ON A TYPE OF SASAKIAN SPACE

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Introduction

A Sasakian space [1] M^n $(n = 2m+1)$ is a Riemannian n-space with a positive definite metric tensor g_{ij} and a unit Killing vector field η which satisfies

$$\eta_{j,kl} = \eta_k g_{lj} - \eta_j g_{lk} \tag{1}$$

where the comma denotes covariant differentiation with respect to the metric tensor. In a recent paper [2] M. C. Chaki and A. N. Roy Chowdhury studied conformally recurrent spaces of second order, or briefly conformally 2-recurrent spaces, that is, non-flat Riemannian spaces V_n $(n > 3)$ defined by

$$C_{kji}{}^{h}{}_{,lm} = a_{lm} C_{kji}{}^{h} \tag{2}$$

where $C_{kji}{}^{h}$ is the conformal curvature tensor:

$$C_{kji}{}^{h} = R_{kji}{}^{h} - \frac{1}{n-2}(g_{ji}R_k{}^h - g_{ki}R_j^h + R_{ji}\delta_k{}^h - R_{ki}\delta_j{}^h) + \frac{R}{(n-1)(n-2)}(g_{ji}\delta_k{}^h - g_{ki}\delta_j{}^h) \tag{3}$$

and a_{lm} is a tensor not identically zero.

The present paper deals with conformally 2-recurrent Sasakian spaces, that is, Sasakian spaces in which (2) is satisfied. It is proved that such an n-space $(n > 3)$ is of constant curvature.

1. Some formulas in a Sasakian Space

Since in a Sasakian space η is a unit vector field

$$\eta^r \eta_r = 1 \tag{1.1}$$

Applying Ricci's identity to η_j we obtain

$$\eta_{j,kl} - \eta_{j,lk} = -\eta_r R_{lkj}{}^r \tag{1.2}$$

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where $R_{kji}{}^{h}$ is the Riemannian curvature tensor:

$$R_{kji}{}^{h} = \frac{\partial}{\partial x^k}\begin{Bmatrix} h \\ ji \end{Bmatrix} - \frac{\partial}{\partial x^j}\begin{Bmatrix} h \\ ki \end{Bmatrix} + \begin{Bmatrix} h \\ kr \end{Bmatrix}\begin{Bmatrix} r \\ ji \end{Bmatrix} - \begin{Bmatrix} h \\ jr \end{Bmatrix}\begin{Bmatrix} r \\ ki \end{Bmatrix}$$

Using (1) we can express (1.2) as

$$\eta_r R_{lkj}{}^{r} = \eta_l g_{jk} - \eta_k g_{jl} \tag{1.3}$$

Contracting (1.3) with g^{jl} we have

$$\eta_r R_k{}^{r} = (n-1)\eta_k \tag{1.4}$$

Thus in a Sasakian space the formulas (1.1), (1.3) and (1.4) hold.

2. Conformally 2-recurrent Sasakian space

Let us suppose that a Sasakian space is conformally 2-recurrent. If possible, let a conformally 2-recurrent Sasakian space be not conformally flat. It has been proved in theorem 1 of [2] that if a conformally 2-recurrent space with positive definite metric is not conformally flat, then its tensor of recurrence a_{lm} is symmetric. Hence if a conformally 2-recurrent Sasakian space be not conformally flat, then

$$C_{kji}{}^{h}{}_{,lm} - C_{kji}{}^{h}{}_{,ml} = 0 \tag{2.1}$$

Applying Ricci's identity to the left-hand side of (2.1) we get

$$C_{kji}{}^{r} R_{mlr}{}^{h} - C_{kjr}{}^{h} R_{mli}{}^{r} - C_{kri}{}^{h} R_{mlj}{}^{r} - C_{rji}{}^{h} R_{mlk}{}^{r} = 0$$

Transvecting this with $\eta_h \eta^m$ and using (1.1) and (1.3) we have

$$C_{kjil} - \eta_r \eta_l C_{kji}{}^{r} + \eta_r \eta_i C_{kjl}{}^{r} + \eta_j \eta_r C_{kli}{}^{r} + \eta_r \eta_k C_{lji}{}^{r} + g_{lj}\eta^r \eta_s C_{rki}{}^{s} - g_{lk}\eta^r \eta_s C_{rji}{}^{s} = 0 \tag{2.2}$$

Contracting this with g^{lj} we get

$$\eta^r \eta_s C_{rki}{}^{s} = 0 \tag{2.3}$$

Substituting this value in (2.2) we have

$$C_{kjil} - \eta_r \eta_l C_{kji}{}^{r} + \eta_r \eta_i C_{kjl}{}^{r} + \eta_j \eta_r C_{kli}{}^{r} + \eta_k \eta_r C_{lji}{}^{r} = 0 \tag{2.4}$$

Now,

$$C_{kji}{}^{r}\eta_r = \frac{1}{n-2}\left\{\left(\frac{R}{n-1} - 1\right)\eta_k g_{ji} - \eta_j g_{ki}) - (\eta_k R_{ji} - \eta_j R_{ki})\right\} \tag{2.5}$$

Using (2.5) we can express (2.4) as

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$$(2.6) \qquad C_{kjil} + \frac{1}{n-2}\left[\left(\frac{R}{n-1} - 1\right)(\eta_k g_{jl} - \eta_j g_{kl}) - (\eta_k R_{jl} - \eta_j R_{kl})\right]\eta_i = 0$$

Contracting this with g^{ik} we get

$$R_{lj} = \left(\frac{R}{n-1} - 1\right) g_{lj} + \left(n - \frac{R}{n-1}\right)\eta_l \eta_j$$

Substituting this value in (2.6) we have $C_{kjil} = 0$. But this is contrary to hypothesis.

Hence if a Sasakian space is conformally 2-recurrent, then it is conformally flat. It has been proved by Okumura [3] that a conformally flat Sasakian space is of constant curvature. We can therefore state the following theorem.

THEOREM. *A conformally 2-recurrent Sasakian space is of constant curvature.*

In theorem 9 of [2], it has been proved that every n-dimensional ($n > 3$) projective 2-recurrent space, that is a Riemannian space in which Weyl's projective curvature tensor:

$$W_{kji}{}^h = R_{kji}{}^h - \frac{1}{n-1}(\delta_k{}^h R_{ji} - \delta_j{}^h R_{ki})$$

satisfies the relation $W_{kji}{}^h{}_{,lm} = a'_{lm} W_{kji}{}^h$ for a non-zero tensor a'_{lm}, is a conformally 2-recurrent space. We have therefore the following corollary of the above theorem:

COROLLARY. *A projective 2-recurrent Sasakian space is of constant curvature.*

References

[1] S. Sasaki, *Lecture note on almost contact manifolds* Tohoku University (1965).

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[3] M. Okumura, 'Some remarks on space with a certain contact structure', *Tohoku Math. J.* 14 (1962), 135—145.

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