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TENSOR PRODUCTS OF POSITIVE DEFINITE QUADRATIC FORMS IV

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Let L, M, N be positive definite quadratic lattices over $\boldsymbol{Z}$. We treated the following problem in [5], [6]:

If $L \otimes M$ is isometric to $L \otimes N$, then is M isometric to N?

We gave a condition (**) in [6] such that the answer is affirmative for an indecomposable lattice L satisfying (**), and we gave some examples. In this paper we introduce a certain apparently weaker condition (**A**) than the condition (**), and we show that the condition (**A**) implies the condition (**) and more on integral orthogonal groups than a result in [6].

By a positive lattice we mean a lattice of a positive definite quadratic space over the rational number field $\boldsymbol{Q}$. Terminology and notations are generally those from [8].

Let L be an indecomposable positive lattice. We consider the following two conditions (**A**), (**B**).

(**A**) For any given positive lattices M, N and for any isometry σ from $L \otimes M$ on $L \otimes N$ which satisfies that $\sigma(L \otimes m) = L \otimes n$ $(m \in M, n \in N)$ implies $m = 0, n = 0$, there is a basis $\{v_1, \cdots, v_n\}$ of L (depending on M, N, σ) such that

(i) $[M : \sum_{i=1}^{n} M_i] < \infty, [N : \sum_{i=1}^{n} N_i] < \infty$ where $M_i = \{m \in M\,; \sigma(L \otimes m) \subset v_i \otimes N\}, N_i = \{n \in N\,; \sigma^{-1}(L \otimes n) \subset v_i \otimes M\}$, and

(ii) $\sigma(v_i \otimes M_i) \subset v_i \otimes N_i$ for $i = 1, 2, \cdots, n$.

(**B**) Let X be an indecomposable positive lattice. Then we have

(i) $L \otimes X$ is indecomposable,

(ii) if X is isometric to $L \otimes X'$, then X' is uniquely determined by X up to isometries, and

(iii) if $X = \otimes^m L \otimes X'$ and $X' \not\cong L \otimes K$ for any positive lattice K, then the orthogonal group $O(X)$ of X is generated by $O(L), O(X')$ and

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interchanges of L's.

Our aim is to prove

THEOREM. *For an indecomposable positive lattice L, the conditions* **(A)**, **(B)** *are equivalent.*

1.

In this section we prove that **(A)** implies **(B)**. Through this section L denotes an indecomposable positive lattice satisfying the condition **(A)**.

1.1. LEMMA 1. *Let M, N, M_i, N_i, σ be those as in the condition* **(A)**. *Then we have $M = \sum M_i$, $N = \sum N_i$, $\sigma(L \otimes M_i) = v_i \otimes N$ and $M \cong N \cong L \otimes K$. Defining μ by $\sigma(v_i \otimes m) = v_i \otimes \mu(m)$ $(m \in M_i)$, we get an isometry μ from M on N such that $\mu(M_i) = N_i$. Especially the condition* **(A)** *implies the condition* (**) *in* [6].

