

BIRATIONAL RIGIDITY OF ORBIFOLD DEGREE 2 DEL PEZZO FIBRATIONS

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Abstract. Varieties fibered into del Pezzo surfaces form a class of possible outputs of the minimal model program. It is known that del Pezzo fibrations of degrees 1 and 2 over the projective line with smooth total space satisfying the so-called K^2 -condition are birationally rigid: their Mori fiber space structure is unique. This implies that they are not birational to any Fano varieties, conic bundles, or other del Pezzo fibrations. In particular, they are irrational. The families of del Pezzo fibrations with smooth total space of degree 2 are rather special, as for *most* families a general del Pezzo fibration has the simplest orbifold singularities. We prove that orbifold del Pezzo fibrations of degree 2 over the projective line satisfying explicit generality conditions as well as a generalized K^2 -condition are birationally rigid.

§1. Introduction

Birational classification of complex algebraic varieties is a central research area in algebraic geometry. Given an algebraic variety, one can produce a somewhat simpler birational model of it by first taking a resolution of singularities and then running the minimal model program (MMP). We work in dimension 3 and over the field of complex numbers, where both these theories are settled. The result of this procedure is either a Mori fiber space or a minimal model, depending on whether the initial variety was uniruled or not. We are interested in explicit classification of Mori fiber spaces, that is, the study of birational relations among Mori fiber spaces as end points of the MMP. A Mori fiber space can be a unique product of the MMP, so-called *Birationally Rigid*, and can have a few, or infinitely many birational models (see Definition 1.2 for a precise definition of rigidity). For example, a smooth quartic in \mathbb{P}^4 is known to be birationally rigid [18], whereas a quartic with a single cA_2 singular point has precisely two birational models [12]. On the other hand, the projective space \mathbb{P}^3 is birational to any Fano variety V_{22} , whose moduli contain an uncountable set (see, e.g., [23]).

A Mori fiber space is a normal projective variety X together with a morphism $\pi: X \rightarrow S$ with connected fibers to a lower-dimensional variety S , where

- X is \mathbb{Q} -factorial with terminal singularities,
- $-K_X$ is π -ample, and
- the relative Picard number $\rho_{X/S}$ is 1.

Based on the dimension of S , either X is a conic bundle over the surface S , or it is a fibration of del Pezzo surfaces over a curve S , or it is a Fano threefold when S is a point.

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DEFINITION 1.1. Let $\pi_X: X \rightarrow S$ and $\pi_Y: Y \rightarrow Z$ be Mori fiber spaces. A birational map $\chi: X \dashrightarrow Y$ is called *square* if it fits into a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\chi} & Y \\ \pi_X \downarrow & & \downarrow \pi_Y \\ S & \xrightarrow{g} & Z, \end{array}$$

where g is birational and, in addition, the induced map on the generic fibers $\chi_\eta: X_\eta \dashrightarrow Y_\eta$ is an isomorphism. In this case, we say that X/S and Y/Z are square-birational.

DEFINITION 1.2 [11, Def. 1.2]. A Mori fiber space $\pi_X: X \rightarrow S$ is said to be *birationally rigid* if existence of a birational map $\chi: X \dashrightarrow Y$ to a Mori fiber space $\pi_Y: Y \rightarrow Z$ implies that there exists a birational self-map $\varphi: X \dashrightarrow X$ such that the birational map $\chi \circ \varphi$ is square.

Birational rigidity of conic bundles has been studied extensively (see, e.g., [31]). The reader is encouraged to consult [11, §4] for an overview and some interesting related questions on birational geometry of conic bundles. Comparatively, the question of (stable) nonrationality of conic bundles has recently had more development [4], [8], [16]. Birational rigidity and nonrationality of Fano threefolds has progressed in the past decades, but is yet to be completed (see [17], [25] for rationality and [6] for birational rigidity and investigate the references therein for further details). The focus of this paper is on birational rigidity of threefold del Pezzo fibrations.

del Pezzo fibrations form 10 classes, according to their degrees $1 \leq d \leq 9$, and there are two classes in degree 8. If the degree is 5 or higher, then the fibration is rational over the base [10], [24]. Degree 4 fibrations admit a conic bundle structure [7], and hence they are not birationally rigid. Their stable rationality was recently studied in [17].

A smooth degree 2 del Pezzo surface can be defined as the zeros of a quartic form in $\mathbb{P}(1, 1, 1, 2)$, and a general such form defines a smooth del Pezzo surface of degree 2. However, when defined over a base curve, this typically has some orbifold singularities of type $\frac{1}{2}(1, 1, 1)$ enforced by the zeros of the coefficient of the quadratic monomial in the defining polynomial of the threefold (see [30, §1] for notation and explanation of this type of singularities). This coefficient degree imposes a discrete invariant that splits the moduli space of such fibrations into an infinite set of families. As a result, whenever we use the term general, we mean general after fixing a family. In low degrees, most results on birational rigidity concentrate on smooth models, for example, stable rationality of very general del Pezzo fibrations in low degrees with smooth total space was recently studied in [22]. Smoothness of the total space is a strong restriction in degrees 1 and 2 as they are satisfied in very few families. However, a general cubic surface (degree 3 del Pezzo) fibration has smooth total space, which is no longer true in degrees 1 and 2.

The following is the main result in the literature for birational rigidity of del Pezzo fibrations over \mathbb{P}^1 with smooth total space, which was later improved slightly by Grinenko [13]–[15] and Sobolev [33], also with the smoothness condition.

THEOREM 1.3 [28, Th. 2.1]. *Let $\pi: X \rightarrow \mathbb{P}^1$ be a del Pezzo fibration of degree 1, 2, or 3, with smooth X , and assume generality in degree 3. If K_X^2 is not in the interior of the Mori cone of effective 1-cycles $\overline{NE}(X)^\circ$, then X is birationally rigid.*

This theorem is the center of attention in this article. The only birational rigidity statement for del Pezzo fibrations of low degree with quotient singularities is proved in [21], which proves Pukhlikov's theorem above for special families of fibrations with such singularities. We generalize these results for the general case with orbifold families of del Pezzo fibrations in degree 2 in the following precise setting. There is a fundamental difference in the geometry of the models in the general case, which makes the proof more complicated: the singular points produce elementary Sarkisov links to other models square-birational to X , and that the construction of the staircase (explained below) for smooth points near the singular point is more complex.

MAIN THEOREM. Let $\pi: X \rightarrow \mathbb{P}^1$ be a del Pezzo fibration of degree 2. Suppose X is a general quasi-smooth hypersurface in a $\mathbb{P}(1, 1, 1, 2)$ -bundle over \mathbb{P}^1 satisfying the K^2 -total condition. Then X is birationally rigid.

1.1 Conditions in the Main Theorem

Here, we spell out assumptions in the theorem above, and explain their necessity.

1.1.1. Quasi-smoothness

Quasi-smooth means that no singularities come from the defining equation of X , and hence all singularities are indeed of type $\frac{1}{2}(1, 1, 1)$ and inherited from the ambient toric variety (see Remark 1.5 for a precise definition). This condition cannot be removed as there exists an example of a nonbirationally rigid degree 2 del Pezzo fibration satisfying other conditions that has a nontoric singular point [2].

1.1.2. The K^2 -total condition

The K^2 -condition is precisely as in Theorem 1.3, that is, $K_X^2 \notin \overline{\text{NE}}(X)^\circ$. For every singular point Q of X , there is a Sarkisov link starting by blowing up X at Q and resulting in a new quasi-smooth model of X , that is, square-birational. This manoeuvre is explicitly described in §4. Let N be the number of singularities of X and denote the singular points by Q_i for $i \in \{1, \dots, N\}$. Denoting by X_I the model acquired by combining the elementary links corresponding to Q_i , $i \in I$, $I \subset \{1, \dots, N\}$, we say X satisfies the K^2 -total condition if for every $I \subset \{1, \dots, N\}$ the model X_I satisfies the K^2 -condition. We are convinced that K^2 -condition on X may be enough, and the totality assumption is redundant. In §4, we give an explicit recipe for constructing all X_I models from X . Given X embedded in a $\mathbb{P}(1, 1, 1, 2)$ -bundle, it is rather easy to check this condition using the recipe. Note that in [32, Cor. 6.4], it is stated that X is birationally rigid if the K^2 -condition holds for any del Pezzo fibration $X' \rightarrow \mathbb{P}^1$ such that there is a square-birational map $g: X \dashrightarrow X'$. Of course, this result is impossible to apply in practice, and usually one works with only X in order to show birational rigidity. However, in our situation, we have explicit descriptions of all necessary models (for this check to be carried out) as described in §4.

1.1.3. Generality

For each singular point, let the fiber containing the point, that is defined by a quartic F in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$, be given by

$$wq(x, y, z) + r(x, y, z) = 0,$$

where q and r are homogeneous polynomials of degrees 2 and 4, respectively. We require that the intersection $q(x, y, z) = r(x, y, z) = 0$ on \mathbb{P}^2 consists of eight distinct points, for all singular points of X . Let $\sigma: \tilde{F} \rightarrow F$ be the blowup of F at the point $(0:0:0:1)$. We also require that \tilde{F} is smooth. This generality condition is necessary in our computations, as otherwise there are several arrangements of singularities and curves of low degree on F to be considered in the “exclusion” process of the proof. In a few cases we have checked, this condition can be dropped. Upon improving the techniques, we believe that the generality condition can be dropped altogether. Note that the models considered in [21] are special cases in our setting as for them $q = 0$. They admit no Sarkisov link and in a sense are “more rigid.”

1.2 Method of the proof

Birational rigidity is often proved via the method of maximal singularities [29]. Roughly speaking, the method goes as follows: assume there is a birational map $\chi: X \dashrightarrow Y$, where $Y \rightarrow Z$ is another Mori fiber space. Then consider \mathcal{H} , the transform of a very ample complete linear system \mathcal{H}' on Y . Essentially, \mathcal{H} is mobile and there exists a rational number $n > 0$ and a divisor A pulled back from the base of the fibration such that $\mathcal{H} \subset | -nK_X + A |$. It follows that the pair $(X, \frac{1}{n}\mathcal{H})$ is not canonical (this is standard; see, e.g., the discussion after Theorem 1.9 in [11]). This implies that there exists a valuation E with center $C \subset X$ such that

$$m_E(\mathcal{H}) > na_E(K_X),$$

where $a_E(K_X)$ is the discrepancy of E with respect to K_X and m_E is the multiplicity of \mathcal{H} along C . This inequality together with the geometry of X is then used to exclude many centers (curves and points on X) to satisfy this conditions by concluding various contradictions. Pukhlikov in [28, §6] studied multiplicities on towers of blowups of center on X and introduced the *construction of the staircase* to achieve the desired contradiction. Corti in [11, §5] refines Pukhlikov’s methods for the situations where the staircase technique is not necessary to reduce the computations without the tower by introducing a new inequality. We use a combination of these two techniques in §§5 and 6 to exclude all smooth centers; we then have to use the staircases together with Corti’s inequality at various stages to obtain our results. We then show that the singular points produce Sarkisov links to other models X_I of the del Pezzo fibration $\pi: X \rightarrow \mathbb{P}^1$. We show that there is at least one X_I with centers only at the smooth points, which allows a combination of techniques of Corti and Pukhlikov to be efficiently used.

1.3 Models

Suppose $X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration of degree 2, and view X as a hypersurface of bi-degree $(d, 4)$ in a toric variety T with Picard group \mathbb{Z}^2 , where Cox ring of T is given by the following data:

- (i) the homogeneous coordinate ring of T is $\text{Cox}(T) = \mathbb{C}[u, v, x, y, z, t]$,
- (ii) with the irrelevant ideal $I = (u, v) \cap (x, y, z, t)$ and

(iii) the grading given by the columns of the matrix

$$\begin{pmatrix} u & v & x & y & z & w \\ 1 & 1 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta$ are integers. As this description is invariant under an action of $\mathrm{SL}(2, \mathbb{Z})$ on the weight matrix, we can rescale to the following:

$$\begin{pmatrix} u & v & x & y & z & w \\ 1 & 1 & 0 & a & b & c \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix},$$

where $0 \leq a \leq b$ are positive integers and $c \in \mathbb{Z}$. We denote this toric variety by $\mathbb{P}(1, 1, 1, 2)_{(0, a, b, c)}$, and say X , the hypersurface, has bi-degree $(l, 4)$ in the new coordinate weights.

REMARK 1.4. Note that, once a, b , and c are fixed, not all values of l can define a suitable hypersurface. For example, if $\frac{l}{4} < a, b, \frac{c}{2}$, where a, b, c are positive integers, then the equation of X is divisible by x , making X reducible. There are various restrictions that one must take into account to have a suitable del Pezzo fibration. We refer to §5 in [1] for various cases that are not suitable. For us, the set of discrete invariants that make sense in the Main Theorem are stated in Proposition 1.7 and in the Appendix.

DEFINITION 1.5. Assume that the discrete invariants define a suitable del Pezzo fibration. Then quasi-smoothness means that the singular locus of the affine cover of X in \mathbb{C}^6 is included entirely in the irrelevant locus defined by I .

REMARK 1.6. A general member of a family of hypersurfaces of bi-degree $(4, l)$ in $\mathbb{P}(1, 1, 1, 2)_{(0, a, b, c)}$ is smooth only if $l = 2c$. Note that this is because such a hypersurface does not intersect the singular curve of the ambient toric variety. This confirms how restrictive the smoothness condition is for these varieties.

For more analysis of this construction, and a complete list of those models that admit a Sarkisov link of Type III or IV, we refer the reader to [1].

The following results describe some $X \subset T$, as above, which satisfy K^2 -total condition in the Main Theorem.

PROPOSITION 1.7. *Let X be a hypersurface of bi-degree $(4, \ell)$ in a $\mathbb{P}_{0, a, b, c}(1, 1, 1, 2)$ -bundle over \mathbb{P}^1 . Suppose*

$$7\ell/2 > 2a + 4b + 3c + 8,$$

then X satisfies K^2 -total condition.

