Bull. Aust. Math. Soc. **96** (2017), 299–306
doi:10.1017/S000497271700017X

SHRINKING PROJECTION ALGORITHMS FOR THE SPLIT COMMON NULL POINT PROBLEM

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(Received 20 December 2016; accepted 3 January 2017; first published online 29 March 2017)

Abstract

We consider the split common null point problem in Hilbert space. We introduce and study a shrinking projection method for finding a solution using the resolvent of a maximal monotone operator and prove a strong convergence theorem for the algorithm.

2010 *Mathematics subject classification*: primary 47H05; secondary 47H09.

Keywords and phrases: maximal monotone operator, resolvent operator, projection, split common null point problem, shrinking projection algorithm.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces and $T : H_1 \to H_2$ a bounded linear operator. Suppose that C and D are nonempty, closed and convex subsets of H_1 and H_2, respectively. The split feasibility problem is to find $z \in H_1$ such that $z \in C \cap T^{-1}D$. Censor and Elfving [7] introduced the split feasibility problem in finite-dimensional Hilbert space to study problems which arise from signal detection and image recovery. The split feasibility problem has many practical applications such as image restoration, computer tomography and radiation therapy treatment planning [3, 6, 8, 10]. It is known that if $C \cap T^{-1}D \neq \emptyset$, the problem is equivalent to $z = P_C(I - rT^*(I - P_D)T)z$, where T^* is the adjoint operator of T, P_C and P_D are the metric projections from H_1 onto C and from H_2 onto D, respectively, and $r > 0$ is a positive constant (see [2] for more details). Byrne [3, 4] developed the so-called CQ algorithm as an iterative method to solve the split feasibility problem in infinite-dimensional Hilbert space:

$$x_{n+1} = P_C(x_n - \gamma T^*(x_n - P_D T x_n)),$$

where $\gamma \in (0, 1/\lambda)$, with λ being the spectral radius of the operator A^*A. Xu [16] studied the convergence of the CQ algorithm and a refinement involving Mann's algorithm. Strong convergence has been established for a number of algorithms (see, for example, [1, 13, 16]). The author recently proved the following result

Vahid Dadashi is supported by the Islamic Azad University—Sari Branch.

(in 'Shrinking projection algorithms for split common null point problem in Hilbert spaces', submitted for publication).

THEOREM 1.1. *Let X be a uniformly convex and smooth Banach space with duality mapping J_X. Suppose that H is a Hilbert space and $T : H \to X$ is a bounded linear operator such that $T \neq 0$ and let T^* denote the adjoint operator of T. Let C and D be nonempty, closed and convex subsets of H and X, such that $C \cap T^{-1}D \neq \emptyset$. Let P_C and P_D denote the metric projections of H onto C and of X onto D, respectively. Let $\{x_n\}$ be the sequence generated by the algorithm*

$$\begin{cases} z_n = P_C(x_n - rT^*J_X(Tx_n - P_DTx_n)) \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n \\ C_{n+1} = \{z \in C_n : \langle y_n - z, x_n - y_n\rangle \geq 0\} \\ x_{n+1} = P_{C_{n+1}}(x_1), \end{cases}$$

where $C_1 = C$ and $x_1 \in H$. If $0 < \frac{1}{2}\|T\| \leq \alpha_n \leq a < 1$ and $r < 1$, then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap T^{-1}D$ and $z_0 = P_{C\cap T^{-1}D}x_1$.

Using the metric resolvents of maximal monotone operators and metric projections, Byrne *et al.* [5] proved a strong convergence theorem for the related split common null point problem. Takahashi and Yao [14] recently proved the following extension.

THEOREM 1.2 [14]. *Let H be a Hilbert space and let X be a uniformly convex and smooth Banach space. Let J_X be the duality mapping on X. Let A and B be maximal monotone operators of H into 2^H and of X into 2^{X^*}, respectively, such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to X$ be a bounded linear operator such that $T \neq 0$ and let T^* denote the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Take $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the algorithm*

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^*J_X(Tx_n - Q_{\mu_n}Tx_n)) \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n\cap D_n}(x_1), \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\}, \{\mu_n\} \subset (0,\infty)$ satisfy the conditions

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \mu_n \quad \text{and} \quad 0 < c \leq \lambda_n\|T\|^2 \leq d < 2$$

for some $a, b, c, d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0\cap T^{-1}(B^{-1}0)}x_1$.

