradiction. By our previous result [5], f must have just three totally ramified values $\{w_i\}_{i=1,2,3}$.

Set

(5.1) $$\delta = \tfrac{1}{72} \min_{k \neq m} \chi(w_k, w_m)$$

and

(5.2) $$\delta_n = M \exp(-\mu_n/2)\,.$$

By our condition (1.2), there exists a positive integer L_3 such that, for $n \geqq L_3$,

(5.3) $$\delta_n < \delta$$

and

(5.4) $$\delta_{n+1} < \tfrac{1}{2}\delta_n\,.$$

Further, by Lemma 4, we can choose, for any $n \geqq L_4 = \max(L_1, L_2, L_3)$, discs $D_{n,k}$ with $|D_{n,k}| < 2\delta_n$ containing $f(\Gamma_{n,k})$. The union $\tilde{D} \equiv D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$ consists of at most three, say α $(1 \leqq \alpha \leqq 3)$, components, which are covered by discs $\{D_{n,k}^{(j)}\}_{j=1,\dots,\alpha}$ with $D_{n,k}^{(1)} \supset D_{n,k}$, $|D_{n,k}^{(1)}| = 12\delta_n$ and $|D_{n,k}^{(j)}| = 12\delta_{n+1}$ for $j \neq 1$. Here we may assume that there are no ramified values of f on $\partial D_{n,k}^{(j)}$. Denote by $\Delta_{n,k}$ the triply connected domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. When the restriction of f to $\Delta_{n,k}$ takes no values outside $\tilde{D}$, then $\alpha = 1$ and the image of $\Delta_{n,k}$ is contained in $D_{n,k}^{(1)}$. In this case, we say that $\Delta_{n,k}$ is degenerate (f).

Suppose that the restriction of f to $\Delta_{n,k}$ takes values outside $\tilde{D}$. Then we see from 4°) of Lemma 3 that each component of $\tilde{D}$ contains just one of the $\{w_i\}_{i=1,2,3}$, so that the center of $D_{n,k}^{(j)}$ can be taken at the point $w_{\lambda_j} \in \{w_i\}_{i=1,2,3}$, the totally ramified value contained in the corresponding component of $\tilde{D}$. This show that $\{D_{n,k}^{(j)}\}_{j=1,\dots,\alpha}$ are mutually disjoint. We choose a triply connected subdomain $\Delta'_{n,k}$ of $\Delta_{n,k}$ corresponding to Δ' of Lemma 3. It is always known that the covering surface generated by f on $\Delta'_{n,k}$ belongs to one of the 25 classes. Each component of $\Delta_{n,k} - \Delta'_{n,k}$ is doubly connected and its image is contained in one of the $\{D_{n,k}^{(j)}\}_{j=1,\dots,\alpha}$. If the covering surface is of class m, $\Delta_{n,k}$ and $\Delta'_{n,k}$ are said to be of class m. Generically these $\Delta_{n,k}$ are said to be non-degenerate (f).

Let $\Delta_{n,k}$ be non-degenerate (f). The boundary curves of $\Delta'_{n,k}$ are denoted by $\check{\gamma}_{n,k}$, $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$, homotopic to $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, respectively. Each of them is a component of the inverse image of some $\partial D_{n,k}^{(j)}$ under f and said to be of w_{λ_j}-type (f). Assuming that $\check{\gamma}_{n,k}$ $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$ are positively oriented, we set for $\zeta_0 \in \hat{C} - \bigcup_{j=1}^{\alpha} D_{n,k}^{(j)}$, $\zeta_j \in D_{n,k}^{(j)}$ $(j = 1$ to $\alpha)$,

$$\check{s}_{n,k} = \sum_{j=1}^{\alpha} O(\check{\gamma}_{n,k};\, \zeta_0, \zeta_j)\ ;$$

$$\hat{s}_{n+1,2k-i} = \sum_{j=1}^{\alpha} O(\hat{\gamma}_{n+1,2k-i};\, \zeta_0, \zeta_j) \qquad (i = 0, 1)\ .$$

5.2. The centers of $D_{n,k}^{(j)}$ are totally ramified values $w_{\lambda_j} \in \{w_i\}_{i=1,2,3}$ for any $\Delta_{n,k}$ being non-degenerate (f), while $D_{n,k}^{(1)}$ might contain no values $\{w_i\}_{i=1,2,3}$ for $\Delta_{n,k}$ being degenerate (f). However $D_{n,k}^{(1)}$ stay always considerably near one of the $\{w_i\}_{i=1,2,3}$.