COROLLARY 1.8. *Suppose $c \geq 2b$ and X does not satisfy the K^2 -total condition. Then X belongs to one of the 20 families with a, b, c, ℓ from the following table. In particular, if $c \geq 2b$, X satisfies the generality conditions, and X is not birationally rigid, then X belongs to one of these 20 families.*

| a | b | c | ℓ |
|-----|-----|-----|--------|
| 0 | 0 | 0 | 0,1,2 |
| 0 | 0 | 1 | 2,3 |
| 0 | 0 | 2 | 4 |
| 0 | 1 | 2 | 4,5 |
| 0 | 1 | 3 | 6 |
| 0 | 2 | 4 | 8 |
| 1 | 1 | 2 | 4,5 |
| 1 | 1 | 3 | 6 |
| 1 | 2 | 4 | 8 |
| 2 | 2 | 4 | 8,9 |
| 2 | 2 | 5 | 10 |
| 2 | 3 | 6 | 12 |
| 3 | 3 | 6 | 12 |
| 4 | 4 | 8 | 16 |

1.4 Convention

We denote numerical equivalence by \equiv and use $\sim_{\mathbb{Q}}$ for linear equivalence of \mathbb{Q} -divisors. All varieties are algebraic, normal, and defined over \mathbb{C} unless stated otherwise. For a rational map $\chi : Y \dashrightarrow X$, we denote the proper transform on Y of an object on X as $\chi_*^{-1}(A)$.

§2. Preliminary results: Singularities of pairs

In this section, we recall the notion of singularities of pairs. We also state some results which let us find relations between multiplicities and singularities.

Let X be an algebraic variety, possibly nonprojective and singular. Let E be a prime divisor on X . Then it is easy to associate a discrete valuation ν_E of $\mathbb{C}(X)$ corresponding to E by

$$\nu_E(f) = \text{mult}_E(f) \text{ for } f \in \mathbb{C}(X).$$

DEFINITION 2.1 [27]. Let $\varphi : \tilde{X} \rightarrow X$ be a projective birational morphism, and let ν be a discrete valuation of $\mathbb{C}(X)$. We say that a triple (\tilde{X}, φ, E) is a *realization* of the discrete valuation ν if E is a prime divisor on \tilde{X} and $\nu_E = \nu$. Then $\varphi(E)$ is called the *center* of the discrete valuation ν_E on X .

Note that if X is projective, then every discrete valuation of the field $\mathbb{C}(X)$ admits a center on X , which does not depend on the realization.

DEFINITION 2.2. Let D be a \mathbb{Q} -divisor on X . The *multiplicity* of a discrete valuation ν at D is

$$\nu(D) = \text{mult}_E \varphi^*(D) \in \mathbb{Q}$$

for some realization (\tilde{X}, φ, E) of ν . If the center of ν on X is of codimension ≥ 2 , then we can write

$$\varphi^*(D) = \varphi_*^{-1}(D) + \nu(D)E + \sum a_i E_i,$$

where E_i are the exceptional divisors of φ that differ from E and $a_i \in \mathbb{Q}$. It is important to note that the multiplicity does not depend on the choice of the realization.

DEFINITION 2.3. Let D be a \mathbb{Q} -divisor on X such that $K_X + D$ is \mathbb{Q} -Cartier. Let $\pi : \tilde{X} \rightarrow X$ be a birational morphism, and let $\tilde{D} = \pi_*^{-1}(D)$ be the proper transform of D . Then

$$K_{\tilde{X}} + \tilde{D} \sim \pi^*(K_X + D) + \sum_E a(E, X, D)E,$$

where E runs through all distinct exceptional divisors of π on \tilde{X} and $a(E, X, D)$ are rational numbers. The number $a(E, X, D) = a(\nu_E, X, D)$ is called the *discrepancy* of the divisor E (discrete valuation ν_E) with respect to the pair (X, D) .

DEFINITION 2.4. Let D be a divisor on X . We say that the pair (X, D) is *terminal* (resp. *canonical*) at a discrete valuation ν with a center on X if $a(E, X, D) > 0$ (resp. $a(E, X, D) \geq 0$) for some realization (\tilde{X}, φ, E) of ν . We say that the pair (X, D) is terminal (resp. canonical) at a subvariety Z if it is terminal (resp. canonical) at every discrete valuation ν on $K(X)$ such that a center of ν on X is Z . We say that the pair (X, D) is terminal (resp. canonical) if it is terminal (resp. canonical) at every subvariety of codimension ≥ 2 . If $D = 0$, we simply say that X has only terminal (resp. canonical) singularities.

DEFINITION 2.5. Let \mathcal{M} be a linear system, not necessarily mobile, on X . We say that the pair $(X, \lambda\mathcal{M})$ is terminal (resp. canonical) if for every subvariety Z of codimension ≥ 2 and for a general $D \in \mathcal{M}$, the pair $(X, \lambda D)$ is terminal (resp. canonical) at Z .

REMARK 2.6. Consider the pair (X, \mathcal{M}) . Let $f : Y \rightarrow X$ be a projective birational morphism, let E_i be the exceptional divisors of f , and let $\tilde{\mathcal{M}}$ be the proper transform of \mathcal{M} on Y . Then

$$K_Y + \tilde{\mathcal{M}} - \sum a(E_i, X, \mathcal{M})E_i \sim f^*(K_X + \mathcal{M}).$$

The pair

$$(Y, \tilde{\mathcal{M}} - \sum a(E_i, X, \mathcal{M})E_i)$$

is called the *log pullback* of the pair (X, \mathcal{M}) . It follows from the definition that the log pullback of the pair has the same singularities as the pair. Note that we view $\tilde{\mathcal{M}} - \sum a(E_i, X, \mathcal{M})E_i$ as a multiple of a linear system with fixed components.

LEMMA 2.7 [9, Th. 1.8]. Let \mathcal{M} be a mobile linear system on \mathbb{C}^2 . Let C be a curve passing through the origin. Suppose the pair $(\mathbb{C}^2, \frac{1}{n}\mathcal{M} - \alpha C)$ is not canonical at 0, then for general $D_1, D_2 \in \mathcal{M}$,

$$\text{mult}_0 D_1 \cdot D_2 > 4n^2\alpha.$$

PROPOSITION 2.8 (Corti’s inequality [11, Th. 3.12]). Let $\sum F_i \subset \mathbb{C}^3$ be a reduced surface. Let \mathcal{M} be a mobile linear system on \mathbb{C}^3 , and let $Z = D_1 \cdot D_2$ be the intersection of general divisors $D_1, D_2 \in \mathcal{M}$. Write $Z = Z_h + \sum Z_i$, where the support of Z_i is contained in F_i and

Z_h intersects $\sum F_i$ properly. Let $\gamma_i > 0$ be rational numbers such that the pair $(\mathbb{C}^3, \frac{1}{n}\mathcal{M} - \sum \gamma_i F_i)$ is not canonical. Then there are positive rational numbers $0 < t_i \leq 1$ such that

$$\text{mult}_0 Z_h + \sum t_i \text{mult}_0 Z_i > 4n^2(1 + \sum \gamma_i t_i \text{mult}_0 F_i).$$

REMARK 2.9. Decomposition of $Z = Z_h + \sum Z_i$ is not unique, but the inequality holds for any choice of the decomposition. Note that it is not a requirement that $\sum F_i$ is a normal crossing divisor, nor do we ask the surfaces F_i to be smooth.

§3. Rigidity of del Pezzo fibrations

3.1 Noether–Fano method for del Pezzo fibrations

Let us first describe the framework of proving birational rigidity for del Pezzo fibrations. Let $\pi : X \rightarrow \mathbb{P}^1$ be a del Pezzo fibration (satisfying K^2 -condition), and let $\chi : X \dashrightarrow Y$ be a birational map to a variety admitting a Mori fiber space $\pi_Y : Y \rightarrow Z$. Let H be a very ample divisor on Y , and let $\mathcal{M} = \chi_*^{-1}(|H|)$; we say that \mathcal{M} is a *mobile linear system associated to* χ . There are nonnegative numbers $n \in \mathbb{Z}$ and $l \in \mathbb{Q}$ such that $\mathcal{M} \subset |-nK_X + nlF|$, where F is a fiber of π .

The core of the Noether–Fano method is the following. Suppose χ is not an isomorphism, then the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical. There are several possibilities for the centers of singularities of this pair: a singular point of X , a curve passing through a singular point of X , any other curve of X , and a nonsingular point of X . To prove X is birationally rigid, one uses the geometric properties of X to exclude a center or untwist χ using that center. Excluding goes by proving that the pair is canonical at the center, and untwisting composes χ with some square-birational map $\varphi : X \dashrightarrow X'$, where $\chi \circ \varphi^{-1}$ has lower Sarkisov degree than χ (see [11, §2.1] for the definition). Thus, the mobile linear system associated to $\chi \circ \varphi^{-1}$ is less complicated, and the model X is replaced by X' .

We do this analysis for orbifold del Pezzo fibrations of degree 2. Let X be a quasi-smooth hypersurface in a $\mathbb{P}(1, 1, 1, 2)$ -bundle over \mathbb{P}^1 . Suppose also that $\pi : X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration of degree 2 and that X satisfies the K^2 -condition.

3.2 Step 1: Curves not passing through singular locus of X

This part is well known by the work of Pukhlikov [28].

PROPOSITION 3.1 [28, §3, Cases 2 and 4]. *Let C be a curve on X which is not a section of π . Suppose also that C does not pass through singular points of X . Then the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C .*

PROPOSITION 3.2 [28, §3, Case 2]. *Let C be a section of π which does not pass through a singular point of X . Then either the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at C or there exists a birational involution φ_C such that*

- *there are numbers $n' < n$ and l' such that $\varphi_C(\mathcal{M}) \subset |-n'K_X + n'l'F|$, and*
- *the pair $(X, \frac{1}{n'}(\varphi_C)(\mathcal{M}))$ is canonical at C .*

COROLLARY 3.3 [28, §3]. *There is a birational map $\varphi : X \dashrightarrow X$ such that for the linear system $\varphi(\mathcal{M}) \subset |-n'K_X + n'l'|$, the pair $(X, \frac{1}{n'}\varphi(\mathcal{M}))$ is canonical at curves in $X \setminus \text{Sing } X$ and $\varphi = \varphi_{C_1} \circ \dots \circ \varphi_{C_m}$, where C_i are sections of X .*

Proof. Apply Proposition 3.1 and then apply Proposition 3.2 as many times as necessary. The process terminates since for every application of Proposition 3.2, we have $n' < n$. \square

In what follows, we write χ instead of $\chi \circ \varphi^{-1}$ to simplify notation, and safely assume it is canonical at curves in the smooth locus. In the remainder of this article, we show that points and curves passing through singular points are either excluded or untwisted, as explained in the following steps.

3.3 Step 2: Singular points of X and curves passing through them

Singular points of X are cyclic quotient singularities. By the famous result of Kawamata on extremal extractions from such points, if the center of noncanonicity contains the singular point, then it is only that point and cannot be a curve (see §4). So it remains to study the singular points as centers. In §4, we construct birational maps to other models of del Pezzo fibration $\varphi_I : X \dashrightarrow X_I$, with the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_I} & X_I \\ \pi \downarrow & & \downarrow \pi_I \\ \mathbb{P}^1 & \xrightarrow{g} & \mathbb{P}^1, \end{array}$$

being commutative. Let $\mathcal{M}_I \subset |-nK_{X_I} + nI F|$ be the linear system corresponding to the map $\chi \circ \varphi_I$. We prove that the pair $(X_I, \frac{1}{n}\mathcal{M}_I)$ may only be noncanonical at nonsingular points of X_I for appropriate choice of I . Furthermore, we show that by construction X_I satisfies the generality conditions in the Main Theorem. This untwists χ for singular points. From that point on, with an abuse of notation, instead of χ , we work with $\chi \circ \varphi_I$ and call it χ .

3.4 Step 3: Non-singular points of X

After Steps 1 and 2, we conclude that if $(X, \frac{1}{n}\mathcal{M})$ is not canonical at a subvariety B , then B is a point and X is smooth at B . This may indeed occur if χ is square-birational. If we assume that χ is not square-birational, then we can use Proposition 5.1 to get a stronger requirement: $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is not canonical at B , where F is a fiber containing B and γ is a number depending on \mathcal{M} . Recall that there are lines in the fibers that contain singular points of X which intersect $-K_X$ with $\frac{1}{2}$. If B does not lie on a such line, in particular, when F does not contain singularities of X , then Corti's inequality together with Proposition 5.1 gives us the necessary contradiction to exclude B . This is the approach of [11], and we cover it in §5. Sections 6–8 are entirely devoted to the remaining case: excluding the possibility that the center is a nonsingular point on a curve in a fiber intersecting $-K_X$ by $\frac{1}{2}$. As the reader expects, this is the most complicated case. We combine the approach of [11] with the *staircase* used for cubic fibrations in [28, §§6 and 7] to exclude these nonsingular points.

§4. Square maps from $\frac{1}{2}(1, 1, 1)$ -points

In this section, we explicitly work out the square-birational maps produced by blowing up the singular points. Let X be the hypersurface of bi-degree $(\ell, 4)$ given by the equation

$$f(u, v)w^2 + q(x, y, z; u, v)w + r(x, y, z; u, v) = 0$$

in the toric variety $T = \mathbb{P}(1_x, 1_y, 1_z, 2w)_{(0, a, b, c)} / \mathbb{P}_{u:v}^1$.