Motivated by these theorems, we use the resolvent of a maximal monotone operator to introduce and study a new hybrid algorithm for solving the split feasibility problem in Hilbert space.

https://doi.org/10.1017/S000497271700017X Published online by Cambridge University Press

2. Preliminaries

Throughout the paper, X is a real Banach space. We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. As in [9], the *normalised duality mapping* J from X into the family of nonempty w^*-compact subsets of its dual X^* is defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\} \quad \text{for } x \in E.$$

Lemma 2.1 [12]. *Let X be a real Banach space and J the duality mapping. Then, for each $x, y \in X$,*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle.$$

The norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U := \{z \in X : \|z\| = 1\}$. The norm is said to be uniformly Gâteaux differentiable if for $y \in U$, the limit is attained uniformly for $x \in U$. The space X is said to have a Fréchet differentiable norm if for each $x \in X$ the limit in (2.1) is attained uniformly for $y \in U$. The space X is said to have a uniformly Fréchet differentiable norm (and X is said to be uniformly smooth) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$. It is known that X is smooth if and only if each duality mapping J is single-valued. It is also well known that if X has a uniformly Gâteaux differentiable norm, J is uniformly norm-to-weak* continuous on each bounded subset of X.

The normed space X is called uniformly convex if for any $\varepsilon \in (0, 2]$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in X$ satisfying $\|x\| = 1$, $\|y\| = 1$ and $\|x - y\| \ge \varepsilon$, then $\|\frac{1}{2}(x + y)\| \le 1 - \delta$.

Definition 2.2. The multifunction $A : X \to 2^{X^*}$ is called a *monotone operator* if for every $x, y \in X$,

$$\langle x^* - y^*, x - y\rangle \ge 0 \quad \text{for all } x^* \in A(x) \text{ and } y^* \in A(y).$$

A monotone operator $A : X \to 2^{X^*}$ is said to be *maximal monotone* when its graph is not properly included in the graph of any other monotone operator on the same space and its *effective domain* is defined by $D(A) = \{x \in X : A(x) \ne \emptyset\}$.

Let C be a closed convex subset of X. The operator P_C is called a metric projection operator if it assigns to each $x \in X$ its nearest point $y \in C$ such that

$$\|x - y\| = \min\{\|x - z\| : z \in C\}.$$

It is known that the metric projection operator P_C is continuous in a uniformly convex Banach space X and uniformly continuous on each bounded subset of X if, in addition, X is uniformly smooth. The element y is called the metric projection of X onto C and denoted by $P_C x$. It exists and is unique at any point of a reflexive strictly convex space.

Lemma 2.3. *Let H be a Hilbert space and C a nonempty, closed and convex subset of H. Then, for $x \in H$, the element z satisfies $z = P_C x$ if and only if*

$$\langle x - z, z - y \rangle \geq 0 \quad \textit{for all } y \in C.$$

Let C be a closed convex subset in a Hilbert space H. The metric projection has the property

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \quad \text{for all } x, y \in H.$$

Therefore, the metric projection is a firmly nonexpansive operator in H.

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space X, define s-$\underline{\mathrm{Lim}}C_n$ and w-$\overline{\mathrm{Lim}}C_n$ as follows: $x \in$ s-$\underline{\mathrm{Lim}}C_n$ if and only if there exists $\{x_n\} \subset X$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$ w-$\overline{\mathrm{Lim}}C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset X$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. We say that $\{C_n\}$ converges to C_0 in the sense of Mosco [11] if $C_0 =$ s-$\underline{\mathrm{Lim}}C_n =$ w-$\overline{\mathrm{Lim}}C_n$, and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [11]. The following lemma was proved by Tsukada [15].