Proof. Take any element $m = \sum m_i$ of M where $m_i \in QM_i$; then $\sigma(v_1 \otimes m) = \sum \sigma(v_1 \otimes m_i)$ and $\sigma(v_1 \otimes m_i) = v_i \otimes n_i$ for some n_i in QN by the definition of M_i. Since $\sigma(v_1 \otimes m) = \sum v_i \otimes n_i$ is an element of $L \otimes N$ and $\{v_i\}$ is a basis of L, we have $n_i \in N$. Hence it implies $v_1 \otimes m_i = \sigma^{-1}(v_i \otimes n_i) \in L \otimes M$ and so $m_i \in M$. As M_i is obviously primitive in M, we have $m_i \in M_i$ and $M = \sum M_i$. Since $\sigma(L \otimes M_i) \subset v_i \otimes N$, M is a direct sum of M_i, and we have $\sigma(L \otimes M) = \sigma(L \otimes \sum M_i) \subset \sum v_i \otimes N = L \otimes N$. This implies $\sigma(L \otimes M_i) = v_i \otimes N$. Hence N is isometric to $L \otimes K$ for some positive lattice K. $\sigma(L \otimes M_i) = v_i \otimes N$ implies $\operatorname{rank} M_i = \operatorname{rank} N / \operatorname{rank} L$. Similarly we have $N = \sum N_i$ (direct sum) and $\operatorname{rank} N_i = \operatorname{rank} M / \operatorname{rank} L = \operatorname{rank} N / \operatorname{rank} L$. Since $v_i \otimes M_i$, $v_i \otimes N_i$ are primitive in $L \otimes M$, $L \otimes N$ respectively, and $\operatorname{rank} v_i \otimes M_i = \operatorname{rank} v_i \otimes N_i$, the part (ii) in (A) implies $\sigma(v_i \otimes M_i) = v_i \otimes N_i$. Define μ by $\sigma(v_i \otimes m) = v_i \otimes \mu(m)$ for $m \in M_i$; then μ is an isomorphism from M on N. We must prove that μ is an isometry. Take elements $m_i \in M_i$, $m_j \in M_j$; then $B(v_i \otimes m_i, v_j \otimes m_j) = B(\sigma(v_i \otimes m_i), \sigma(v_j \otimes m_j)) = B(v_i \otimes \mu(m_i), v_j \otimes \mu(m_j))$ where B denotes the bilinear form associated with quadratic spaces in general. Hence we have $B(v_i, v_j)B(m_i, m_j) = B(v_i, v_j)B(\mu(m_i), \mu(m_j))$, and $B(m_i, m_j) = B(\mu(m_i), \mu(m_j))$ for $B(v_i, v_j) \neq 0$. Suppose $B(v_i, v_j) = 0$; then $B(L \otimes M_i, L \otimes M_j) = B(v_i \otimes N, v_j \otimes N) = 0$ implies $B(M_i, M_j) = 0$. Since the situations are symmetric with respect to M, N, we have $\sigma^{-1}(L \otimes N_i) = v_i \otimes M$, $\sigma^{-1}(v_i \otimes N_i) = v_i \otimes M_i$, $\sigma^{-1}(v_i \otimes n) = v_i \otimes \mu^{-1}(n)$ for $n \in N_i$. Therefore

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$B(v_i, v_j) = 0$ implies $B(N_i, N_j) = B(\mu(M_i), \mu(M_j)) = 0$. Thus μ is an isometry. $\mu(M_i) = N_i$ is obvious by definition.

COROLLARY. *The condition* (**A**) *implies* (ii) *in the condition* (**B**).

Proof. This follows from Theorem in § 1 in [6].

1.2. In 1.1 we proved that the condition (**A**) implies the condition (**) in [6]. Let X, Y be positive lattices and let σ be an isometry from $L \otimes X$ on $L \otimes Y$. Then the proof of Theorem in § 1 in [6] shows that there are orthogonal decompositions $X = \perp_{i=1}^{t} M_{0,i} \perp M$, $Y = \perp_{i=1}^{t} N_{0,i} \perp N$ such that $\sigma(L \otimes M_{0,i}) = L \otimes N_{0,i}$, $\sigma(L \otimes M) = L \otimes N$, and $\sigma = \alpha_i \otimes \beta_i$ on $L \otimes M_{0,i}$ where $\alpha_i \in O(L)$, $\beta_i : M_{0,i} \cong N_{0,i}$, and $\sigma(L \otimes m) = L \otimes n$ $(m \in M, n \in N)$ implies $m = 0, n = 0$. Hence we have

LEMMA 2. *Let X, Y be indecomposable positive lattices and σ be an isometry from $L \otimes X$ on $L \otimes Y$. If there are non-zero elements $x \in X, y \in Y$ such that $\sigma(L \otimes x) = L \otimes y$, then we have $\sigma = \alpha \otimes \beta$ where $\alpha \in O(L)$, $\beta : X \cong Y$. If $\sigma(L \otimes x) = L \otimes y (x \in X, y \in Y)$ implies $x = 0$, $y = 0$, then we have $X \cong Y \cong L \otimes K$ for some positive lattice K.*

1.3. LEMMA 3. *Let M, N be indecomposable positive lattices, and suppose $M \otimes N = K_1 \perp K_2$ $(K_1 \neq 0, K_2 \neq 0)$. Then an isometry α of $M \otimes N$ defined by $\alpha|_{K_1} = \mathrm{id}_{K_1}$, $\alpha|_{K_2} = -\mathrm{id}_{K_2}$ is not in $O(M) \otimes O(N)$.*