4.1 Simple case: X has one singular point

First, for simplicity, suppose $f(u, v) = u$. Then the fiber F given by $u = 0$ contains the singular point of X . We now describe each map of the following diagram:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\psi} & \bar{U} \\ \sigma \downarrow & & \downarrow \bar{\sigma} \\ U & \xrightarrow{\varphi} & U \end{array}$$

The variety U is the open subset of X given by $v \neq 0$. It can be viewed as the locus $\{f(u, 1)w^2 + q(x, y, z; u, 1)w + r(x, y, z; u, 1) = 0\}$ inside $\mathbb{P}(1_x, 1_y, 1_z, 2_w) \times \mathbb{C}_u^1$. The map σ is the Kawamata blowup of U along the $\frac{1}{2}(1, 1, 1)$ -point [20]. We may describe \tilde{U} as a hypersurface in \tilde{V} , where \tilde{V} is a quasi-projective toric variety with $\text{Cox } \tilde{V} = \mathbb{C}[x, y, z, w, u, \bar{u}]$. The grading of this ring is given by the matrix

$$\begin{pmatrix} u & x & y & z & w & \bar{u} \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 2 & 1 & 1 & 1 & 0 & -2 \end{pmatrix}$$

and irrelevant ideal $(u, x, y, z) \cap (w, \bar{u}) \cap (x, y, z, w)$ (see [3, §3.2] for a general treatment of Cox rings of blowups of rank 2 toric varieties). Note that we are looking at the open set $\{v \neq 0\}$, and hence the rank drops to 2 and the ideal simplifies. We may well-form [3, §2] the matrix above and rewrite it as

$$\begin{pmatrix} u & x & y & z & w & \bar{u} \\ 0 & 1 & 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The equation of \tilde{U} is

$$uw^2 + q(x, y, z; \bar{u}u)w + \bar{u}r(x, y, z; \bar{u}u) = 0,$$

where $q(x, y, z; u) = q(y, x, z; u, 1)$ and $r(x, y, z; u) = r(x, y, z; u, 1)$. To produce a Sarkisov link on \tilde{U} , we run a 2-ray game on the ambient space (\tilde{V} to begin with) and restrict it to the threefold [5]. Note that the 2-ray game corresponds to the variation of Geometric Invariant Theory (GIT), inevitably changing $(u, x, y, z) \cap (w, \bar{u})$ part in the irrelevant ideal. The component (x, y, z, w) in the ideal remains unchanged, and preserves the fibers (the 2-ray game is relative over the base of the fibration). The first step of the game on \tilde{V} does not produce a 2-ray game on \tilde{U} , caused by the fact that the equation of the hypersurface belongs to the irrelevant component (w, \bar{u}) , and hence we are forced to do an unprojection (preserving \tilde{U}). This embeds \tilde{U} in the toric variety defined by the Cox ring $\mathbb{C}[x, y, z, w, u, \bar{u}]$ graded by

$$\begin{pmatrix} u & t & x & y & z & w & \bar{u} \\ 0 & 2 & 1 & 1 & 1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The irrelevant ideal is $(u, t, x, y, z) \cap (w, \bar{u}) \cap (x, y, z, w, t)$, and \tilde{U} is a complete intersection of two hypersurfaces:

$$uw + q(x, y, z; \bar{u}u) = \bar{u}t \quad \text{and} \quad -wt = r(x, y, z; \bar{u}u).$$

The toric 2-ray game now restricts to a 2-ray game on \tilde{U} . The first step is the flop of eight lines, which corresponds to the eight solutions of $\{q(x, y, z, 0) = r(x, y, z, 0) = 0\} \subset \mathbb{P}^2$, mapping \tilde{U} to \bar{U} by crossing the $(1, 0)$ wall. Recall that the solutions of $\{q(x, y, z, 0) = r(x, y, z, 0) = 0\}$ are distinct because of the generality conditions in the Main Theorem.

The map $\bar{\sigma}$ is defined as

$$(x, y, z, w, t, u, \bar{u}) \mapsto (u^{\frac{1}{2}}x, u^{\frac{1}{2}}y, u^{\frac{1}{2}}z, w, ut, u\bar{u}).$$

It contracts the proper transform of F into a $\frac{1}{2}(1, 1, 1)$ -point. Thus, we see that the equations of the image of $\bar{\sigma}$ are

$$\bar{u}t = w + q(x, y, z; \bar{u}) \quad \text{and} \quad wt + r(x, y, z; \bar{u}) = 0.$$

We can eliminate the variable w , and thus we have only one equation

$$\bar{u}t^2 - q(x, y, z; \bar{u})w + r(x, y, z; \bar{u}) = 0.$$

Clearly, this variety is isomorphic to U . Thus, the map φ is a birational automorphism. We may write the map φ in coordinates as

$$(x, y, z, w, u) \mapsto \left(x, y, z, -w + \frac{q(x, y, z; u)}{u}; u \right).$$

In particular, we see that φ is a local involution.

4.2 Birational models of X : Square-birational maps

Note that the description of maps and models in the simple case in §4.1 works out for any calculations around the local ring near the point in the image of the singular point at the base curve. We use this to describe all models X_I .

Embed X into V , a toric $\mathbb{P}(1, 1, 1, 2, 2)$ -bundle over \mathbb{P}^1 , as a complete intersection as follows. Let N be the degree of $f \in \mathbb{C}[u, v]$. Let the weight matrix of the toric variety V be

$$\begin{pmatrix} v & u & x & y & z & w & s \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & a & b & c & c+N \end{pmatrix}$$

with irrelevant ideal $(u, v) \cap (x, y, z, w, s)$, and suppose the equations of X are

$$s = f(u, v)w + q(x, y, z; u, v), \quad sw + r(x, y, z; u, v) = 0.$$

Note that the variable s can be eliminated altogether using the first equation, and then what we are left with is the model presented in §1.3.

The quasi-smoothness of X implies that f has N distinct zeros, and hence it can be written in the form $f(u, v) = l_1(u, v) \dots l_N(u, v)$, a product of distinct linear forms.

Following the Sarkisov link described in §4.1, we can construct $X_{\{1\}}$, the variety obtained at the end of the Sarkisov link starting from the Kawamata blowup of the singular point in the fiber $l_1 = 0$:

The threefold $X_{\{1\}}$ is defined as a complete intersection defined by equations

$$l_1s = l_2 \dots l_N w + q(x, y, z; u, v) \quad \text{and} \quad sw + r(x, y, z; u, v) = 0$$

in the toric fivefold V_1 with grading

$$\begin{pmatrix} v & u & x & y & z & w & s \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & a & b & c+1 & c+N-1 \end{pmatrix}.$$

Note that the irrelevant ideal remains unchanged. Similarly, $X_I \subset V_{|I|}$ is defined by

$$\prod_{i \in I} l_i s = \prod_{j \notin I} l_j w + q(x, y, z; u, v) \quad \text{and} \quad sw + r(x, y, z; u, v) = 0,$$

where $V_{|I|}$ is graded by

$$\begin{pmatrix} v & u & x & y & z & w & s \\ 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & a & b & c+|I| & c+N-|I| \end{pmatrix}.$$

At last, we denote by $\pi_I : X_I \rightarrow \mathbb{P}_{u,v}^1$ the restriction of the projection $V_{|I|} \rightarrow \mathbb{P}_{u,v}^1$.

While this defines the birational maps between, say, X_I and $X_{I \cup \{j\}}$ for $j \notin I$, the description of this birational map as an elementary Sarkisov link is identical to the case in §4.1, after a suitable change of coordinates.

REMARK 4.1. The extremal contraction $\tilde{X} \rightarrow X$ is constructed so that the image of the exceptional locus is a terminal quotient singularity [20, Th. 5], and locally is the Kawamata blowup of that point. This is a unique extremal extraction according to Kawamata. Then the 2-ray game is followed by a simple Atiyah flop, in particular, it does not change the singularities of \tilde{X} . In short, the links we described here are unique elementary Sarkisov links starting at $\frac{1}{2}(1, 1, 1)$ -points, and, as it can be seen from the coordinate description of the birational map, the links we constructed are birational involutions in analytic neighborhood of the central fiber. In particular, it follows that $\pi_I : X_I \rightarrow \mathbb{P}_{u,v}^1$ is a del Pezzo fibration.

REMARK 4.2. It is clear from the symmetric equation of X_I described above that $X_I \cong X_J$, where $J = \{1, \dots, N\} \setminus I$. Furthermore, in particular, $X_{\{1, \dots, N\}} \cong X$.

4.3 Canonicity at $\frac{1}{2}(1, 1, 1)$ -point

Let $\mathcal{M} \subset |-nK_X + lF|$ be the mobile linear system associated to a birational map $\chi : X \dashrightarrow Y$ to a total space Y of a Mori fiber space. Let \mathcal{M}_I be the proper transform of \mathcal{M} on X_I . We now prove that there is an $I \subset \{1, \dots, N\}$ for which the linear system \mathcal{M}_I is canonical at $\frac{1}{2}(1, 1, 1)$ -points and at curves passing through them.

PROPOSITION 4.3 [20, Th. 5]. *Let $g : \tilde{X} \rightarrow X$ be the blowup at a $\frac{1}{2}(1, 1, 1)$ -point Q , let D be an effective \mathbb{Q} -divisor on X , and let E_Q be the exceptional divisor of g . Then a pair (X, D) is canonical at Q if and only if it is canonical at ν_{E_Q} , that is, $a(E_Q, X, D) \geq 0$.*

COROLLARY 4.4. *Suppose the pair (X, D) is not canonical at a curve C passing through the $\frac{1}{2}(1, 1, 1)$ -point Q . Then the pair (X, D) is not canonical at Q .*

Because of this corollary, we only concentrate on $\frac{1}{2}(1, 1, 1)$ being the center of noncanonicity. Note that it is possible for the pair $(X, \frac{1}{n}\mathcal{M})$ to be noncanonical at a singular point of X . What is true, and proved here, is that there always exist a model X_I of X for which the pair $(X_I, \frac{1}{n}\mathcal{M}_I)$ is canonical at singular points.

PROPOSITION 4.5. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a del Pezzo fibration as in the Main Theorem. Let $\mathcal{M} \subset |-nK_X + lF|$ be the mobile linear system associated to a birational map $\chi : X \dashrightarrow Y$ to the total space Y of a Mori fiber space. Denote singular points of X by Q_1, \dots, Q_N . Let $I \subset \{1, \dots, N\}$ be the maximal subset of indices such that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at Q_i for $i \in I$. Let \mathcal{M}_I be the proper transform of \mathcal{M} on X_I . Then the pair $(X_I, \frac{1}{n}\mathcal{M}_I)$ is canonical at all singular points of X_I .*

Proof. We prove it for $I = \{1\}$, and the rest follows by induction. Suppose $\varphi : X \dashrightarrow X' = X_I$ is the birational transform described earlier starting from the blowup of the $\frac{1}{2}$ point Q_1 , which factors as

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\psi} & \widetilde{X}_I \\ f \downarrow & & \downarrow f_I \\ X & \xrightarrow{\varphi} & X_I \end{array}$$

with f and f_I being the Kawamata blowups of the singular points and ψ the flopping map between \widetilde{X} and \widetilde{X}_I . We have the following relations:

$$\mathcal{M} \sim -nK_X + lF \quad \text{and} \quad \mathcal{M}_I \sim -nK_{X_I} + l_I F_I,$$

$$\widetilde{\mathcal{M}} = f^* \mathcal{M} - \nu E \quad \text{and} \quad \widetilde{\mathcal{M}}_I = g^* \mathcal{M} - \nu_I E_I,$$

$$f^* F = f_*^{-1} F + E \quad \text{and} \quad f_I^* F_I = (f_I)_*^{-1} F_I + E_I,$$

where E and E_I are the exceptional divisors of f and f_I , and $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_I$ are the strict transforms of \mathcal{M} and \mathcal{M}_I with multiplicities ν, ν_I positive integers.

Let $g : V \rightarrow \widetilde{X}$ and $h : V \rightarrow \widetilde{X}_I$ be the resolution of the flopping map ψ . Denote

$$F_V = (f \circ g)^*(F) = (f_I \circ h)^*(F_I), \quad \text{and} \quad \mathcal{M}_V = (f \circ g)_*^{-1}(\mathcal{M}) = (f_I \circ h)_*^{-1}(\mathcal{M}_I).$$

Then we have the following equivalence:

$$lF_V + \left(\frac{1}{2} - \frac{\nu}{n}\right) g_*^{-1}(E) + \sum a_i E_i \sim K_V + \frac{1}{n} \mathcal{M}_V \sim l_I F_V + \left(\frac{1}{2} - \frac{\nu_I}{n}\right) h_*^{-1}(E_I) + \sum b_i E_i$$

for some rational a_i and b_i . Since ψ is small, the divisors E_i in the sums on the left-hand side and on the right-hand side are the same. We rearrange the terms:

$$(l - l_I) F_V \sim \left(\frac{1}{2} - \frac{\nu_I}{n}\right) h_*^{-1}(E_I) + \left(\frac{\nu}{n} - \frac{1}{2}\right) g_*^{-1}(E) + \sum (b_i - a_i) E_i. \tag{1}$$

On the other hand, we have

$$F_V \sim g_*^{-1}(E) + h_*^{-1}(E_I) + \sum c_i E_i \tag{2}$$

for some c_i .

It follows from linear independence of $E_i, g_*^{-1}(E), h_*^{-1}(E_I)$ that equivalences (1) and (2) are proportional, in particular, we have

$$\nu + \nu_I = n.$$

It implies that at most one of $(X, \frac{1}{n}\mathcal{M})$ and $(X_I, \frac{1}{n}\mathcal{M}_I)$ can be noncanonical at those particular singular points. \square

REMARK 4.6. Note that $\mathcal{M}_I \subset |-nK_{X_I} + l_I F_I|$ for some l_I , that is, n does not change. This is the case since the maps φ_I induce isomorphisms on general fibers.

LEMMA 4.7. *All the varieties X_I satisfy the generality conditions in the Main Theorem.*

Proof. The maps φ_I are compositions of maps φ_i , where the map φ_i is a birational involution of the analytical neighborhood of the fiber containing Q_i . Thus, the fibers, containing singularities on X_I , are isomorphic to the fibers containing singularities on X . Note that quasi-smoothness of X_I can be checked explicitly. The ambient space is singular along the surface $\{x = y = z = 0\}$, which is a $\mathbb{P}(2, 2)$ -bundle over \mathbb{P}^1 . This is cut out by X at the solutions to $sw = (\prod_{i \in I} l_i)s - (\prod_{j \notin I} l_j)w = 0$, which is a finite set of points. Locally near each singular point s or w , as well as a local parameter in the base, appears as linear term (because l_i 's are distinct), and hence can be eliminated so that the analytic type of singularity at each point is $\frac{1}{2}(1, 1, 1)$. There are no other singularities by the generality assumptions on X . \square

§5. Supermaximal singularities

In this section, we establish a framework for excluding smooth points. We follow the approach of Pukhlikov [28]. We exclude the points that lie on π -vertical curves intersecting $-K_X$ by $\frac{1}{2}$ in §6–8, and all the other points in Proposition 5.4.

By Corollary 3.3, we may assume that the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at curves in the smooth locus. In §§3 and 4, we have shown that there is a model X_I for which the pair $(X_I, \frac{1}{n}\mathcal{M}_I)$ is canonical at all the singularities of X_I . By Corollary 4.4, the pair $(X_I, \frac{1}{n}\mathcal{M}_I)$ is also canonical at curves passing through the singular locus of X_I . Thus, if the pair $(X_I, \frac{1}{n}\mathcal{M}_I)$ is not canonical at B , then B is a point and X is smooth at B . This allows us to use the following result.