PROPOSITION. *Let $\Delta_{n,k}$ be degenerate (f). Then $D_{n,k}^{(1)}$, the disc covering $f(\Delta_{n,k})$, is contained in one of the $\{D(w_i, 24\delta_n)\}_{i=1,2,3}$.*

Proof of Proposition. Suppose that $D^{(1)}_{n,k} \not\subset \bigcup_{i=1}^{3} D(w_i, 24\delta_n)$. Since $|D^{(1)}_{n,k}| = 12\delta_n$,

$$D^{(1)}_{n,k} \subset \left\{\bigcup_{i=1}^{3} D(w_i, 12\delta_n)\right\}^c,$$

so that

$$f(\varDelta_{n,k}) \subset \left\{\bigcup_{i=1}^{3} D(w_i, 12\delta_n)\right\}^c.$$

By (5.4) and this inclusion

$$D^{(1)}_{n+1,2k-j} \subset \left\{\bigcup_{i=1}^{3} D(w_i, 12\delta_{n+1})\right\}^c \qquad (j = 0, 1)\,.$$

This shows that $D^{(1)}_{n+1,2k-j}$ $(j = 0, 1)$ contain no totally ramified values $\{w_i\}_{i=1,2,3}$ and $\varDelta_{n+1,2k-j}$ must be degenerate (f). Therefore

$$f(\bar{\varDelta}_{n,k} \cup \bar{\varDelta}_{n+1,2k-1} \cup \bar{\varDelta}_{n+1,2k}) \subset D^{(1)}_{n+1} \cup D^{(1)}_{n+1,2k-1} \cup D^{(1)}_{n+1,2k}\,,$$

which imply

$$|f(\bar{\varDelta}_{n,k} \cup \bar{\varDelta}_{n+1,2k-1} \cup \bar{\varDelta}_{n+1,2k})| < 12\delta_n + 24\delta_{n+1}\,.$$

By repeating this procedure, we have

$$\begin{aligned} |f((\varGamma_{n,k}) - E)| &\leqq 12\delta_n + 24(\delta_{n+1} + \delta_{n+2} + \cdots) \\ &< 36\delta_n < \tfrac{1}{2}\min_{k\neq m} \chi(w_k, w_m) < \sqrt{2}\,, \end{aligned}$$

where $(\varGamma_{n,k})$ denotes the domain bounded by $\varGamma_{n,k}$ (see (5.1), (5.3), (5.4)).

We may assume that f is bounded in $(\varGamma_{n,k})$, because if necessary, we take a certain linear transformation of f in place of f. Since E is of linear measure zero, $(\varGamma_{n,k}) \cap E$ must be removable for any bounded analytic function (cf. A. S. Besicovitch [1]). This contradicts our assumption that each point of E is an essential singularity of f.

5.3. Now assume that infinitely many of $\varDelta_{n,k}$ are non-degenerate (f). Then there are $\varDelta_{n,k}$'s being non-degenerate (f) with $n \geqq L_4$. We take such a fixed $\varDelta_{n,k}$. Let the boundary curves $\hat{\gamma}_{n+1,2k}$ and $\hat{\gamma}_{n+1,2k-1}$ of $\varDelta'_{n,k}$ be of w_λ-type (f) and of $w_{\lambda'}$-type (f), respectively. Here we may assume that $\hat{s}_{n+1,2k-1} \geqq \hat{s}_{n+1,2k}$ and that $\lambda \geqq \lambda'$ if $\hat{s}_{n+1,2k-1} = \hat{s}_{n+1,2k}$. From Table 1 we see that $\hat{s}_{n+1,2k} = 1$ or 2.

The adjacent domain $\varDelta_{n+1,2k}$ will be either

(A) degenerate (f)

or

(B) non-degenerate (f).