Proof. Assume $\alpha = \sigma \otimes \mu$, $\sigma \in O(M)$, $\mu \in O(N)$; then $\alpha^2 = \sigma^2 \otimes \mu^2 = 1$ implies (i) $\sigma^2 = 1$, $\mu^2 = 1$ or (ii) $\sigma^2 = -1$, $\mu^2 = -1$. Suppose $\sigma^2 = 1$, $\mu^2 = 1$, and put $M_\pm = \{x \in M ; \sigma x = \pm x\}$, $N_\pm = \{x \in N ; \mu(x) = \pm x\}$; then we have $[M : M_+ \perp M_-] < \infty$, $[N : N_+ \perp N_-] < \infty$. Fix a primitive element $n \in N$ such that $\mu(n) = \delta n$ $(\delta = \pm 1)$. For any element $x = x_+ + x_-$ in M $(x_+ \in \boldsymbol{Q}M_+, x_- \in \boldsymbol{Q}M_-)$, we have $x \otimes n = x_+ \otimes n + x_- \otimes n$, and $\alpha(x_+ \otimes n) = \delta x_+ \otimes n$, $\alpha(x_- \otimes n) = -\delta x_- \otimes n$. $x \otimes n \in M \otimes N = K_1 \perp K_2$ implies $x_+ \otimes n \in K_1$ if $\delta = 1$, $x_+ \otimes n \in K_2$ if $\delta = -1$, and so $x_+ \otimes n \in M \otimes N$. This means $x_+ \in M$ and $x_- \in M$. Hence we have $M = M_+ \perp M_-$. Since M is indecomposable, we have $M = M_+$ or M_- and $\sigma = \pm 1$. Similarly we have $\mu = \pm 1$. This contradicts $\alpha = \sigma \otimes \mu \neq \pm 1$. Suppose $\sigma^2 = -1$, $\mu^2 = -1$. Considering M as $\boldsymbol{Z}[\sigma] \cong \boldsymbol{Z}[\sqrt{-1}]$-module, M is isomorphic to $\oplus \boldsymbol{Z}[\sqrt{-1}]$ as a $\boldsymbol{Z}[\sqrt{-1}]$-module. Hence there is a submodule M_1 such that $M = M_1 \oplus \sigma(M_1)$. Similarly there is a submodule N_1 of N such that

$N = N_1 \oplus \mu(N_1)$. Taking a basis $\{m_i\}$ of M_1 and a basis $\{n_i\}$ of N_1, we have a basis $\{m_i \otimes n_j, m_i \otimes \mu(m_j), \sigma(m_i) \otimes n_j, \sigma(m_i) \otimes \mu(n_j)\}$ of $M \otimes N$. Since $\alpha(m_i \otimes n_j) = \sigma(m_i) \otimes \mu(n_j)$, $\alpha(m_i \otimes \mu(n_j)) = -\sigma(m_i) \otimes n_j$, we have $\{m_i \otimes n_j + \sigma(m_i) \otimes \mu(n_j), m_i \otimes \mu(n_j) - \sigma(m_i) \otimes n_j\}$ as a basis of K_1 and $\{m_i \otimes n_j - \sigma(m_i) \otimes \mu(n_j), m_i \otimes \mu(n_j) + \sigma(m_i) \otimes n_j\}$ as a basis of K_2. This implies that $m_i \otimes n_j$ is not contained in $K_1 \perp K_2 = M \otimes N$. This is a contradiction.

1.4. LEMMA 4. *Let L be an indecomposable positive lattice satisfying the condition* (**A**). *Then we have*

(i) *$L \otimes L$ is indecomposable, and*

(ii) *$O(L \otimes L) = O(L) \otimes O(L) \cup O(L) \otimes O(L)\mu$, where $\mu \in O(L \otimes L)$ is an isometry defined by $\mu(x \otimes y) = y \otimes x$ for $x, y \in L$.*