PROPOSITION 5.1 [26, Prop. 2.7]. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a del Pezzo fibration. Suppose that we are given a nonsquare-birational map $\chi : X \dashrightarrow Y$ to a Mori fiber space $\pi_Y : Y \rightarrow Z$, and let $\mathcal{M} \subset |-nK_X + lnF|$ be a mobile linear system associated to χ . Suppose in addition that the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at curves on X and $l \geq 0$. Then there exist divisorial valuations ν_i centered at points P_1, \dots, P_k of X contained in distinct π -fibers and there exist positive rational numbers $\gamma_1, \dots, \gamma_k$ with the following properties:*

- $(X, \frac{1}{n}\mathcal{M} - \sum \gamma_j F_j)$ is not canonical at ν_1, \dots, ν_k , where F_i is the π -fiber containing P_i .
- $\sum_{j=1}^k \gamma_j > l$.

We combine this proposition with Corti's inequality to derive a contradiction from the existence of a nonsquare-birational map $\chi : X \dashrightarrow Y$. Note that $l \geq 0$ since X_I satisfies the K^2 -condition. This is why we require the total K^2 -condition in the Main Theorem.

We now introduce the notions and notations necessary for application of Corti's inequality. First, we simplify the notations by dropping the index I , that is, from now on, we denote X_I by X , π_I by π , and so forth.

Let $D_1, D_2 \in \mathcal{M}$ be general members, and consider the effective 1-cycle $Z = D_1 \cdot D_2$. For some point P , we find an upper bound on $\text{mult}_P Z$ using the degrees and a lower bound using Corti's inequality.

We decompose $Z = Z^v + Z^h$ into the vertical and horizontal components with relation to the fibration π . For the vertical components, we have the decomposition

$$Z^v = \sum_{t \in \mathbb{P}^1} Z_t^v + Z',$$

where the support of Z_t^v is contained in F_i from Proposition 5.1 and the support of Z' is disjoint from F_1, \dots, F_k . We define the degree of a vertical 1-cycle C^v by the number $\deg C^v = C^v \cdot (-K_X)$ and the degree of a horizontal 1-cycle C^h by the number $\deg C^h = C^h \cdot F$. Let f be the class of a line in a fiber, then $\deg f = 1$.

Now, we look for the pair, and the point, at which we apply Corti's inequality.

LEMMA 5.2 [21, Cor. 4.8]. *There is an index i such that*

$$\deg Z_i^v < 4n^2\gamma_i.$$

We say that ν_i is a *supermaximal singularity* if $\deg Z_i^v < 4n^2\gamma_i$, and clearly Lemma 5.2 implies that a supermaximal singularity exists. Fix such a supermaximal singularity ν_i . To simplify the notations, from now on, denote F_i as F , γ_i as γ , P_i as P , Z_i^v as Z_v , Z^h as Z_h , and ν_i as ν .

LEMMA 5.3 [28, Prop. 1.1]. *Let P be a nonsingular point, and let C be a vertical curve on X . Suppose $\deg C \geq 1$, then $\text{mult}_P C \leq \deg C$.*

Proof. For a sufficiently big integer a , the base locus of the linear system $| -K_X + aF |$ consists of singular points of X . Let $\mathcal{H} \subset | -K_X + aF |$ be the subsystem of divisors passing through P . Then \mathcal{H} has a base curve L if and only if L is the vertical curve of degree $\frac{1}{2}$ passing through P . In particular, if $\deg C \geq 1$, then C is not in the base locus of \mathcal{H} . Thus, for general $H \in \mathcal{H}$, we have

$$\text{mult}_P C \leq H \cdot C = -K_X \cdot C = \deg C. \quad \square$$

PROPOSITION 5.4 [28, §4]. *Let P be a nonsingular point on X . Suppose the degree of every vertical curve passing through P is at least 1. Then P cannot be the center of a supermaximal singularity.*

Proof. By definition of the degree of a horizontal cycle, we have

$$\text{mult}_P Z_h \leq Z_h \cdot F = \deg Z_h = D^2 \cdot F = 2n^2$$

for a general divisor $D \in \mathcal{M}$.

Suppose P is the center of a supermaximal singularity, and denote by F the fiber containing P . A general member $H \in \mathcal{H}|_F$ does not share components with Z_v since $\mathcal{H}|_F$ does not have fixed components. Thus, we see that

$$\text{mult}_P Z_v \leq D \cdot Z_v = \deg Z_v < 4n^2\gamma.$$

On the other hand, the pair $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is not canonical at the point P by Proposition 5.1. Hence, by Corti's inequality, there is a number $0 < t \leq 1$ such that

$$\text{mult}_P Z_h + t \text{mult}_P Z_v \geq 4n^2(1 + \gamma t \text{mult}_P F) > 4(1 + \gamma t)n^2.$$

Combining the bounds, we get a contradiction. □

REMARK 5.5. Let F be a fiber not containing a singular point of X . Then every curve in F has a positive integer degree and Proposition 5.4 applies for every $P \in F$. Let F be a fiber containing a $\frac{1}{2}(1, 1, 1)$ -point. Then there are curves on F which intersect $-K_X$ by $\frac{1}{2}$. If F satisfies the generality condition in the Main Theorem, then there are exactly eight such curves. If P lies on such a curve, then the bound on multiplicity of Z_v at P becomes $\text{mult}_P Z_v < 8n^2\gamma$ which is not enough to lead to a contradiction. Sections 6–8 are entirely devoted to dealing with this issue.

§6. Construction of the staircase in the generic case

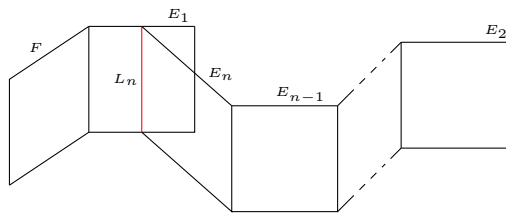
Let $X^{(0)}$ be a projective threefold, let $F^{(0)} \subset X^{(0)}$ be a surface, let $L_0 \subset F^{(0)}$, $L_0 \cong \mathbb{P}^1$, and suppose $X^{(0)}$ and $F^{(0)}$ are smooth in the neighborhood of L_0 . Let us associate the following construction to L_0 which goes by the name *staircase*. The notion of the staircase has been introduced by Pukhlikov in [28, §6]. The staircase associated to L_0 is the following chain of morphisms which we define inductively:

$$X^{(M)} \xrightarrow{\sigma_M} X^{(M-1)} \xrightarrow{\sigma_{M-1}} \dots \xrightarrow{\sigma_1} X^{(0)}.$$

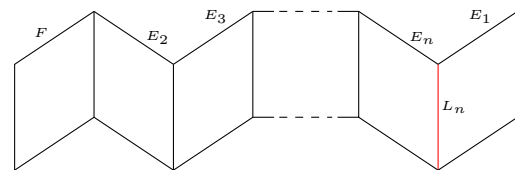
The morphism $\sigma_i : X^{(i)} \rightarrow X^{(i-1)}$ is the blowup at L_{i-1} , and we denote $E^{(i)}$ to be its exceptional divisor. Clearly, for every i , we have that $E^{(i)} \cong \mathbb{F}_m$, for some m . If $m > 0$, then let L_i be the exceptional section of $E^{(i)}$. If $m = 0$, then let L_i be a line from the ruling of $E^{(i)} \cong \mathbb{P}^1 \times \mathbb{P}^1$ horizontal with respect to the \mathbb{P}^1 -fibration $\sigma_i|_{E^{(i)}}$. Denote the proper transform of $E^{(i)}$ on $X^{(j)}$ as $E^{(i,j)}$ and the proper transform of $F^{(0)}$ on $X^{(j)}$ as $F^{(j)}$. Let $f_i \in A^2(X^{(i)})$ be the class of a fiber of the ruled surface $E^{(i)}$.

THEOREM 6.1. Let $X^{(0)}$ be a projective threefold, let $F^{(0)}$ be a surface in it, and suppose L_0 is a smooth rational curve in $F^{(0)}$. Suppose $X^{(0)}$ and $F^{(0)}$ are smooth in the neighborhood of L_0 . Moreover, assume that $L_0 \cdot K_{X^{(0)}} = 0$ and $L_0 \cdot F^{(0)} = -1$. Then, for the staircase associated to L_0 , the following assertions are true:

- (i) $E^{(1)} \cong \mathbb{P}^1 \times \mathbb{P}^1$,
- (ii) $E^{(i)} \cong \mathbb{F}_1$, for $i \geq 2$,
- (iii) $\sigma_1^*(L_0) \equiv L_1 - f_1$,
- (iv) $\sigma_i^*(L_{i-1}) \equiv L_i$, for $i \geq 2$,
- (v) $L_i \subset E^{(1,i)}$, for all i ,
- (vi) $E^{(i,i+1)}|_{E^{(i+1)}}$ is disjoint from L_{i+1} , for $i \geq 2$,
- (vii) $F^{(1)}$ either contains L_1 or is disjoint from it, and $\nu_{E^{(1)}}(F^{(0)}) = 1$, and
- (viii) $F^{(i)}$ is disjoint from L_i , for $i \geq 2$, and $\nu_{E^{(i)}}(F^{(0)}) = i - 1 + \delta$, where $\delta = \text{mult}_{L_1} F^{(1)}$.



The staircase when $\delta = 0$



The staircase when $\delta = 1$

We use the following result in the proof of the theorem.

LEMMA 6.2 [19, Lem. 2.2.14]. *Let $\sigma : Y \rightarrow X$ be the blowup of a threefold X at a smooth projective curve C . Suppose also that X is smooth along C . Let E be the exceptional divisor of σ , then E is a projectivization of the normal bundle $N_{C/X}$. Denote the class of the fiber of E as f , then the following equalities hold:*

- (i) $E^2 = -\sigma^*(C) + \text{deg}(N_{C/X})f$,
- (ii) $E^3 = \text{deg}(N_{C/X})$,
- (iii) $E \cdot f = -1$,
- (iv) $E \cdot \sigma^*(D) = (C \cdot D)f$, for some divisor D on X ,
- (v) $f \cdot \sigma^*(D) = 0$, for some divisor D on X ,
- (vi) $E \cdot \sigma^*(Z) = 0$, for any $Z \in A^2(X)$, and
- (vii) $\text{deg}(N_{C/X}) = 2g(C) - 2 - K_X \cdot C$.

To prove Theorem 6.1, we combine the following lemmas.

LEMMA 6.3. *With the assumptions of Theorem 6.1, we have $\text{deg} N_{L_0/X^{(0)}} = \mathcal{O}_{L_0}(-1) \oplus \mathcal{O}_{L_0}(-1)$. In particular, $E^{(1)} \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Recall that L_0 is a smooth rational curve, and thus Lemma 6.2 implies

$$\text{deg} N_{L_0/X^{(0)}} = 2g(L_0) - 2 - K_{X^{(0)}} \cdot L_0 = -2.$$

Since $F^{(0)} \cdot L_0 = -1$, we have $(N_{F^{(0)}/X^{(0)}})|_{L_0} \cong \mathcal{O}_{L_0}(-1)$. There is the exact sequence

$$0 \rightarrow N_{L_0/F^{(0)}} \xrightarrow{e} N_{L_0/X^{(0)}} \xrightarrow{s} (N_{F^{(0)}/X^{(0)}})|_{L_0} \rightarrow 0.$$

For some a and b , we have $N_{L_0/X^{(0)}} \cong \mathcal{O}_{L_0}(a) \oplus \mathcal{O}_{L_0}(b)$, and without loss of generality, we may assume that $a \leq b$. Because $\text{deg} N_{L_0/X^{(0)}} = -2$, we have $a + b = -2$, and since $(N_{F^{(0)}/X^{(0)}})|_{L_0} \cong \mathcal{O}_{L_0}(-1)$, we have $N_{L_0/F^{(0)}} \cong \mathcal{O}_{L_0}(-1)$.

Let p_a be the projection $\mathcal{O}_{L_0}(a) \oplus \mathcal{O}_{L_0}(b) \rightarrow \mathcal{O}_{L_0}(a)$. Suppose $p_a \circ e : \mathcal{O}_{L_0}(-1) \rightarrow \mathcal{O}_{L_0}(a)$ is a nontrivial map, then $b \geq a \geq -1$ and therefore $a = b = -1$. Hence, $a = -1$ and $b = -1$.

Suppose $p_a \circ e$ is trivial, then the map $s|_{\mathcal{O}_{L_0}(b)}$ is nontrivial and $b \leq -1$. Thus, we have $b = -1$, and hence $a = b = -1$. □

LEMMA 6.4. *With the assumptions of Theorem 6.1, we have*

- (i) $\sigma_1^*(L_0) \equiv L_1 - f_1$,
- (ii) $E^{(1)} \cdot L_1 = -1$,
- (iii) $K_{X^{(1)}} \cdot L_1 = -1$, and
- (iv) $E^{(2)} \cong \mathbb{F}_1$.

Proof. By Lemma 6.2, we have

$$0 = E^{(1)} \cdot \sigma_1^*(L_0) = E^{(1)}|_{E^{(1)}} \cdot \sigma_1^*(L_0) = (-\sigma_1^*(L_0) - 2f_1) \cdot \sigma_1^*(L_0),$$

and thus $\sigma_1^*(L_0)^2 = -2f_1 \cdot \sigma_1^*(L_0)$. Clearly, $\sigma_1^*(L_0) \equiv L_1 + af_1$ for some a , and hence

$$-2 = \sigma_1^*(L_0)^2 = (L_1 + af_1)^2 = 2a.$$

Therefore, $\sigma_1^*(L_0) \equiv L_1 - f_1$.

We use (i) and Lemma 6.2 to prove (ii):

$$L_1 \cdot E^{(1)} = \sigma_1^*(L_0) \cdot E^{(1)} + f_1 \cdot E^{(1)} = f_1 \cdot E^{(1)} = -1.$$

Thus, (iii) holds:

$$L_1 \cdot K_{X^{(1)}} = L_0 \cdot K_{X^{(0)}} + f_1 \cdot E^{(1)} = -1.$$

By Lemma 6.2, we compute

$$\text{deg } N_{L_1/X^{(1)}} = -2 - K_{X^{(1)}} \cdot L_1 = -1.$$

On the other hand, $N_{E^{(1)}/X^{(1)}}|_{L_1} \cong \mathcal{O}_{L_1}(-1)$ by (ii). Using the same argument as in Lemma 6.3, we conclude that $E^{(2)} \cong \mathbb{F}_1$. □

LEMMA 6.5. *Suppose $E^{(i)} \cong \mathbb{F}_1$ and $K_{X^{(i-1)}} \cdot L_{i-1} = -1$, then*

- (i) $\sigma_i^*(L_{i-1}) \equiv L_i$,
- (ii) $E^{(i)} \cdot L_i = 0$,
- (iii) $K_{X^{(i)}} \cdot L_i = -1$, and
- (iv) $E^{(i+1)} \cong \mathbb{F}_1$.