Lemma 2.4 [15]. *Let X be a reflexive and strictly convex Banach space and $\{C_n\}$ a sequence of nonempty, closed and convex subsets of X. If $C_0 = \text{M-lim}_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in X$, $P_{C_n}x$ converges weakly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the metric projections of X onto C_n and C_0, respectively. Moreover, if X has the Kadec–Klee property, the convergence is strong.*

Definition 2.5. Let X be a reflexive Banach space and $A : X \to 2^{X^*}$ be a maximal monotone operator. By [9, Ch. 5, Theorem 3.4], the relation $0 \in \lambda A\tilde{x} + J(\tilde{x} - x)$ has a unique solution $\tilde{x} = x_\lambda \in D(A)$ for every $x \in X$. The operator $J_\lambda^A : X \to D(A)$ defined by $J_\lambda^A(x) = x_\lambda$ is called the *metric resolvent* of A of order λ and x_λ satisfies $\lambda^{-1} J(x - x_\lambda) \in A(x_\lambda)$. In the following, we denote the metric resolvent J_λ^A by J_λ.

In a Hilbert space, the metric resolvent is also a firmly nonexpansive operator.

3. Main results

In this section, using a shrinking projection method, we prove a strong convergence theorem for the solution of the split common null point problem in Hilbert space.

Theorem 3.1. *Let X be a uniformly convex Banach space with a Gâteaux differentiable norm and duality mapping J_X. Let H be a Hilbert space and $T : H \to X$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Let A and B be maximal monotone operators of H into 2^H and of X into 2^{X^*}, respectively, such that*

https://doi.org/10.1017/S000497271700017X Published online by Cambridge University Press

$F := A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let J_λ be the resolvent of A for $\lambda > 0$ and Q_μ the metric resolvent of B for $\mu > 0$. Generate the sequence $\{x_n\}$ by the algorithm

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^* J_X(Tx_n - Q_{\mu_n}Tx_n)) \\ y_n = \alpha_n x_n + (1-\alpha_n)z_n, \\ C_n = \{z \in C_{n-1} : \langle y_n - z, x_n - y_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n}(x_1), \end{cases} \tag{3.1}$$

where $C_1 = H$ and $x_1 \in H$. If $0 < \|T\| \leq 2\alpha_n \leq a < 2$, $0 < b \leq \mu_n$ and $0 < c \leq \lambda_n < 1$, then $\{x_n\}$ converges strongly to a point $z_0 \in F$, where $z_0 = P_F x_1$.

Proof. We first prove that the sequence $\{x_n\}$ generated by algorithm (3.1) is well defined. It is easy to check that C_n is closed and convex for each $n \in \mathbb{N}$. We claim that $F \subset C_n$ for each $n \in \mathbb{N}$. Clearly, $F \subset C_1$. Assume that $F \subset C_{n-1}$ for some $n \in \mathbb{N}$. If $p \in F$, then $0 \in A(p)$ and $0 \in B(Tp)$. Since Q_{μ_n} is the metric resolvent of B,

$$\frac{1}{\mu_n} J_X(Tx_n - Q_{\mu_n}Tx_n) \in B(Q_{\mu_n}Tx_n), \tag{3.2}$$

and hence, by the monotonicity of B,

$$0 \leq \langle Q_{\mu_n}Tx_n - Tp, J_X(Tx_n - Q_{\mu_n}Tx_n)\rangle.$$

From the definition of z_n,

$$x_n - z_n - \lambda_n T^* J_X(Tx_n - Q_{\mu_n}Tx_n) \in \lambda_n A z_n. \tag{3.3}$$

From the monotonicity of A,

$$0 \leq \langle x_n - z_n - \lambda_n T^* J_X(Tx_n - Q_{\mu_n}Tx_n), z_n - p\rangle,$$

and hence

$$\begin{aligned} \langle x_n - z_n, z_n - p\rangle &\geq \lambda_n\langle T^* J_X(Tx_n - Q_{\mu_n}Tx_n), z_n - p\rangle \\ &= \lambda_n\langle J_X(Tx_n - Q_{\mu_n}Tx_n), Tz_n - Tp\rangle \\ &= \lambda_n\langle J_X(Tx_n - Q_{\mu_n}Tx_n), Tz_n - Tx_n\rangle \\ &\quad + \lambda_n\langle J_X(Tx_n - Q_{\mu_n}Tx_n), Tx_n - Q_{\mu_n}Tx_n\rangle \\ &\quad + \lambda_n\langle J_X(Tx_n - Q_{\mu_n}Tx_n), Q_{\mu_n}Tx_n - Tp\rangle \\ &\geq \lambda_n\langle J_X(Tx_n - Q_{\mu_n}Tx_n), Tz_n - Tx_n\rangle + \lambda_n\|Tx_n - Q_{\mu_n}Tx_n\|^2. \end{aligned} \tag{3.4}$$