Proof. Take an isometry σ of $L \otimes L$. If there are non-zero elements x, y in L such that $\sigma(L \otimes x) = L \otimes y$, then Lemma 2 implies $\sigma \in O(L) \otimes O(L)$. Suppose that $\sigma(L \otimes x) = L \otimes y$ implies $x = y = 0$; then there is a basis $\{v_i\}$ of L such that $\sigma(L \otimes L_i) = v_i \otimes L$, putting $L_i = \{x \in L\,; \sigma(L \otimes x) \subset v_i \otimes L\}$. Hence we have rank $L_i = 1$, and put $L_i = \boldsymbol{Z}[u_i]$. It yields $\mu\sigma(L \otimes u_i) = L \otimes v_i$. Therefore $\mu\sigma \in O(L) \otimes O(L)$ follows from Lemma 2. Thus we have $O(L \otimes L) = O(L) \otimes O(L) \cup \mu O(L) \otimes O(L)$. This completes the proof of (ii). Suppose that $L \otimes L = K_1 \perp K_2\, (K_1 \neq 0, K_2 \neq 0)$. Define an isometry α of $L \otimes L$ by $\alpha = \text{id.}$ on K_1, $\alpha = -\text{id.}$ on K_2. Then Lemma 3 and (ii) in this lemma imply $\alpha = (\sigma_1 \otimes \sigma_2)\mu$ where $\sigma_1, \sigma_2 \in O(L)$. From $\alpha^2 = 1$ follows that, for $x_1, x_2 \in L$, $x_1 \otimes x_2 = (\sigma_1 \otimes \sigma_2)\mu(\sigma_1(x_2) \otimes \sigma_2(x_1)) = \sigma_1\sigma_2(x_1) \otimes \sigma_2\sigma_1(x_2)$. This yields $\sigma_1\sigma_2 = \pm 1$. Hence we may assume $\alpha = (\sigma \otimes \sigma^{-1})\mu$ $(\sigma \in O(L))$, taking $-\alpha$ instead of α if necessary. Take a basis $\{e_i\}$ of L and decompose $\sigma(e_i) \otimes e_j$ as $\sigma(e_i) \otimes e_j = (\sigma(e_i) \otimes e_j + \alpha(\sigma(e_i) \otimes e_j))/2 + (\sigma(e_i) \otimes e_j - \alpha(\sigma(e_i) \otimes e_j))/2$. Then $(\sigma(e_i) \otimes e_j + \alpha(\sigma(e_i) \otimes e_j))/2 \in \boldsymbol{Q}K_1$, $(\sigma(e_i) \otimes e_j - \alpha(\sigma(e_i) \otimes e_j))/2 \in \boldsymbol{Q}K_2$ and $L \otimes L = K_1 \perp K_2$ imply $(\sigma(e_i) \otimes e_j + \alpha(\sigma(e_i) \otimes e_j))/2 \in K_1$. Therefore we have $(\sigma(e_i) \otimes e_j + \sigma(e_j) \otimes e_i)/2 \in L \otimes L$. This is a contradiction because $\{e_i\}$ is a basis of L.

1.5. LEMMA 5. *$\otimes^m L$ is indecomposable provided that the orthogonal group $O(\otimes^m L)$ is generated by $O(L)$ and interchanges of L's and that $\otimes^{m-1} L$ is indecomposable.*

Proof. By Lemma 4 we may assume $m \geq 3$. Suppose $\otimes^m L = K_1 \perp K_2$ $(K_1 \neq 0, K_2 \neq 0)$ and define an isometry α of $O(\otimes^m L)$ by $\alpha = \text{id.}$ on K_1,