Proof. Since $\text{deg } N_{L_{i-1}/X^{(i-1)}} = -1$, Lemma 6.2 implies

$$0 = E^{(i)} \cdot \sigma_i^*(L_{i-1}) = E^{(i)}|_{E^{(i)}} \cdot \sigma_i^*(L_{i-1}) = (-\sigma_i^*(L_{i-1}) - f_i) \cdot \sigma_i^*(L_{i-1}).$$

Therefore, $\sigma_i^*(L_{i-1})^2 = -f \cdot \sigma_i^*(L_{i-1}) = -1$, and hence $\sigma_i^*(L_{i-1})$ is equivalent to the exceptional section L_i . Thus, assertions (i) and (ii) hold.

By (i), we have

$$K_{X^{(i)}} \cdot L_i = K_{X^{(i)}} \cdot \sigma_i^*(L_{i-1}) = K_{X^{(i-1)}} \cdot L_{i-1} + E^{(i)} \cdot \sigma_i^*(L_{i-1}) = -1.$$

Lemma 6.2 implies

$$\text{deg } N_{L_i/X^{(i)}} = -2 - K_{X^{(i)}} \cdot L_i = -1,$$

while $N_{L_i/E^{(i)}} \cong \mathcal{O}_{L_i}(-1)$ since $L_i^2 = -1$. Hence, computing $N_{L_i/X^{(i)}}$ as in Lemma 6.3, we conclude that $E^{(i+1)} \cong \mathbb{F}_1$. □

Proof of Theorem 6.1. Lemma 6.3 implies (i). Lemma 6.4 implies (iii). Lemmas 6.4 and 6.5 imply (ii) and (iv) by induction.

We prove (v) by induction. Obviously, $L_1 \subset E^{(1)}$. The equality $L_1 \cdot E^{(1)} = -1$ holds by Lemma 6.4. We show that if $L_{i-1} \cdot E^{(1,i-1)} = -1$, then $L_i \subset E^{(1,i)}$ and $L_i \cdot E^{(1,i)} = -1$. Indeed, by Lemma 6.2, we have

$$E^{(1,i)} \cdot E^{(i)} \equiv \sigma_i^*(E^{(1,i-1)}) \cdot E^{(i)} - (E^{(i)})^2 \equiv (E^{(1,i-1)} \cdot L_{i-1})f_i + L_i + f_i \equiv L_i.$$

The curve L_i is the only effective curve in its numerical equivalence class that lies on $E^{(i)}$ since $E^{(i)} \cong \mathbb{F}_1$ by assertion (ii). Thus, we see that $E^{(i)} \cap E^{(1,i)} = L_i$, in particular, $L_i \subset E^{(1,i)}$. Moreover,

$$E^{(1,i)} \cdot L_i = (E^{(1,i)}|_{E^{(i)}} \cdot L_i)_{E^{(i)}} = L_i^2 = -1.$$

Thus, (v) holds.

By induction, Lemmas 6.4 and 6.5 imply that $E^{(i)} \cdot L_i = 0$, for all $i \geq 2$. Thus, we compute

$$E^{(i-1,i)} \cdot E^{(i)} \equiv \sigma_i^*(E^{(i-1)}) \cdot E^{(i)} - (E^{(i)})^2 \equiv (E^{(i-1)} \cdot L_{i-1})f_i + L_i + f_i \equiv L_i + f_i.$$

Hence, by computing the intersection,

$$E^{(i-1,i)} \cdot L_i = E^{(i-1,i)}|_{E^{(i)}} \cdot L_i = ((L_i + f_i) \cdot L_i)_{E^{(i)}} = 0.$$

We conclude that either $E^{(i-1,i)}$ is disjoint from L_i or $L_i \subset E^{(i-1,i)}$. The latter does not occur since $E^{(1,i-1)}$ intersects $E^{(i-1)}$ transversally and $L_i \subset E^{(1,i)}$. Therefore, we have shown that (vi) holds.

By Lemma 6.2, we have

$$F^{(1)}|_{E^{(1)}} \equiv -(E^{(1)})^2 + (F^{(0)} \cdot L_0)f_1 \equiv \sigma_1^*(L_0) + 2f_1 - f_1 \equiv L_1.$$

Using this equivalence, we compute $F^{(1)} \cdot L_1 = (L_1 \cdot L_1)_{E^{(1)}} = 0$. Thus, $F^{(1)}$ is either disjoint from L_1 or it contains L_1 . Since $\text{mult}_{L_0} F^{(0)} = 1$, we have $\nu_{E^{(1)}}(F^{(0)}) = 1$, that is, (vii) holds.

If $F^{(1)}$ and L_1 are disjoint, then $F^{(i)}$ is disjoint from L_i . Suppose $L_1 \subset F^{(1)}$, then $F^{(1)}$ and $E^{(1)}$ are smooth surfaces intersecting along a smooth curve L_1 transversally at every point. Thus, $F^{(2)}$ and $E^{(1,2)}$ are disjoint, in particular, $L_2 \subset E^{(1,2)}$ and $F^{(2)}$ are disjoint. It follows that L_i is disjoint from $F^{(i)}$ for all $i \geq 2$.

To finish the proof of (viii), we compute

$$\begin{aligned} (\sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_i)^*(F^{(0)}) &= F^{(i)} + E^{(1,i)} + (1 + \delta)E^{(2,i)} + \dots + (i - 2 + \delta)E^{(i-1,i)} \\ &\quad + (i - 1 + \delta)E^{(i)}. \end{aligned}$$

□

6.1 The staircase over a del Pezzo fibration

We would like to apply Theorem 6.1 to $\pi_I: X_I \rightarrow \mathbb{P}^1$. As before, we drop the index I to simplify the notations. Suppose $\pi: X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration of degree 2 such that X is a quasi-smooth complete intersection in a $\mathbb{P}(1, 1, 1, 2, 2)$ -bundle over \mathbb{P}^1 . Suppose $Q \in X$ is a $\frac{1}{2}(1, 1, 1)$ -point and F is the fiber containing Q . Let $\sigma_Q: X^{(0)} \rightarrow X$ be the Kawamata blowup of X at Q , and let E_Q be the exceptional divisor of σ_Q . The fiber F , containing Q , can be embedded into $\mathbb{P}(1, 1, 1, 2)$, and the equation of the image is of the form

$$wq(x, y, z) + r(x, y, z) = 0.$$

Let $L \subset F$ be a *half-line*, that is, a curve L such that $L \cdot (-K_F) = \frac{1}{2}$. Let L_0 and $F^{(0)}$ be the proper transforms of L and F , respectively, on $X^{(0)}$. We may construct the staircase associated to L_0 . We also say that the staircase is associated to the half-line L .

If X satisfies the generality conditions, then $F^{(0)}$ and $L^{(0)}$ are smooth along L_0 . Thus, to show that the triple $X^{(0)}, F^{(0)}, L_0$ satisfies the assumptions of Theorem 6.1, we only need to consider the intersections.

LEMMA 6.6. *The following equalities hold:*

- (i) $L_0 \cdot E_Q = 1$,
- (ii) $L_0 \cdot F^{(0)} = -1$, and
- (iii) $L_0 \cdot K_{X^{(0)}} = 0$.

Proof. Let $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$, and let H be the generator of $\text{Cl}(\mathbb{P})$. Because $F \subset \mathbb{P}$, we can express L in \mathbb{P} as the intersection $H_1 \cdot H_2$, for some $H_i \in |H|$. Let $\sigma_{\mathbb{P}}: \mathbb{P}_Q \rightarrow \mathbb{P}$ be the blowup at the point Q , and let $E_{\mathbb{P}}$ be its exceptional divisor. Clearly, $\sigma_{\mathbb{P}}: (\sigma_{\mathbb{P}})_*^{-1}(F) \rightarrow F$ is the blowup at Q , and thus we may identify $(\sigma_{\mathbb{P}})_*^{-1}(F)$ with $F^{(0)}$ and $(\sigma_{\mathbb{P}})_*^{-1}(L)$ with L_0 . Let

$H_{i,Q}$ be the proper transform of H_i on \mathbb{P}_Q , then $L_0 = H_{1,Q} \cdot H_{2,Q}$. Denote the exceptional divisor of $\sigma_{\mathbb{P}}$ as $E_{\mathbb{P}}$, then

$$L_0 \cdot E_Q = L_0 \cdot E_Q|_{F^{(0)}} = L_0 \cdot E_{\mathbb{P}}|_{F^{(0)}} = L_0 \cdot E_{\mathbb{P}} = H_{1,Q} \cdot H_{2,Q} \cdot E_{\mathbb{P}} = H_{1,Q}|_{H_{2,Q}} \cdot E_{\mathbb{P}}|_{H_{2,Q}}.$$

The surface H_2 is isomorphic to $\mathbb{P}(1, 1, 2)$, and $\sigma_{\mathbb{P}}|_{H_{2,Q}}$ is the blowup at the singular point. It follows that $H_{2,Q} \cong \mathbb{F}_2$, $E_{\mathbb{P}}|_{H_{2,Q}}$ is the exceptional section, and $H_{1,Q}|_{H_{2,Q}}$ is a fiber. Thus, we have (i)

$$L_0 \cdot E_Q = H_{1,Q}|_{H_{2,Q}} \cdot E_{\mathbb{P}}|_{H_{2,Q}} = 1.$$

To prove (iii), it is enough to show that $F^{(0)} = F - \frac{2}{2}E_Q = F - E_Q$. Indeed, we find the intersection

$$L_0 \cdot F^{(0)} = L_0 \cdot (F - E_Q) = -L_0 \cdot E_Q = -1.$$

The equality (iii) follows from (i)

$$K_{X^{(0)}} \cdot L_0 = (\sigma^* K_X + \frac{1}{2}E_Q) \cdot L_0 = K_X \cdot L + \frac{1}{2}E_Q \cdot L_0 = 0.$$

We now prove the claim. Recall that X is a complete intersection in $\mathbb{P}(1, 1, 1, 2, 2)$ given by the equations

$$g(u, v)s = f(u, v)w + q(x, y, z; u, v), \quad \text{and} \quad sw + r(x, y, z; u, v) = 0.$$

The point Q lies on $(x = y = z = 0)$, and up to a change of coordinates on the base, we may assume that it is in the fiber $u = 0$. Thus, up to exchanging variables w and s , we may assume that Q is given by $x = y = z = w = u$. Consider a neighborhood U_V of Q given by $s = v = 1$, then the equations of $U = X \cap U_V$ are

$$u(1 + u(\dots)) = w(1 + u(\dots)) + q(x, y, z; u, 1), \quad \text{and} \quad w + r(x, y, z; u, 1) = 0.$$

We may now eliminate the second equation to acquire the embedding $U \subset T_U \cong \mathbb{C}_{x,y,z}^3 / \mu_2 \times \mathbb{C}_u$ given by the equation

$$u(1 + u(\dots)) + r(x, y, z; u, 1)(1 + u(\dots)) = q(x, y, z; u, 1).$$

We consider an analytic neighborhood of 0 in $\mathbb{C}_{x,y,z}^3 / \mu_2$ and the corresponding analytic neighborhoods U_{an} and $T_{U,an}$. The equation of U_{an} is $u = 0$, and hence the equation of $F_{an} = F \cap U_{an}$ is

$$r(x, y, z; 0, 1) = q(x, y, z; 0, 1),$$

where $r(x, y, z; 0, 1) \neq 0$ by the generality conditions. It follows that $F^{(0)} = F - \frac{2}{2}E_Q = F - E_Q$ since $\text{deg} r(x, y, z; 0, 1) = 2$. □

We now discuss the termination condition of the staircase. Recall that there is a mobile linear system $\mathcal{M} \subset |-nK_X + lF|$ and discrete valuations ν such that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at a discrete valuation ν , and ν is a supermaximal singularity. We have shown in §§3 and 4 that the center of ν is a nonsingular point P . By Remark 5.5 and Proposition 5.4, the point P lies on a curve L such that $L \cdot K_{X_I} = -\frac{1}{2}$. In particular, P is in a fiber containing a $\frac{1}{2}(1, 1, 1)$ -point Q . We want to track the center of ν as we go up the staircase, in particular, we want to know how far up the staircase we must go.

PROPOSITION 6.7 [28, Prop. 7.1]. *Let $X^{(0)}$ be a threefold, and let $F^{(0)}$ be a surface in it. Suppose $L_0 \cong \mathbb{P}^1$, $L_0 \subset F^{(0)}$. Suppose also $X^{(0)}$ and $F^{(0)}$ are smooth in the neighborhood of L_0 . Let $\sigma_i : X^{(i)} \rightarrow X^{(i-1)}$ be the associated staircase. Let ν be a discrete valuation of $K(X^{(0)})$, and suppose that a center of ν on $X^{(0)}$ is a point on L_0 . Then there is a positive integer M such that, for every $i < M$, the center of ν on $X^{(i)}$ is a point on L_i and the center of ν on $X^{(M)}$ is one of the following:*

- A) a fiber of a ruled surface $E^{(M)}$,
- B) a point not on L_M and not on $E^{(M-1,M)}$ and $M \geq 2$, or
- C) a point on $E^{(M)} \cap E^{(M-1,M)}$ and $M \geq 3$.

The idea of the proof is the following: the discrepancy of $E^{(i)}$ is increasing, and therefore the center of ν should be away from $E^{(i)}$ for $i \gg 0$. If the center of ν on $E^{(1)}$ is a point, we can choose L_1 to be the line from horizontal ruling passing through that point, and thus we never get cases B and C for $M = 1$. The intersection of $E^{(1,2)}$ and $E^{(2)}$ is L_2 , and thus we never get the case C for $M = 2$.

§7. Multiplicities on the staircase

The plan is to associate a staircase to a half-line, to apply Corti’s inequality upstairs, and to derive a contradiction. Thus, we need to find bounds on multiplicities of the cycles upstairs. We compute them in this section.

Let X be a projective threefold, let B be a smooth, irreducible subvariety of codimension 2, and suppose X is smooth along B . Let $\sigma : \tilde{X} \rightarrow X$ be the blowup along B . Then, by definition of the proper transform for any cycle Z such that $B \not\subset \text{Supp } Z$, we have $\sigma^*(Z) = \sigma_*^{-1}(Z) + Z_E$, where Z_E is the cycle with the support on the exceptional divisor E of σ . The following well-known results give us some information about Z_E .