From the definition of y_n,

$$\begin{aligned} &\langle y_n - p, x_n - y_n\rangle \\ &\quad = \langle \alpha_n(x_n - p) + (1-\alpha_n)(z_n - p), (1-\alpha_n)(x_n - z_n)\rangle \\ &\quad = \alpha_n(1-\alpha_n)\langle x_n - z_n + z_n - p, x_n - z_n\rangle + (1-\alpha_n)^2\langle z_n - p, x_n - z_n\rangle \\ &\quad = \alpha_n(1-\alpha_n)\|x_n - z_n\|^2 + \alpha_n(1-\alpha_n)\langle z_n - p, x_n - z_n\rangle + (1-\alpha_n)^2\langle z_n - p, x_n - z_n\rangle \\ &\quad = \alpha_n(1-\alpha_n)\|x_n - z_n\|^2 + (1-\alpha_n)\langle z_n - p, x_n - z_n\rangle. \end{aligned} \tag{3.5}$$

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By (3.4), (3.5) and the assumptions of the theorem,

$$\begin{aligned}\langle y_n - p, x_n - y_n\rangle &\geq \alpha_n^2(1-\alpha_n)\|x_n - z_n\|^2 + \lambda_n(1-\alpha_n)\langle J_X(Tx_n - Q_{\mu_n}Tx_n), Tz_n - Tx_n\rangle \\ &\quad + \lambda_n(1-\alpha_n)\|Tx_n - Q_{\mu_n}Tx_n\|^2 \\ &\geq (1-\alpha_n)[\alpha_n^2\|x_n - z_n\|^2 - \lambda_n\|T\|\,\|Tx_n - Q_{\mu_n}Tx_n\|\,\|z_n - x_n\| \\ &\quad + \lambda_n\|Tx_n - Q_{\mu_n}Tx_n\|^2] \\ &\geq (1-\alpha_n)[\alpha_n^2\|x_n - z_n\|^2 - 2\alpha_n\lambda_n\|Tx_n - Q_{\mu_n}Tx_n\|\,\|z_n - x_n\| \\ &\quad + \lambda_n^2\|Tx_n - Q_{\mu_n}Tx_n\|^2] \\ &= (1-\alpha_n)(\alpha_n\|x_n - z_n\| - \lambda_n\|Tx_n - Q_{\mu_n}Tx_n\|)^2 \geq 0, \end{aligned} \tag{3.6}$$

which implies that $p \in C_n$. By mathematical induction, we see that $F \subset C_n$ for every $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well defined.

Since F is nonempty, closed and convex, there exists a unique element $z_0 \in F \subset C_n$ such that $z_0 = P_F x_1$. Since $x_{n+1} = P_{C_n}(x_1)$,

$$\|x_{n+1} - x_1\| \leq \|x_1 - z_0\|, \tag{3.7}$$

for every $n \in \mathbb{N}$, and hence the sequence $\{x_n\}$ is bounded.