Case (A). Let $\hat{\Delta}_{n+1,2k}$ be the triply connected domain bounded by $\hat{\gamma}_{n+1,2k}$, $\Gamma_{n+2,4k-1}$ and $\Gamma_{n+2,4k}$. By virtue of the maximum principle, Proposition implies

$$f(\hat{\Delta}_{n+1,2k}) \subset D(w_\lambda, 6\delta_{n+p}) ,$$

where $p=0$ or $p=1$ according to $f(\hat{\gamma}_{n+1,2k}) \subset \partial D^{(1)}_{n,k}$ or $f(\hat{\gamma}_{n+1,2k}) \subset \partial D^{(j)}_{n,k}$ $(j \neq 1)$. We choose the component $J_{n+1,2k}$ of the inverse image $f^{-1}(R(w_\lambda, 24\delta_{n+2}, 6\delta_{n+p}))$ in $\hat{\Delta}_{n+1,2k}$ having $\hat{\gamma}_{n+1,2k}$ as a boundary curve, where $R(w_\lambda, 24\delta_{n+2}, 6\delta_{n+p}) = \{\zeta \,|\, 24\delta_{n+2} < \chi(\zeta, w_\lambda) < 6\delta_{n+p}\}$. From Lemma 2, it is easy to see that the boundary of $J_{n+1,2k}$ outside $\hat{\gamma}_{n+1,2k}$ is mapped onto $C(w_\lambda, 24\delta_{n+2})$ under f. We shall show that the boundary of $J_{n+1,2k}$ outside $\hat{\gamma}_{n+1,2k}$ consists of

(A_1) one boundary curve $\kappa_{n+1,2k}$ separating $\Gamma_{n+2,4k-1} \cup \Gamma_{n+2,4k}$ from $\hat{\gamma}_{n+1,2k}$

or

(A_2) two boundary curves $\kappa_{n+2,4k-1}$ and $\kappa_{n+2,4k}$ separating $\Gamma_{n+2,4k-1}$ and $\Gamma_{n+2,4k}$ from $\Gamma_{n+2,4k} \cup \hat{\gamma}_{n+1,2k}$ and $\Gamma_{n+2,4k-1} \cup \hat{\gamma}_{n+1,2k}$,

respectively.

In fact, we assume contrary that $J_{n+1,2k}$ has boundary curves β_i $(i=1, \cdots, h)$ other than the above, then each β_i is homotopic to zero. Set

$$s_{i,j} = O(\kappa_{i,j}; \zeta_0, w_\lambda) \quad \text{and} \quad t_i = O(\beta_i; \zeta_0, w_\lambda)$$

for $\zeta_0 \in \hat{C} - \bar{D}(w_\lambda, 24\delta_{n+2})$, where $\kappa_{i,j}$ and β_i are positively oriented. Applying the argument principle to f in $J_{n+1,2k}$, we have

$$\hat{s}_{n+1,2k} = s_{n+1,2k} + \sum_{i=1}^{h} t_i \qquad \text{in the case } (A_1)$$

or

$$\hat{s}_{n+1,2k} = s_{n+2,4k-1} + s_{n+2,4k} + \sum_{i=1}^{h} t_i \qquad \text{in the case } (A_2)\,.$$

Since $\hat{s}_{n+1,2k} = 1$ or 2, $s_{i,j} \geqq 1$ and

$$t_i = O(-\beta_i; w_\lambda, \zeta_0) = \nu(w_\lambda, f, (-\beta_j)) \geqq \nu_\lambda \geqq 2\,,$$

which is a contradiction.

Case (A_1). The domain $J_{n+1,2k}$ is doubly connected. By the Hurwitz formula, f have no ramified values on $J_{n+1,2k}$. Hence $J_{n+1,2k}$ is conformally equivalent to

$$R^* = \left\{\zeta \,\middle|\, \left\{\frac{24\delta_{n+2}}{\sqrt{1-24^2\delta_{n+2}^2}}\right\}^{(\hat{s}_{n+1,2k})^{-1}} < |\zeta| < \left\{\frac{6\delta_{n+j}}{\sqrt{1-36\delta_{n+j}^2}}\right\}^{(\hat{s}_{n+1,2k})^{-1}}\right\}.$$

We have

$$\mu(J_{n+1,2k}) = \mu(R^*) . \tag{5.5}$$

As well-known, $\mu(J_{n+1,2k})$ is dominated by the harmonic modulus of the extremal domain of Teichmüller, i.e.

$$\mu(J_{n+1,2k}) \leqq \log 16\left(\frac{r_2}{r_1} + 1\right) = \log 16\left(\frac{2}{\xi_{n+1}} - 1\right), \tag{5.6}$$

where $r_1 = \prod_{p=1}^{n+1} \eta_p$ and $r_2 = \prod_{p=1}^{n} \eta_p(1 - 2\eta_{n+1})$ (cf. O. Lehto and K. I. Virtanen [6] 55–62).