LEMMA 7.1. *Suppose B is a smooth curve, then $Z_E \equiv (Z \cdot B)_S f$, where f is the class of a fiber of the ruled surface E and S is any normal surface containing $\text{Supp } Z$ and B which is smooth at every point of $\text{Supp } Z \cap B$.*

LEMMA 7.2 [21, Lem. 2.14]. *Let F be the hyperplane given by the equation $z = 0$ in \mathbb{C}^3 , let B be a smooth irreducible curve in F , and let C be an irreducible curve which does not lie in F . Let $f \in A^2(X)$ be the class of a fiber of the ruled surface E . Then $\sigma^*C \equiv \sigma_*^{-1}C + kf$, where $k \leq C \cdot F$.*

Suppose cycle Z is the intersection $D_1 \cdot D_2$. Then we can express σ^*Z in terms of the intersection of proper transforms of D_1 and D_2 and a cycle supported on the exceptional divisor.

LEMMA 7.3. *Let D_1 and D_2 be general members in a mobile linear system \mathcal{M} on a smooth variety X and suppose $Z = D_1 \cdot D_2$. Let \tilde{D}_i be the proper transform of D_i on \tilde{X} . Then*

$$\tilde{D}_1 \cdot \tilde{D}_2 \equiv \sigma^*(Z) + \Delta_E,$$

where $\text{Supp } \Delta_E \subset E$.

Suppose also that B is a smooth curve. Let $m = \nu_E(\mathcal{M})$, and let $f \in A^2(\tilde{X})$ be the class of a fiber of the ruled surface E . Then

$$\Delta_E \equiv m^2 E^2 - 2m(D_1 \cdot B)f.$$

We use the following notations for the proper transforms on the staircase. Let A be a cycle, a divisor, or a linear system on X . We denote its proper transform on $X^{(i)}$ as $A^{(i)}$. For divisors and cycles on $X^{(j)}$, we add upper index. For example, $E^{(1,3)}$ is the proper transform of $E^{(1)}$ on $X^{(3)}$. By σ^* we mean the appropriate composition of σ_i^* . For example, $E^{(1,3)} = \sigma^*(E^{(1)}) - E^{(2,3)} - 2E^{(3)}$, here $\sigma^* = \sigma_3^* \circ \sigma_2^*$.

Recall the notations of §5. Let $Z = D_1 \cdot D_2$ for general members $D_1, D_2 \in \mathcal{M}$. Let Z_h be the horizontal part of Z , and let Z_v be the part of Z which lies in F . Let γ be the number such that the pair $(X, \frac{1}{n}\mathcal{M} - \gamma F)$ is not canonical at ν and $\deg Z_v < 4n^2\gamma$. Let L be the curve passing through P and satisfying $L \cdot K_X = -\frac{1}{2}$. The cycle Z_v can be decomposed as

$$Z_v = kL + \Delta,$$

where $k \geq 0$ and Δ does not contain L .

LEMMA 7.4. *The inequality $\Delta \cdot L < 4\gamma n^2$ holds.*

Proof. Let \mathcal{H} be the subsystem of $|-K_F|$ of divisors containing L . We claim that $\dim \mathcal{H} = 1$. Pick a point $P \in L$ such that X is smooth at P . Then it is easy to see that the linear system of divisors in $|-K_F|$ containing P has dimension 1. On the other hand, any divisor in $|-K_F|$ containing P also contains L . Let H be a general element of \mathcal{H} . Let $L' = H - L$. Since \mathcal{H} does not have fixed points except points on L , the linear system $|L'|$ is mobile, in particular, L' is numerically effective. The valuation ν is a supermaximal singularity, and hence we have $\deg Z_v < 4\gamma n^2$, and therefore

$$\Delta \cdot L \leq \Delta \cdot L + \Delta \cdot L' = \Delta \cdot H \leq Z_v \cdot H = \deg Z_v < 4\gamma n^2. \quad \square$$

Denote $\nu_Q = \nu_{E_Q}$. Since the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at Q , we have $\nu_Q(D) \leq \frac{n}{2}$ for general $D \in \mathcal{M}$.

LEMMA 7.5. *Suppose D is a general member in \mathcal{M} , then*

$$\begin{aligned} D^{(0)} \cdot L_0 &= \frac{n}{2} - \nu_Q(D), \quad \text{and} \\ D^{(i)} \cdot L_i &= \frac{n}{2} - \nu_Q(D) + \lambda_1 \quad \text{for } i \geq 1, \end{aligned}$$

where $\lambda_1 = \text{mult}_{L_0} \mathcal{M}^{(0)}$.

Proof. Lemma 6.6 implies

$$\begin{aligned} D^{(0)} \cdot L_0 &= \sigma_Q^*(D) \cdot L_0 - \nu_Q(D)E_Q \cdot L_0 = D \cdot L - \nu_Q(D) \\ &= -nK_X \cdot L - \nu_Q(D) = \frac{n}{2} - \nu_Q(D). \end{aligned}$$

On the other hand, by Lemma 6.2 and (iii) of Theorem 6.1,

$$D^{(1)} \cdot L_1 = (\sigma_1^*D^{(0)} - \lambda_1 E^{(1)})(\sigma_1^*L_0 + f_1) = D^{(0)} \cdot L_0 + \lambda_1.$$

By Theorem 6.1, we have $\sigma_i^*L_i = L_{i+1}$ for $i \geq 1$. Thus, the equality $D^{(1)} \cdot L_1 = D^{(i)} \cdot L_i$ holds for all $i \geq 1$. □

Denote $Z_i = D_1^{(i)} \cdot D_2^{(i)}$, and we have the decomposition $Z_0 = Z_v^{(0)} + Z_h^{(0)} + Z_Q$, where $Z_Q \subset E_Q$. We can disregard the part Z_Q because it is away from P .

For every Z_i , we have the part of the cycle in $E^{(i)}$, and let us denote it $\Gamma^{(i)}$. Recall that $E^{(1)} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $E^{(i)} \cong \mathbb{F}_1$ for $i \geq 2$. The map $\sigma_i|_{E^{(i)}}$ is the corresponding \mathbb{P}^1 -fibration. We say that a curve B on $E^{(i)}$ is vertical if $\sigma_i(B)$ is a point and horizontal otherwise. We can decompose the cycle $\Gamma^{(i)}$ into L_i with multiplicity, the rest of the horizontal part, and the vertical part:

$$\Gamma^{(i)} = k_i L_i + C_h^{(i)} + C_v^{(i)}.$$

Since $E^{(i,i+h)}$ is disjoint from L_{i+h} for any $h \geq 2, i \geq 2$, we have $\sigma^* C^{(i,i+1)} \equiv C^{(i,i+h)}$ for any $h \geq 2, i \geq 2$. Similarly $\Delta^{(i)} \equiv \sigma^* \Delta^{(2)}$ for $i \geq 3$ since $F^{(i)}$ is disjoint from L_i for $i \geq 2$. Thus, we can decompose

$$\begin{aligned} Z_0 &= Z_h^{(0)} + Z_v^{(0)} = Z_h^{(0)} + \Delta^{(0)} + k_0 L_0, \\ Z_1 &= Z_h^{(1)} + \Delta^{(1)} + C_h^{(1)} + C_v^{(1)} + k_1 L_1, \\ Z_2 &= Z_h^{(2)} + \Delta^{(2)} + C_h^{(1,2)} + C_v^{(1,2)} + C_h^{(2)} + C_v^{(2)} + k_2 L_2, \\ Z_i &= Z_h^{(i)} + \sigma^*(\Delta^{(2)}) + C_h^{(1,i)} + C_v^{(1,i)} + \sigma^* C_h^{(2,3)} + \sigma^* C_v^{(2,3)} \\ &\quad + \dots + C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_i L_i. \end{aligned}$$

Recall that additional upper indices mean the proper transforms.

Denote the $\text{mult}_{L_{i-1}} \mathcal{M}^{(i-1)}$ as λ_i . Recall that by f_i we denote the class of the fiber of the ruled surface $E^{(i)}$. Thus, $C_v^{(i)} \equiv d_v^{(i)} f_i$ and $C_h^{(i)} \equiv d_h^{(i)} L_i + \beta_i f_i$ for some $d_v^{(i)}, d_h^{(i)}$, and β_i . Moreover, $d_h^{(i)} \leq \beta_i$, because $C_h^{(i)}$ does not contain the exceptional section. Recall that $\delta = 1$ if $L_1 \subset F^{(1)}$ and $\delta = 0$ otherwise. We now describe how the classes in components of Z_i change as we climb up the staircase.

LEMMA 7.6. *We have the following relations for the proper transforms and the pullbacks of the cycles:*

$$\begin{aligned} \Delta^{(1)} &\equiv \sigma_1^* \Delta^{(0)} - (\Delta^{(0)} \cdot L_0)_{F^{(0)}} f_1, \\ \Delta^{(2)} &\equiv \sigma_2^* \Delta^{(1)} - \delta (\Delta^{(0)} \cdot L_0)_{F^{(0)}} f_2, \\ C_h^{(1,i+1)} &\equiv \sigma_{i+1}^* C_h^{(1,i)} - \beta_1 f_{i+1} \quad \text{for } i \geq 1, \\ C_v^{(1,i+1)} &\equiv \sigma_{i+1}^* C_v^{(1,i)} - d_v^{(1)} f_{i+1} \quad \text{for } i \geq 1, \\ C_h^{(i,i+1)} &\equiv \sigma_{i+1}^* C_h^{(i)} - (\beta_i - d_h^{(i)}) f_{i+1} \quad \text{for } i \geq 2, \\ C_v^{(i,i+1)} &\equiv \sigma_{i+1}^* C_v^{(i)} - d_v^{(i)} f_{i+1} \quad \text{for } i \geq 1, \\ Z_h^{(i+1)} &\equiv \sigma_{i+1}^* Z_h^{(i)} - \alpha_{i+1} f_{i+1} \quad \text{for some } \alpha_{i+1} \leq 2n^2. \end{aligned}$$

Proof. First equivalence follows from Lemma 7.1 directly. If $\delta = 0$, that is, if $F^{(1)}$ and hence $\Delta^{(1)}$ is disjoint from L_1 , then we have $\Delta^{(2)} \equiv \sigma_2^* \Delta^{(1)}$. If $\delta = 1$, that is, if $L_1 \subset F^{(1)}$, then the equivalence follows from Lemma 7.1.

Note that

$$\begin{aligned} (C_h^{(1,i)} \cdot L_i)_{E^{(1,i)}} &= (C^{(1)} \cdot L_1)_{E^{(1)}} = \beta_1, \\ (C_v^{(1,i)} \cdot L_i)_{E^{(1,i)}} &= (C^{(1)} \cdot L_1)_{E^{(1)}} = d_v^{(1)}, \end{aligned}$$

$$\begin{aligned} (C_h^{(i)} \cdot L_i)_{E^{(i)}} &= \beta_i - d_h^{(i)}, \\ (C_v^{(i)} \cdot L_i)_{E^{(i)}} &= d_v^{(i)}. \end{aligned}$$

Thus, all equivalences but the last follow from Lemma 7.1. By Lemma 7.2, we have

$$\alpha_i \leq Z_h^{(i)} \cdot E^{(i)} \leq Z_h \cdot F.$$

Let D_1, D_2 be general divisors in \mathcal{M} , then

$$Z_h \cdot F = D_1 \cdot D_2 \cdot F = 2n^2,$$

since $D_i \in |-nK_X + lnF|$. □

LEMMA 7.7. *We have the following relations for vertical degrees on $X^{(i)}$. For $i = 1$,*

$$\beta_1 + d_v^{(1)} = \alpha_1 + (\Delta^{(0)} \cdot L_0)_{F^{(0)}} - \lambda_1(n - 2\nu_Q(D)) - 2\lambda_1^2 - k_1 - d_h^{(1)},$$

for $i = 2$,

$$\beta_2 + d_v^{(2)} = \alpha_2 + \delta(\Delta^{(0)} \cdot L_0)_{F^{(0)}} - \lambda_1(n - 2\nu_Q(D)) - \lambda_2^2 - 2\lambda_1\lambda_2 + \beta_1 + d_v^{(1)},$$

and for $i \geq 3$,

$$\beta_i + d_v^{(i)} = \alpha_{i-1} + (\beta_1 + d_v^{(1)}) + d_v^{(i-1)} + (\beta_{i-1} - d_h^{(i-1)}) - \lambda_i(n - 2\nu_Q(D)) - 2\lambda_1\lambda_i - \lambda_i^2,$$

where D is a general member in \mathcal{M} .

Proof. By Lemma 7.3, we have the equivalence

$$\begin{aligned} Z_1 &\equiv \sigma^*(Z_0) + \lambda_1^2(E^{(1)})^2 - 2\lambda_1(D^{(0)} \cdot L_0)f_1 \\ &\equiv \sigma^*(Z_0) - \lambda_1^2\sigma^*(L_0) - (\lambda_1(n - 2\nu_Q(D)) + 2\lambda_1^2)f_1. \end{aligned}$$

On the other hand, from the decomposition of Z_1 and Lemmas 6.4(i) and 7.6, we have

$$\begin{aligned} Z_1 &= Z_h^{(1)} + \Delta^{(1)} + C_h^{(1)} + C_v^{(1)} + k_1L_1 \\ &\equiv \sigma_1^*(Z_h^{(0)} + \Delta^{(0)} + (k_1 + d_h^{(1)})L_0) + (\beta_1 + d_h^{(1)})f_1 + C_v^{(1)} - (\alpha_1 + (\Delta^{(0)} \cdot L_0)_{F^{(0)}} - k_1)f_1. \end{aligned}$$

Combining these equivalences, we conclude that, in $A^2(X^{(1)})/\sigma_1^*A^2(X^{(0)})$, we have

$$\begin{aligned} (\beta_1 + d_h^{(1)} + d_v^{(1)})f_1 &\equiv C_h^{(1)} + C_v^{(1)} \\ &\equiv -(\lambda_1(n - 2\nu_Q(D)) + 2\lambda_1^2)f_1 + (\alpha_1 + (\Delta^{(0)} \cdot L_0)_{F^{(0)}} - k_1)f_1. \end{aligned}$$

Similarly, by Lemma 7.3,

$$\begin{aligned} Z_2 &\equiv \sigma^*(Z_1) + \lambda_2^2(E^{(2)})^2 - 2\lambda_2(D_1^{(1)} \cdot L_1)f_2 \\ &\equiv \sigma^*(Z_1) - \lambda_2^2\sigma^*(L_1) - (\lambda_2(n - 2\nu_Q(D)) + \lambda_2^2 + 2\lambda_1\lambda_2)f_2. \end{aligned}$$

From the decomposition of Z_2 and Lemma 7.7, we see that

$$\begin{aligned} Z_2 &= Z_h^{(2)} + \Delta^{(2)} + C_h^{(1,2)} + C_v^{(1,2)} + C_h^{(2)} + C_v^{(2)} + k_2L_2 \\ &\equiv \sigma^*(Z_h^{(1)} + \Delta^{(1)} + C_h^{(1)} + C_v^{(1)} + k_2L_1) + C_h^{(2)} + C_v^{(2)} \\ &\quad - (\alpha_2 + \delta\Delta^{(0)} \cdot L_0 + \beta_1 + d_v^{(1)})f_2. \end{aligned}$$

Combining these equalities, we conclude that, in $A^2(X^{(2)})/\sigma_1^*A^2(X^{(1)})$, we have

$$C_h^{(2)} + C_v^{(2)} \equiv (d_v^{(2)} + \beta_2)f_2 \equiv (\alpha_2 + \delta(\Delta^{(0)} \cdot L_0)_{F^{(0)}} + \beta_1 + d_v^{(1)})f_2 - (\lambda_2(n - 2\nu_Q(D)) + 2\lambda_1\lambda_2 + \lambda_2^2)f_2.$$

Hence, we find $d_v^{(2)} + \beta_2$.

