SOME REMARKS ON COMPLETE INTEGRAL CLOSURE

WILLIAM HEINZER

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1. Introduction

This paper continues an investigation of the complete integral closure of an integral domain which was begun in [2]. We recall that if D is an integral domain with quotient field K then an element x of K is said to be *almost integral* over D if there exists a nonzero element y of D such that yx^n is an element of D for each positive integer n. The set D^* of elements of K almost integral over D is called the *complete integral closure* of D and D is said to be *completely integrally closed* if $D^* = D$.

An element x of K is almost integral over D if and only if the ring extension $D[x]$ is contained in a finite D-module contained in K. Thus if x is integral over D, then x is almost integral over D. In particular, if D is completely integrally closed then D is integrally closed and hence an intersection of valuation rings. Since a valuation ring is completely integrally closed if and only if it has rank $\leqq 1$, Krull was led to conjecture that D is completely integrally closed if and only if D is an intersection of valuation rings of rank $\leqq 1$ [8]. Nakayama in [10] showed this conjecture to be false. However, under an additional finiteness condition on a set of valuation rings representing the domain the conjecture is known to be true ([9] p. 549). The problem of characterizing those completely integrally closed domains which are the intersection of rank one valuation rings remains open. For example, it appears to be unknown whether a completely integrally closed domain having finite valuative dimension [1] is an intersection of rank one valuation rings.

The complete integral closure D^* of a domain D need not be completely integrally closed [2]. Krull observes in [8] that if D is a valuation ring on a field K then D^* is either the unique rank one valuation ring of K containing D or else $D^* = K$, the second case occurring when D is contained in no rank one valuation ring of K. In particular, when D is a valuation ring,

[1] An integral domain J is said to have *valuative dimension* n if there is a valuation ring between J and the quotient field of J which has rank n and no such valuation ring having rank greater than n [7].

D^* is completely integrally closed. The problem of determining the complete integral closure of a Prüfer domain [2] is considered in [2] and the following question is there raised. If D is a Prüfer domain, is D^* completely integrally closed? Our purpose here is to show that D^* need not be completely integrally closed when D is a Prüfer domain.

2. The example

Jaffard in [6] p. 78 has shown that if G is a lattice-ordered abelian group, then there exists an integral domain having G as its group of divisibility. We recall that if D is an integral domain with quotient field K, if K^* is the set of nonzero elements of K, and if U is the multiplicative group of units of D, then the quotient group K^*/U, ordered by defining $x' \geqq y'$ if and only if $x/y \in D$, is called the *group of divisibility* of D. (Here x' denotes the equivalence class of K^*/U containing the element x of K^*).

Our example of a Prüfer domain whose complete integral closure is not completely integrally closed is obtained by employing Jaffard's construction on the following lattice-ordered abelian group.

Let Z denote the ring of integers and N the set of positive integers. We consider the additive group $H = Z \oplus Z$, ordered lexicographically $((a, b) \geqq (c, d)$ if and only if $a > c$ or $a = c$ and $b \geqq d)$. Let

$$G = \{f : N \to H | f(n) = (a_n, b_n)$$

and $a_n = 0$ for all but a finite number of positive integers $n\}$. G is an abelian group under coordinatewise addition: $(f+g)(n) = f(n)+g(n)$ for any $f, g \in G$ and $n \in N$. We order G by defining $f \geqq g$ if and only if $f(n) \geqq g(n)$ in H for each positive integer n. It is easy to check that G so defined is a lattice-ordered abelian group.

Let D be an integral domain constructed using Jaffard's theorem which has G as its group of divisibility and let K^* be the set of nonzero elements of the quotient field of D. We observe that an element x of K^* is almost integral over D if and only if the corresponding element x' of G is such that $x'(n) = (a_n, b_n)$ where $a_n \geqq 0$ for all n and $b_n \geqq 0$ for all but a finite number of positive integers n. This follows directly from the fact that x is almost integral over D if and only if there exists an element y' of G such that $y' > 0$ and $y'+mx' \geqq 0$ for all positive integers m; we omit the details of the proof. Hence D^*, the complete integral closure of D, consists of zero and all $x \in K^*$ such that $x'(n) = (a_n, b_n)$ with $a_n \geqq 0$ for all n and $b_n \geqq 0$ for all but a finite number of positive integers n.

[2] D is called a Prüfer domain if every finitely generated ideal of D is invertible. Equivalently, D is Prüfer if and only if the localization D_P is a valuation ring for each maximal ideal P of D.

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We show next that an element x of K^* is almost integral over D^* provided $x'(n) = (a_n, b_n)$ is such that $a_n \geqq 0$ for all n. It will then follow that D^* is not completely integrally closed. For example, any element of K^* which is associated with the element f of G defined by $f(n) = (0, -1)$ for all n will be almost integral over D^* but not in D^*. To show that an element x of K^* is almost integral over D^* we need only find a nonzero element y of D^* such that $yx^m \in D^*$ for each positive integer m. Let y' and x' be the elements of G associated with y and x and let $y'(n) = (c_n, d_n)$, $x'(n) = (a_n, b_n)$. Then $yx^m \in D^*$ if and only if

$$(y'+mx')(n) = (c_n+ma_n, d_n+mb_n)$$

is such that $c_n+ma_n \geqq 0$ for all n and $d_n+mb_n \geqq 0$ for all but a finite number of positive integers n. We choose y' by defining $c_n = 0$ for all n and $d_n = \sup \{0, -nb_n\}$. Then $d_n+mb_n \geqq 0$ for all $n \geqq m$ and hence for all but a finite number of positive integers n. Thus $yx^m \in D^*$ and we conclude that an element x of K^* is almost integral over D^* provided $x'(n) = (a_n, b_n)$ with $a_n \geqq 0$ for all n.