Hence, by (4.1), (5.2), (5.5) and (5.6), we have

$$\log 16\left(\frac{2}{\xi_{n+1}} - 1\right) \geqq \log \left\{\frac{\delta_{n+1}}{8\delta_{n+2}}\right\}^{(\hat{s}_{n+1,2k})^{-1}},$$

$$\left\{16\left(\frac{2}{\xi_{n+1}} - 1\right)\right\}^2 \geqq \frac{1}{8}\sqrt{\frac{\xi_{n+1}}{\xi_{n+2}}},$$

so

$$\xi_{n+2} \geqq \frac{\xi_{n+1}^5}{2^{22}(2-\xi_{n+1})^4}.$$

This inequality contradicts our assumption (1.2), for a sufficiently large n, which imply that (A_1) cannot occur.

Case (A_2). The domain $J_{n+1,2k}$ is triply connected. In this case, $\hat{s}_{n+1,2k} = 2$. From Table 1 we see that $\Delta_{n,k}$ is of classes 9, 19, 22 or 23 and $\lambda = 3$. The domain $\Delta_{n+2,4k}$ is degenerate (f). In fact, assume that $\Delta_{n+2,4k}$ is non-degenerate (f). Then f takes the value w_λ in the ring domain $R'_{n+2,4k}$ bounded by $\kappa_{n+2,4k}$ and $\check{\gamma}_{n+2,4k}$, and by virtue of the argument principle

$$7 \leqq \nu_\lambda \leqq \nu(w_\lambda, f, R'_{n+2,4k}) = s_{n+2,4k} + \check{s}_{n+2,4k} \leqq 5,$$

which is a contradiction. Let f be restricted to the domain $\hat{\Delta}_{n+2,4k}$ bounded by $\kappa_{n+2,4k}$, $\Gamma_{n+3,8k-1}$ and $\Gamma_{n+3,8k}$ and let $J_{n+2,4k}$ be the component of the inverse image of $R(w_\lambda, 24\delta_{n+2}, 24\delta_{n+1})$, one of whose boundary curves is $\kappa_{n+2,4k}$. Since $s_{n+2,4k} = 1$, (A_1) is only possible for $J_{n+2,4k}$, that is, $J_{n+2,4k}$ is doubly connected. In the same way as above, we conclude that (A_2) cannot occur.

In conclusion, $\Delta_{n+1,2k}$ must be non-degenerate (f), i.e., of the case (B).

5.4. *Case* (B). Suppose that both of $\Delta_{m,n}$ and $\Delta_{m+1,2n}$ are non-degenerate (f) and $\hat{\gamma}_{m+1,2n}$ is of w_λ-type (f). By the argument principle

$$\nu_\lambda \leqq \nu(w_\lambda, f, R_{m+1,2n}) = \hat{s}_{m+1,2n} + \check{s}_{m+1,2n}\ , \tag{5.7}$$

where $R_{m+1,2n}$ denotes the domain bounded by $\hat{\gamma}_{m+1,2n}$ and $\check{\gamma}_{m+1,2n}$. The inequality (5.7) will be useful in this paragraph.

The (B) is divided into the following four cases.

(B_1) $\Delta_{n,k}$ is of classes 1 or 2.

(B_2) $\Delta_{n,k}$ is of classes 3, 4, $\cdots$, 22 or 23.

(B_3) $\Delta_{n,k}$ is of class 24.

(B_4) $\Delta_{n,k}$ is of class 25.

Case (B_1). The adjacent domain $\Delta_{n+1,2k}$ must be of classes 1, 2, 24 or 25. From Table 1 we see

$$\hat{s}_{n+1,2k} = 1\ , \qquad \check{s}_{n+1,2k} = 1 \text{ or } 2\ .$$

These equalities and (5.7) give

$$\nu_\lambda \leqq 3\ . \tag{5.8}$$

On the other hand, since $\lambda = 2$ or 3, we have $\nu_\lambda = 4$ or 5. This contradicts (5.8).