We treat the general case the same. Once again by Lemma 7.3, we have

$$Z_i \equiv \sigma^*(Z^{(i-1)}) + \lambda_i^2(E^{(i)})^2 - 2\lambda_i(D_1^{(i-1)} \cdot L_{i-1})f \equiv \sigma^*(Z^{(i-1)}) - \lambda_i^2\sigma^*(L_{i-1}) - (\lambda_i(n - 2\nu_Q(D)) + \lambda_i^2 + 2\lambda_1\lambda_i)f_i.$$

Once again from the decomposition of Z_i and Lemma 7.6, we see that

$$Z_i \equiv Z_h^{(i)} + C_h^{(1,i)} + C_v^{(1,i)} + \sigma^*(\Delta^{(2)} + C_h^{(2,3)} + C_v^{(2,3)} + \dots + C_h^{(i-2,i-1)} + C_v^{(i-2,i-1)}) + C_h^{(i-1,i)} + C_v^{(i-1,i)} + C_h^{(i)} + C_v^{(i)} + k_iL_i = \sigma^*(Z_h^{(i-1)} + \Delta^{(1+\delta)} + C_h^{(1,i-1)} + C_v^{(1,i-1)} + C_h^{(2,3)} + C_v^{(2,3)} + \dots + C_h^{(i-2,i-1)} + C_v^{(i-2,i-1)} + C_h^{(i-1)} + C_v^{(i-1)} + k_iL_{i-1}) + C_h^{(i)} + C_v^{(i)} - (\alpha_i + \beta_1 + d_v^{(1)} + (\beta_{i-1} - d_h^{(i-1)}) + d_v^{(i-1)})f_i.$$

Combining these equalities, we conclude that, in $A^2(X^{(i)})/\sigma_1^*A^2(X^{(i-1)})$, we have

$$C_h^{(i)} + C_v^{(i)} \equiv (\alpha_{i-1} + (\Delta^{(0)} \cdot L_0)_{F^{(0)}} + (\beta_{i-1} - d_h^{(i-1)}) + d_v^{(i-1)} + \beta_1 + d_v^{(1)})f_i - (\lambda_i(n - 2\nu_Q(D)) + 2\lambda_1\lambda_i + 2\lambda_i^2)f_i.$$

Hence, we find $d_v^{(i)} + \beta_i$. □

COROLLARY 7.8. *The vertical degrees are bounded as follows:*

$$\beta_1 + d_v^{(1)} < 2n^2 - 2\lambda_1^2 + 4n^2\gamma, \\ \beta_M + d_v^{(M)} < (M - 1)(2n^2 - 2\lambda_1^2 + 4n^2\gamma) + \sum_{i=2}^M (2n^2 - \lambda_i^2 - 2\lambda_1\lambda_i) + 4\delta n^2\gamma \text{ for } M \geq 2.$$

Proof. Since the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical at Q , the inequality $n \geq 2\nu_Q(D)$ holds. For $i = 1$, by Lemma 7.7, we have

$$\beta_1 + d_v^{(1)} = \alpha_1 + (\Delta^{(0)} \cdot L_0)_{F^{(0)}} - \lambda_1(n - 2\nu_Q(D)) - 2\lambda_1^2 - k_1 - d_h^{(1)}.$$

Combining this with the inequalities $\alpha_1 \leq 2n^2$, $k_1, d_h^{(1)} \geq 0$, and $(\Delta^{(0)} \cdot L_0)_{F^{(0)}} \leq (\Delta \cdot L)_F < 4\gamma n^2$, we get

$$\beta_1 + d_v^{(1)} < 2n^2 + 4\gamma n^2 - 2\lambda_1^2.$$

We prove the second bound by induction. Suppose $M = 2$. Then, using the same bounds, we get

$$\beta_2 + d_v^{(2)} = \alpha_2 + \delta(\Delta^{(0)} \cdot L_0)_{F^{(0)}} + d_v^{(1)} + \beta_1 - \lambda_2(n - 2\nu_Q(D)) - \lambda_1^2 - 2\lambda_1\lambda_2 < (2n^2 + 4\gamma n^2 - \lambda_1^2) + (2n^2 - \lambda_2^2 - 2\lambda_1\lambda_2) + 4\delta\gamma n^2.$$

Now, suppose the inequality holds for $M - 1$. Then, similar to the previous case, we have

$$\begin{aligned} \beta_i + d_v^{(i)} &= \alpha_i + (\beta_1 + d_v^{(1)}) + d_v^{(i-1)} + (\beta_{i-1} - d_h^{(i-1)}) - \lambda_i(n - 2\nu_Q(D)) - 2\lambda_1\lambda_i - \lambda_i^2 \\ &< (2n^2 - 2\lambda_1\lambda_i - \lambda_i^2) + (\beta_1 + d_v^{(1)}) + d_v^{(i-1)} + \beta_{i-1}. \end{aligned}$$

Combining this with the bound on $\beta_1 + d_v^{(1)}$ and the assumption of induction, we get the statement of the corollary. \square

COROLLARY 7.9. *The following bounds on multiplicities hold.*

(i) *Let B be a fiber of a ruled surface $E^{(i)}$, then*

$$\begin{aligned} \text{mult}_B Z_1 &= \text{mult}_B C_v^{(1)} \leq 2n^2 - 2\lambda_1^2 + 4n^2\gamma, \\ \text{mult}_B Z_i &= \text{mult}_B C_v^{(i)} \leq \sum_{i=2}^M (4n^2 - 2\lambda_1^2 - \lambda_i^2 - 2\lambda_1\lambda_i) + 4(M - 1 + \delta)n^2\gamma, \quad \text{for } i \geq 2. \end{aligned}$$

(ii) *Let B be a point on $E^{(i)}$, and suppose B is not on the exceptional section, then*

$$\begin{aligned} \text{mult}_B (C_v^{(1)} + C_h^{(1)}) &\leq (2n^2 - 2\lambda_1^2 + 4n^2\gamma), \\ \text{mult}_B (C_v^{(i)} + C_h^{(i)}) &\leq \sum_{i=2}^M (4n^2 - 2\lambda_1^2 - \lambda_i^2 - 2\lambda_1\lambda_i) + 4(M - 1 + \delta)n^2\gamma, \quad \text{for } i \geq 2. \end{aligned}$$

Proof. Clearly, $\text{mult}_B C_v^{(i)}$ is bounded by a vertical degree $d_v^{(i)}$ whether B is a point or a curve. Thus, the inequality holds if B is a curve.

Similarly, if B is a point, then $\text{mult}_B C_h^{(i)} \leq d_h^{(i)}$, and hence $\text{mult}_B (C_v^{(i)} + C_h^{(i)}) \leq d_v^{(i)} + d_h^{(i)}$. Since $C_h^{(i)}$ does not contain the exceptional section, we have $d_h^{(i)} \leq \beta_i$. Therefore, by Corollary 7.8, the inequalities hold. \square

§8. Method of supermaximal singularity upstairs

In the previous section, we found an upper bound on the multiplicity of components of Z_M at the center of ν on $X^{(M)}$. In this section, we show that it contradicts Corti’s inequality.

LEMMA 8.1. *The pair*

$$\begin{aligned} \left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(\frac{1}{n}(n - \lambda_1) + \gamma \right) E^{(1,M)} \right. \\ \left. - \sum_{j=2}^M \left(\frac{1}{n} \sum_{i=2}^j (2n - \lambda_1 - \lambda_i) + (j - 1 + \delta)\gamma \right) E^{(j,M)} - \gamma F^{(M)} \right) \end{aligned}$$

is not canonical at ν on $X^{(M)}$.

Proof. By Theorem 6.1, we have $L_i \subset E^{(1,i)}$ for all i and $L_i \not\subset E^{(j,i)}$ for $i > j > 1$, and therefore the following equivalence holds:

$$K_{X^{(M)}} - E^{(1,M)} - \sum_{i=2}^M 2(i - 1)E^{(i,M)} - \frac{1}{2}E_Q \sim \sigma^*(K_X).$$

We disregard E_Q in this equivalence and in the next ones since E_Q is away from P . For a general member $D \in \mathcal{M}$, we have

$$D^{(1)} + \lambda_1 E^{(1)} \sim \sigma^*(D) \quad \text{and}$$

$$D^{(M)} + \sum_{j=1}^M \left(\sum_{i=2}^j (\lambda_i + \lambda_1) E^{(j,M)} \right) \sim \sigma^*(D), \quad \text{for } j \geq 2.$$

By Theorem 6.1(vii) and (viii), we have

$$F^{(M)} + E^{(1,M)} + \sum_{j=2}^M (j - 1 + \delta) E^{(j,M)} = \sigma^*(F).$$

Thus, the pair in the statement of the lemma is the log pullback of the pair

$$\left(X, \frac{1}{n} \mathcal{M} - \gamma F \right).$$

Hence, by Remark 2.6, the pair is not canonical at ν . □

8.1 End of the proof of the Main Theorem

Suppose $X^{(M)}$ is as in Proposition 6.7, that is, the center of ν on $X^{(M)}$ is not a point on the exceptional section of $E^{(M)}$. We consider the three possibilities for the center of ν , and in each case, we obtain a contradiction.

8.2 Case A

Suppose the center B of ν on $X^{(M)}$ is a fiber of the ruled surface $E^{(M)}$. Then the only divisor in the boundary which contains B is $E^{(M)}$. First, suppose $M = 1$, then the pair

$$\left(X^{(1)}, \frac{1}{n} \mathcal{M}^{(1)} - \left(\frac{1}{n} (n - \lambda_1) + \gamma \right) E^{(1)} \right)$$

is not canonical at ν . By Lemma 2.7,

$$\text{mult}_B Z_1 > 4n^2 - 4n\lambda_1 + 4n^2\gamma.$$

Combining this inequality with Corollary 7.9, we get

$$2n^2 - 2\lambda_1^2 + 4n^2\gamma > 4n^2 \frac{n - \lambda_1}{n} + 4n^2\gamma$$

or, equivalently,

$$0 > 2n^2 - 4n\lambda_1 + 2\lambda_1^2 = 2(n - \lambda_1)^2,$$

which is a contradiction.

Now, suppose $M \geq 2$. By Lemma 8.1,

$$\left(X^{(M)}, \frac{1}{n} \mathcal{M}^{(M)} - \left(\frac{1}{n} \sum_{i=2}^M (2n - \lambda_1 - \lambda_i) + (M - 1 + \delta)\gamma \right) E^{(M)} \right)$$

is not canonical at ν . By Lemma 2.7, we have

$$\text{mult}_B Z_M > \sum_{i=2}^M (8n^2 - 4n\lambda_1 - 4n\lambda_i) + 4(M - 1 + \delta)n^2\gamma.$$

Combining this inequality with Corollary 7.9, we get

$$\begin{aligned} \sum_{i=2}^M (4n^2 - 2\lambda_1^2 - \lambda_i^2 - 2\lambda_1\lambda_i) + 4(M - 1 + \delta)n^2\gamma \\ > \sum_{i=2}^M (8n^2 - 4n\lambda_1 - 4n\lambda_i) + 4(M - 1 + \delta)n^2\gamma. \end{aligned}$$

or, equivalently,

$$(M - 1)\lambda_1^2 + \sum_{i=2}^M (2n - \lambda_1 - \lambda_i)^2 < 0,$$

which is again a contradiction.

8.3 Case B

Suppose the center B of ν on $X^{(M)}$ is a point which is not on $E^{(M-1,M)}$. Then the only divisors in the boundary containing B are $E^{(M)}$ and possibly $F^{(M)}$ if $M = 2$.

First, suppose $M = 2$ and $B \in F^{(2)}$, then $\delta = 1$. Then the pair

$$\left(X^{(2)}, \frac{1}{n}\mathcal{M}^{(2)} - \left(\frac{1}{n}(2n - \lambda_1 - \lambda_2) + 2\gamma \right) E^{(2)} - \gamma F^{(2)} \right)$$

is not canonical at ν , and the components of Z_2 which may contain B are: $Z_h^{(2)}$, $C_v^{(2)}$, $C_h^{(2)}$, and $\Delta^{(2)}$. By Corti’s inequality, there are numbers $0 < t, t_F \leq 1$ such that

$$\begin{aligned} \text{mult}_B Z_h^{(2)} + t \text{mult}_B (C_v^{(2)} + C_h^{(2)}) + t_F \text{mult}_B \Delta^{(2)} \\ \geq 4n^2 + t(8n^2 - 4n\lambda_1 - 4n\lambda_2 + 8n^2\gamma) + 4t_F n^2\gamma. \end{aligned}$$

On the other hand,

$$\text{mult}_B \Delta^{(2)} \leq \Delta^{(2)} \cdot E^{(2)} = (\Delta^{(0)} \cdot L_0)_{F^{(0)}} \leq 4n^2\gamma,$$

and we already mentioned the bounds on the other cycles. Combining the bounds, we get

$$2n^2 + t(4n^2 - 2\lambda_1^2 - \lambda_2^2 - 2\lambda_1\lambda_2 + 8n^2\gamma) + 4t_F n^2\gamma > 4n^2 + t(8n^2 - 4n\lambda_1 + 8n^2\gamma) + 4t_F n^2\gamma,$$

or, equivalently,

$$2n^2 + t\lambda_1^2 + t(2n - \lambda_1 - \lambda_i)^2 < 0,$$

which is again a contradiction.