We have therefore shown that the integral domain D has the property that D^* is not completely integrally closed. But as Ohm remarks in [11], an integral domain constructed by means of Jaffard's theorem has the additional property that each finitely generated ideal is principal.[3] In particular, finitely generated ideals are invertible so that such a domain is Prüfer. Thus D provides the desired example of a Prüfer domain whose complete integral closure D^* is not completely integrally closed. In fact, D has the QR-property[4], and yet D^* is not completely integrally closed.

3. Some further observations

We now wish to make some comments about the integral domain D constructed in the previous section. We first remark that $D = \bigcap_{i=1}^{\infty} V_i$ where $\{V_i\}_{i=1}^{\infty}$ is the set of rank 2 valuation rings defined as follows:

$$V_i = \{0\} \cup \{x \in K^* | x'(i) = (a_i, b_i) \geqq (0, 0) \text{ in } H\}.$$

(Here, as usual, x' denotes the equivalence class in $G = K^*/U$ which contains x.) It is straightforward to check that V_i so defined is a rank 2 valuation ring and that $D = \bigcap_{i=1}^{\infty} V_i$. The rank one valuation overring of V_i is

$$W_i = \{0\} \cup \{x \in K^* | x'(i) = (a_i, b_i) \text{ with } a_i \geqq 0\}$$

[3] A proof of this fact is given in [5].

[4] An integral domain J is said to have the *QR-property* if every ring between J and its quotient field is a quotient ring of J [4].

and D^{**}, the complete integral closure of D^*, is equal to $\bigcap_{i=1}^{\infty} W_i$. Thus D^{**} is completely integrally closed.[5]

We next wish to show that D has infinite Krull dimension. Let G^+ denote the set of positive elements of the ordered group G and let $\varphi : K^* \to K^*/U = G$ be the canonical homomorphism of K^* onto the group of divisibility of D. We consider a free ultrafilter $\mathscr{U}$ on the set N of positive integers ([1] p. 25). For $f \in G^+$ let $M_f = \{n \in N | f(n) = (a_n, b_n) \text{ with } b_n > 0\}$. If $P' = \{f \in G^+ | M_f \in \mathscr{U}\}$, then P' is a nonempty subset of G and the fact that $\mathscr{U}$ is an ultrafilter implies that P' has the following properties: [6]

1. $f \in P'$ and $g \geqq f$ implies that $g \in P'$
2. $f, g \in P'$ implies that $\inf \{f, g\}$ is also in P'
3. $f, g \notin P'$ implies that $f+g \notin P'$.

It is easily checked that the set $P = \varphi^{-1}(P') \cup \{0\}$ is a maximal ideal of D. Moreover, the fact that $\mathscr{U}$ is a free ultrafilter implies that P does not contain the center of the valuation ring W_i on D. We show that D is infinite dimensional by showing that the valuation ring D_P is contained in no rank one valuation ring of K and hence has no minimal nonzero prime ideal. We have already observed that $D^{**} = \bigcap_{i=1}^{\infty} W_i$. Each W_i is a rank one discrete valuation ring and a nonzero element of K is a nonunit in only finitely many of the W_i. It follows that D^{**} is a Krull domain. And since D^{**} is an overring of D contained in K each finitely generated ideal of D^{**} is principal. Therefore D^{**} is a Dedekind domain and $\{W_i\}_{i=1}^{\infty}$ is the set of nontrivial valuation rings between D^{**} and K. But every overring of D which is completely integrally closed contains D^{**}. Hence $\{W_i\}$ is the set of rank one valuation rings between D and K. Since P does not contain the center of W_i on D, D_P is not contained in W_i. We conclude that D_P is contained in no rank one valuation ring of K. It follows that D_P, and therefore, D, has infinite Krull dimension.

The above observations lead us to make the following conjecture. If J is a finite-dimensional Prüfer domain, then J^* is completely integrally closed. In this connection it would be interesting to know if a finite-dimensional completely integrally closed Prüfer domain is an intersection of rank one valuation rings. More generally, if J is a completely integrally closed domain having finite valuative dimension, is J an intersection of rank one valuation rings? Nakayama's example is not finite-dimensional and it

[5] We have in fact been unable to find an integral domain J for which J^{**} is not completely integrally closed.

[6] Properties (1) and (2) follow from the fact that a filter is closed with respect to oversets and finite intersections, and property (3) is a consequence of the fact that the complement of an ultrafilter is closed under finite unions.

appears that no simple modification of Nakayama's construction will provide a finite-dimensional completely integrally closed domain which is not an intersection of rank one valuation rings.

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Louisiana State University
Bâton Rouge, Louisiana