Case (B_2). By (5.7), we have

$$\nu_\lambda \leqq 2 + 4 = 6\ ,$$

so that

$$\lambda = 1 \text{ or } 2\ .$$

This implies that $\Delta_{n,k}$ cannot be of classes 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21, 22 and 23.

In the case $\lambda = 1$, $\Delta_{n,k}$ is of class 20 and $\hat{s}_{n+1,2k} = 1$. Hence $\Delta_{n+1,2k}$ is of classes 4, 20 or 25.

In the case $\lambda = 2$, $\Delta_{n,k}$ is of classes 3, 5, 18 or 19 and $\hat{s}_{n+1,2k} = 1$. Hence $\Delta_{n+1,2k}$ is of classes 3, 5, 6, 12, 18, 19, 24 or 25. We see that $\Delta_{n+1,2k}$

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is of class 24 in the following way. Assume that $\Delta_{n+1,2k}$ is of classes 3, 5, 6, 12, 18, 19 or 25, then (5.7) gives $3 \leqq \nu_2$ and $\nu_2 \leqq 2$, which is impossible.

In either case, at least one of $\{\hat{\gamma}_{n+2,4k-1}, \hat{\gamma}_{n+2,4k}\}$, say $\hat{\gamma}_{n+2,4k}$, is of w_3-type and $\hat{s}_{n+2,4k} = 1$. Assuming that $\Delta_{n+2,4k}$ is non-degenerate (f), we are led to a contradiction $7 \leqq \nu_3 \leqq 1 + 4 = 5$. However $\Delta_{n+2,4k}$ is not degenerate (f). Both cases cannot occur.

Case (B_3). In this case, $\hat{s}_{n+1,2k} = 1$ and $\lambda = 3$. By (5.7), we have $7 \leqq \nu_3$ and $\nu_3 \leqq 5$, which is impossible.

Case (B_4). We have shown that no $\Delta_{n,k}$'s of other classes than 25 class appear. It follows that $\Delta_{n+1,2k}$ and $\Delta_{n+2,4k}$ are also of class 25. By (5.7) we have

$$2 \leqq \nu_\lambda \leqq \nu(w_\lambda, f, R_{n+1,2k}) = 1 + 1 = 2$$

and

$$2 \leqq \nu_{\lambda'} \leqq \nu(w_{\lambda'}, f, R_{n+2,4k}) = 1 + 1 = 2\,,$$

which contradict (1.1), because $\lambda' \neq \lambda$.

Thus the case (B) also cannot occur. Consequently, there exists a positive integer N $(\geqq L_4)$ such that every $\Delta_{n,k}$ $(n \geqq N, k = 1, 2, \cdots, 2^n)$ is degenerate (f).

5.5. Finally, we take a fixed $n(\geqq N)$. Since $\Delta_{n+p,q}$ is degenerate (f), we have $f(\bar{\Delta}_{n+p,q}) \subset D^{(1)}_{n+p,q}$. For any $z \in (\Gamma_{n,k}) - E$, there is a chain of $\{\Delta_{n+p,q}\}$ connecting $\Delta_{n,k}$ to z. The diameter of the chain $\leqq 12\,(\delta_n + \delta_{n+1} + \cdots + \delta_{n+m} + \cdots) \leqq 24\delta_n$, because $|D^{(1)}_{n+p,q}| = 12\delta_{n+p}$ and $\delta_{n+p+1} < (1/2)\delta_{n+p}$ (see (5.4)). Hence

$$f((\Gamma_{n,k}) - E) \subset D(w_0, 24\delta_n)\,,$$

where $w_0 \in f(\bar{\Delta}_{n,k})$, that is,

$$|f((\Gamma_{n,k}) - E)| < 48\delta_n < 48\delta < \sqrt{2}\,.$$

We may assume that f is bounded in $(\Gamma_{n,k})$, because if necessary, we take a certain linear transformation of f in place of f. The Cantor set E is of linear measure zero, so that $(\Gamma_{n,k}) \cap E$ is removable for f. This contradicts our assumption that each point of E is an essential singularity of f.

The proof of Theorem is thus complete.

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