Let $D = \cap_{n=1}^{\infty} C_n$. Since $F \subset C_n$ for every $n \in \mathbb{N}$, we see that $D \neq \emptyset$. By Lemma 2.4, $x_n = P_{C_n}x_1 \to P_D x_1 = w_0$. We claim that $w_0 \in F$. Since $w_0 \in C_n$,

$$0 \leq \langle y_n - w_0, x_n - y_n\rangle = -\|x_n - y_n\|^2 + \langle x_n - w_0, x_n - y_n\rangle,$$

therefore,

$$\|x_n - y_n\|^2 \leq \langle x_n - w_0, x_n - y_n\rangle \leq \|x_n - w_0\|\,\|x_n - y_n\|.$$

Consequently, $\|x_n - y_n\| \to 0$. Hence, $y_n \to w_0$ and

$$\|x_n - z_n\| = \frac{1}{1-\alpha_n}\|x_n - y_n\| \to 0. \tag{3.8}$$

From (3.6),

$$0 \leq (1-\alpha_n)(\alpha_n\|x_n - z_n\| - \lambda_n\|Tx_n - Q_{\mu_n}Tx_n\|)^2 \leq \langle y_n - w_0, x_n - y_n\rangle \to 0,$$

so that

$$\|Tx_n - Q_{\mu_n}Tx_n\| \to 0, \tag{3.9}$$

and hence

$$\|J_X(Tx_n - Q_{\mu_n}Tx_n)\| \to 0. \tag{3.10}$$

Therefore,

$$\|T^*J_X(Tx_n - Q_{\mu_n}Tx_n)\| \leq \|T\|\,\|Tx_n - Q_{\mu_n}Tx_n\| \to 0. \tag{3.11}$$

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Since $\{x_n\}$ converges strongly to w_0 and T is a bounded linear operator, $\{Tx_n\}$ converges strongly to Tw_0. From (3.9), $\{Q_{\mu_n}Tx_n\}$ converges strongly to Tw_0. By (3.2) and the monotonicity of B,

$$0 \leq \left\langle v - Q_{\mu_n}Tx_n, w - \frac{1}{\mu_n}J_X(Tx_n - Q_{\mu_n}Tx_n)\right\rangle, \tag{3.12}$$

for each $(v, w) \in B$. Taking the limit in (3.12) as $n \to \infty$ and using (3.10) shows that $\langle v - Tw_0, w - 0\rangle \geq 0$, and the maximal monotonicity of B implies that $w_0 \in T^{-1}(B^{-1}0)$. Similarly, by (3.3) and monotonicity of A,

$$0 \leq \langle x_n - z_n - \lambda_n T^* J_X(Tx_n - Q_{\mu_n}Tx_n) - w, z_n - v\rangle, \tag{3.13}$$

for each $(v, w) \in A$. Taking the limit in (3.13) as $n \to \infty$ and using (3.8) and (3.11) shows that $\langle 0 - w, w_0 - v\rangle \geq 0$, and the maximal monotonicity of A implies that $w_0 \in A^{-1}0$. Therefore, $w_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0) = F$.

We now show that $w_0 = P_F(x_1)$. From (3.7), $\lim_{n\to\infty} \|x_n - x_1\| \leq \|x_1 - z_0\|$. Therefore, from $z_0 = P_F(x_1)$, $w_0 \in F$ and

$$\|x_1 - z_0\| \leq \|w_0 - x_1\| = \lim_{n\to\infty} \|x_n - x_1\| \leq \|x_1 - z_0\|.$$

This, together with the uniqueness of $P_F(x)$, implies that $w_0 = z_0 = P_F(x)$ and hence $\{x_n\}$ converges strongly to $P_F(x)$. This completes the proof. □

COROLLARY 3.2. *Suppose that H is a Hilbert space and A and B are maximal monotone operators of H into 2^H such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the resolvent of B for $\mu > 0$. Generate the sequence $\{x_n\}$ by the algorithm*

$$\begin{cases} z_n = J_{\lambda_n}((1 - \lambda_n)x_n + \lambda_n Q_{\mu_n} x_n) \\ y_n = \alpha_n x_n + (1 - \alpha_n)z_n, \\ C_n = \{z \in C_{n-1} : \langle y_n - z, x_n - y_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n}(x_1), \end{cases}$$

where $C_1 = H$ and $x_1 \in H$. If $\frac{1}{2} \leq \alpha_n \leq a < 1$, $0 < b \leq \mu_n$ and $0 < c \leq \lambda_n < 1$, then $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap B^{-1}0$, where $z_0 = P_{A^{-1}0\cap B^{-1}0}x_1$.

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https://doi.org/10.1017/S000497271700017X Published online by Cambridge University Press