Note that the proper transform on $F^{(2)}$ of the half-line L which gave us so much trouble earlier is a part of $C_h^{(2)}$, and thus its contribution to multiplicity is bounded.

Suppose $M \geq 2$ and $B \notin F^{(2)}$. Then the pair

$$\left(X^{(M)}, \frac{1}{n}\mathcal{M}^{(M)} - \left(\frac{1}{n} \sum_{i=2}^M (2n - \lambda_1 - \lambda_i) + (M - 1 + \delta)\gamma \right) E^{(M)} \right)$$

is not canonical at ν , and the components of Z_M which may contain B are $Z_h^{(M)}$, $C_v^{(M)}$, and $C_h^{(M)}$. By Corti's inequality, there is a number $0 < t \leq 1$ such that

$$\text{mult}_B Z_h^{(M)} + t \text{mult}_B (C_v^{(M)} + C_h^{(M)}) \geq 4n^2 + t \sum_{j=2}^M (8n^2 - 4\lambda_1 n - 4\lambda_j n) + 4t(M - 1 + \delta)n^2\gamma.$$

Combining this inequality with the bounds from Corollary 7.9, we get

$$\begin{aligned} 2n^2 + t \sum_{j=2}^M (4n^2 - 2\lambda_1^2 - \lambda_j^2 - 2\lambda_1 \lambda_j) + 4t(M - 1 + \delta)n^2\gamma \\ > 4n^2 + t \sum_{j=2}^M (8n^2 - 4\lambda_1 n - 4\lambda_j n) + 4t(M - 1 + \delta)n^2\gamma, \end{aligned}$$

or, equivalently,

$$2n^2 + t(M - 1)\lambda_1^2 + \sum_{i=2}^M (2n - \lambda_1 - \lambda_i)^2 < 0,$$

contradiction.

8.4 Case C

Suppose the center B of ν on $X^{(M)}$ is a point on the intersection $E^{(M)} \cap E^{(M-1,M)}$. These are the only divisors of the boundary containing B , that is, $B \notin F^{(M)}$. Denote $M^- = M - 1$ for compactness of formulae. The components of Z_M which may contain B are: $Z_h^{(M)}$, $C_v^{(1,M)}$, $C_h^{(1,M)}$, $C_h^{(M^-,M)}$, and $C_v^{(M^-,M)}$. Furthermore, the pair

$$\begin{aligned} \left(X^{(M)}, \frac{1}{n} \mathcal{M}^{(M)} - \left(\frac{1}{n} \sum_{i=2}^{M^-} (2n - \lambda_1 - \lambda_i) + (M^- - 1 + \delta)\gamma \right) E^{(M^-,M)} \right. \\ \left. - \left(\frac{1}{n} \sum_{i=2}^M (2n - \lambda_1 - \lambda_i) + (M - 1 + \delta)\gamma \right) E^{(M)} \right) \end{aligned}$$

is not canonical at ν . By Corti's inequality, there are numbers $0 < t, t^- \leq 1$ such that

$$\begin{aligned} \text{mult}_B Z_h^{(M)} + t \text{mult}_B (C_v^{(M)} + C_h^{(M)}) + t^- \text{mult}_B (C_v^{(M^-,M)} + C_h^{(M^-,M)}) \\ \geq 4n^2 + t \sum_{j=2}^M (8n^2 - 4\lambda_1 n - 4\lambda_j n) + t^- \sum_{j=2}^{M^-} (8n^2 - 4\lambda_1 n - 4\lambda_j n) \\ + 4t(M - 1 + \delta)n^2\gamma + 4t^-(M^- - 1 + \delta)n^2\gamma. \end{aligned}$$

By combining the inequality with the bounds from Corollary 7.9, we get

$$\begin{aligned} 2n^2 + t \sum_{j=2}^M (4n^2 - 2\lambda_1^2 - \lambda_j^2 - 2\lambda_1 \lambda_j) + t^- \sum_{j=2}^{M^-} (4n^2 - 2\lambda_1^2 - \lambda_j^2 - 2\lambda_1 \lambda_j) \\ + 4t(M - 1 + \delta)n^2\gamma + 4t^-(M^- - 1 + \delta)n^2\gamma > \end{aligned}$$

$$\begin{aligned}
 &> 4n^2 + t \sum_{j=2}^M (8n^2 - 4\lambda_1 n - 4\lambda_j n) + t^- \sum_{j=2}^{M^-} (8n^2 - 4\lambda_1 n - 4\lambda_j n) \\
 &\quad + 4t(M - 1 + \delta)n^2\gamma + 4t^-(M^- - 1 + \delta)n^2\gamma,
 \end{aligned}$$

or, equivalently,

$$2n^2 + t(M - 1)\lambda_1^2 + t \sum_{i=2}^M (2n - \lambda_1 - \lambda_i)^2 + t^-(M^- - 1)\lambda_1^2 + t^- \sum_{i=2}^{M^-} (2n - \lambda_1 - \lambda_i)^2 < 0,$$

which is a contradiction.

In addition, this ends the process of exclusion of points in fibers containing singular points of X , and therefore the proof of the Main Theorem is complete. \square

Appendix. Explicit computations on K^2 -condition

A.1 On preservation of the K^2 -condition

Here, we prove a result in favor of K^2 -condition being preserved in the running of the Sarkisov links at singular points of X .

Throughout this section, we assume that $\pi : X \rightarrow \mathbb{P}^1$ is a del Pezzo fibration and that X is a general quasi-smooth hypersurface of degree $(4, \ell)$ in a $\mathbb{P}(1, 1, 1, 2)$ -bundle $T = \mathbb{P}(1, 1, 1, 2)_{(0, a, b, c)}$. We may assume $0 \leq a \leq b$ without loss of generality, and allow c to take value in \mathbb{Z} . Let u, v be the coordinates on the base \mathbb{P}^1 , and let x, y, z, w be the coordinates on the fiber.

Denote by M_T the divisor with the equation $\{x = 0\}$, and let $M = M_T|_X$. Denote by F a fiber of π on X , and let F_T be a fiber of the $\mathbb{P}(1, 1, 1, 2)$ -bundle. Thus, we have that $X \sim 4M_T + \ell F_T$. Clearly, for the cone of effective divisors on the toric level, we have that

$$\text{Eff}^1(T) = \langle M_T, F_T \rangle \quad \text{if } c \geq 0.$$

By the choice of T , we have $K_T = -5M_T - (2 + a + b + c)F_T$ and by adjunction

$$K_X = -M + (\ell - 2 - a - b - c)F.$$

The following relations for the intersection numbers are rather easy to calculate.

LEMMA A.1. *With notation as above, we have that*

$$\begin{aligned}
 M_T^3 \cdot F_T &= 1/2, \\
 2M_T^4 &= -a - b - c/2, \\
 M^2 \cdot F &= M_T^2 \cdot F_T \cdot X = 2, \\
 M^3 &= M_T^3 \cdot X = \ell/2 - 2a - 2b - c.
 \end{aligned}$$

Proof. The first equality is just computation on $\mathbb{P}(1, 1, 1, 2)$. The second equality follows from the fact that the intersection $\{x = 0\} \cap \{y = 0\} \cap \{z = 0\} \cap \{w = 0\}$ is empty. The rest is computed by intersection with X . \square

PROPOSITION A.2. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a del Pezzo fibration of degree 2. Suppose X is a quasi-smooth hypersurface of bi-degree $(4, \ell)$ in a $\mathbb{P}(1, 1, 1, 2)$ -bundle T over \mathbb{P}^1 . Suppose $c \geq 2b$, then for any I , the variety X_I is also a hypersurface in T of bi-degree $(4, \ell)$.*

Proof. It is enough to prove the statement of the proposition for $X' = X_{\{i\}}$, then the general case follows for any I by induction.

In §4.2, we represented X' as a complete intersection in V' a $\mathbb{P}(1, 1, 1, 2, 2)$ -bundle over \mathbb{P}^1 . We change the coordinates in V' to have a more convenient first equation.

Suppose the singularity for which we constructed a Sarkisov link is given by $x = y = z = u = 0$. Then we can decompose the coefficient at w^2 in the following way:

$$f(u, v) = uv^{N-1} + u^2g(u, v).$$

The equations of X' in V' are

$$us = (v^{N-1} + ug(u, v))w + q(x, y, z; u, v) \quad \text{and} \quad sw + r(x, y, z; u, v) = 0.$$

Let us change the coordinates $s = s_{\text{new}} + g(u, v)w$. Then the new equations are

$$us = v^{N-1}w + q(x, y, z; u, v) \quad \text{and} \quad sw + g(u, v)w^2 + r(x, y, z; u, v) = 0.$$

Since $c \geq 2b$, we may also decompose

$$q(x, y, z; u, v) = uq_1(x, y, z; u, v) + v^{N-1}q_2(x, y, z; u, v).$$

Indeed, the degree of the first equation is $(2, c + N)$, and thus

$$q \in \langle u, v \rangle^{c+N-2b} \subset \langle u, v \rangle^N,$$

where $\langle u, v \rangle$ is the ideal generated by u, v . Then we change coordinates

$$\begin{aligned} s &= s_{\text{new}} + q_1(x, y, z; u, v), \\ w &= w_{\text{new}} - q_2(x, y, z; u, v). \end{aligned}$$

The new first equation is $us = v^{N-1}w$. Clearly, this equation defines a $\mathbb{P}(1, 1, 1, 2)$ -bundle over \mathbb{P}^1 . Let $(u_T, v_T, x_T, y_T, z_T, w_T)$ be the coordinates on T . Consider a map from T into V' defined as follows:

$$\begin{aligned} u &= u_T, & v &= v_T, & x &= x_T, & y &= y_T, & z &= z_T, \\ w &= uw_T, & s &= v^{N-1}w_T. \end{aligned}$$

It is easy to see that this map is well defined and gives an embedding of T into V' . The equation of T in V' is $us = v^{N-1}w$, that is, the first equation of X' . Thus, we have X' embedded into T , substituting coordinates of T into the second equation we conclude that the equation of X' has degree $(4, \ell)$ that is the same degree as that of the equation of X in T . \square

REMARK A.3. Let X be as in Proposition A.2, and suppose X satisfies the K^2 -condition. Then all X_I live in the same family of hypersurfaces by Proposition A.2. Yet, we cannot say whether X_I also satisfies the K^2 -condition.

The general expectation is that K^2 -condition is an open property in the moduli. Given this, let U be an open subset in a given family for which the K^2 -condition is satisfied. Then, by Proposition A.2, X satisfies the total K^2 -condition if and only if $X_I \in U$ for all $I \subset \{1, \dots, N\}$, which suggests that K^2 -total condition must also be open in moduli.

A.2 Sufficient conditions for K^2 -total condition

LEMMA A.4. Let D_1, D_2, D_3 be divisors on X . Fix $I \subset \{1, \dots, N\}$, and let G_1, G_2, G_3 be divisors on X_I of the same bi-degrees. Then $D_1 \cdot D_2 \cdot D_3 = G_1 \cdot G_2 \cdot G_3$.

Proof. It is enough to check it for the case of $\deg G_1 = \deg G_2 = (1, 0)$ and $\deg G_3 = (1, 0)$ or $\deg G_3 = (0, 1)$. We are going to compute the intersections on the ambient scroll $V|_I$. Let $M|_I$ be the divisor given by $x = 0$, and let $F|_I$ be a fiber of the projection onto $\mathbb{P}^1_{u,v}$. Then, similar to Lemma A.1, we compute that $M^4|_I \cdot F|_I = 1/4$ and $4M^5|_I = -a - b - \ell/2$. Now, suppose $\deg G_3 = (0, 1)$, then

$$G_1 \cdot G_2 \cdot G_3 = M^2|_I \cdot F|_I \cdot (2, \ell - c) \cdot (4, \ell) = 2.$$

If instead $\deg G_3 = (1, 0)$, then

$$G_1 \cdot G_2 \cdot G_3 = M^3|_I \cdot (2, \ell - c) \cdot (4, \ell) = \ell/2 - 2a - 2b - c.$$

These intersections coincide with the ones we computed on X in Lemma A.1 □

LEMMA A.5. For X_I for any I we have the following. Let M_I denote the divisor given by $x = 0$, and let F_I be a fiber of π_I . Then the divisor $M_I + bF_I$ is nef.

Proof. The base locus of $|M_I + bF_I|$ is contained in $\{x = y = z = 0\}$. On the other hand,

$$\{x = y = z = 0\} = \left\{ \prod_{i \in I} l_i = w = 0 \right\} \cup \left\{ \prod_{j \notin I} l_j = s = 0 \right\} = \text{Sing } X_I,$$

which is of cardinality N . Thus, the base locus of $|M_I + bF_I|$ is finite, and hence $M_I + bF_I$ is nef. □

Proof of Proposition 1.7. First, let us compute K_{X_I} . By adjunction, we have

$$\begin{aligned} \deg K_{X_I} &= \deg K_{V|_I} + (2, c + N) + (4, 2c + N) = \\ &= \left(-1, 3c + 2N - 2 - a - b - (c + |I|) - (c + N - |I|) \right). \end{aligned}$$

Since $d = 2c + N$, we have

$$\deg K_{X_I} = (-1, \ell - 2 - a - b - c) = \deg K_X.$$

If the intersection of $K^2_{X_I}$ with a nef divisor is nonpositive, then $K^2_{X_I}$ is not in the interior of the Mori cone. By Lemma A.5, the divisor $D_I = M_I + bF_I$ is nef, and by Lemma A.4, the intersection of $K^2_{X_I}$ with D_I is the same on all models X_I , and thus it is enough to compute it on X .

We compute

$$\begin{aligned} K^2_X \cdot D &= (-1, \ell - a - b - c - 2)^2 \cdot (1, b) = M^3 + (4 + 2a + 2b + 2c - 2\ell + b)F \cdot M^2 = \\ &= 8 + 2a + 4b + 3c - \frac{7}{2}\ell. \end{aligned}$$

If this value is nonpositive, all X_I satisfy the K^2 -condition. This finishes the proof. □

Proof of Corollary 1.8. By assumption, $c \geq 2b$, and thus Proposition 1.7 and the bounds $0 \leq 4a \leq 4b \leq 2c \leq \ell$ imply that

$$a \leq 4, \quad 2b \leq 4 + a, \quad \text{and } 2c \leq 4 + a + 2b.$$

Using these inequalities, we fill the table. □

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