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$\alpha = -\text{id.}$ on K_2. By the assumption we have $\alpha = (\otimes \sigma_i)\mu$ where $\sigma_i \in O(L)$ and μ is an isometry defined by $\mu(x_1 \otimes \cdots \otimes x_m) = x_{\mu(1)} \otimes \cdots x_{\mu(m)}$ (μ is considered as a permutation). $\alpha^2 = 1$ implies $\alpha^2(x_1 \otimes \cdots \otimes x_m) = \alpha(\sigma_1(x_{\mu(1)}) \otimes \cdots \otimes \sigma_m(x_{\mu(m)})) = \sigma_1(\sigma_{\mu(1)}(x_{\mu^2(1)})) \otimes \cdots \otimes \sigma_m(\sigma_{\mu(m)}(x_{\mu^2(m)})) = x_1 \otimes \cdots \otimes x_m$ for any $x_i \in L$. Hence we have $\mu^2 = 1$. Suppose $\mu(1) = 1$; then $\alpha(x_1 \otimes \cdots) = \sigma_1(x_1) \otimes \cdots$, and we have $\alpha \in O(L) \otimes O(\otimes^{m-1} L)$. This contradicts Lemma 3. Suppose $\mu(1) = j \geq 2$. Define an isometry μ_j by $\mu_j (x_1 \otimes x_2 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_m) = x_j \otimes x_2 \otimes \cdots \otimes x_1 \otimes \cdots \otimes x_m$; then $\mu_j \alpha \mu_j^{-1} (x_1 \otimes \cdots \otimes x_j \otimes \cdots) = \mu_j \alpha(x_j \otimes \cdots \otimes x_1 \otimes \cdots) = \mu_j(\sigma_1(x_1) \otimes \cdots \otimes \sigma_j(x_j) \otimes \cdots) = \sigma_j(x_j) \otimes \cdots \otimes \sigma_1(x_1) \otimes \cdots$. Hence we have $\mu_j \alpha \mu_j^{-1} \in O(\otimes^2 L) \otimes O(\otimes^{m-2} L)$ for $j = 2$. This contradicts Lemma 3 since $\mu_j \alpha \mu_j^{-1} = \text{id.}$ on $\mu_j(K_1)$, $\mu_j \alpha \mu_j^{-1} = -\text{id.}$ on $\mu_j(K_2)$. Suppose $\mu(1) = j \geq 3$. Defining an isometry μ' by $\mu'(x_1 \otimes x_2 \otimes \cdots \otimes x_j \otimes \cdots) = x_1 \otimes x_j \otimes \cdots \otimes x_2 \otimes \cdots$, we have $\mu' \mu_j \alpha \mu_j^{-1} \mu'^{-1} (x_1 \otimes x_2 \otimes \cdots \otimes x_j \cdots) = \sigma_j(x_2) \otimes \sigma_1(x_1) \otimes \cdots$. Thus $\mu' \mu_j \alpha \mu_j^{-1} \mu'^{-1} \in O(\otimes^2 L) \otimes O(\otimes^{m-2} L)$. This is also a contradiction as in the case of $j = 2$.

1.6. To prove that the condition (**A**) implies the condition (**B**), it is sufficient to show

LEMMA. *Let K be an indecomposable positive lattice such that $K \not\cong L \otimes K'$ for any lattice K'. Then we have*

(i) *$\otimes^m L \otimes K$ is indecomposable, and*

(ii) *$O(\otimes^m L \otimes K)$ is generated by $O(L)$, $O(K)$ and interchanges of L's.*

Proof. We use the induction with respect to m. Suppose $m = 1$; then Lemma 2 implies (ii), and (ii) and Lemma 3 imply (i). Suppose that (i), (ii) are true for $m = t$. Assume that there is an isometry $\sigma \in O\otimes^{t+1} L \otimes K)$ which is not in the subgroup generated by $O(L)$, $O(K)$ and interchanges of L's. Put $M = \otimes^t L \otimes K$; then $O(M)$ is generated by $O(L)$, $O(K)$ and interchanges of L's, and M is indecomposable. If there are non-zero elements $m, m' \in M$ such that $\sigma(L \otimes m) = L \otimes m'$, then Lemma 2 implies $\sigma \in O(L) \otimes O(M)$. This contradicts our assumption on σ. Hence for such an isometry σ follows that $\sigma(L \otimes m) = L \otimes m'$ ($m, m' \in M$) implies $m = m' = 0$. Hence the condition (**A**) and Lemma 1 yield $\sigma(L \otimes M_1) = v_1 \otimes M$ where $\{v_i\}$ is some basis of L and $M_1 = \{m \in M;\ \sigma(L \otimes m) \subset v_1 \otimes M\}$. Defining an isometry μ_2 by $\mu_2(x \otimes y \otimes z) = y \otimes x \otimes z$

$(x, y \in L, z \in \otimes^{t-1} L \otimes K)$, we have $\mu_2\sigma(L \otimes M_1) = L \otimes v_1 \otimes (\otimes^{t-1} L) \otimes K$. Since $\mu_2\sigma$ is not contained in the subgroup generated by $O(L)$, $O(K)$ and interchanges of L's, $\mu_2\sigma(L \otimes m) = L \otimes m'$ $(m \in M_1 \subset M, m' \in v_1 \otimes (\otimes^{t-1} L \otimes K) \subset M)$ implies $m = m' = 0$ as above. Applying the condition (**A**) to $\mu_2\sigma, M_1, v_1 \otimes (\otimes^{t-1} L) \otimes K$ instead of σ, M, N respectively, we have $\mu_2\sigma(L \otimes M_{1,1}) = v'_1 \otimes v_1 \otimes (\otimes^{t-1} L) \otimes K$ where $\{v'_i\}$ is a basis of L and $M_{1,1} = \{m \in M_1 ; \mu_2\sigma(L \otimes m) \subset v'_1 \otimes v_1 \otimes (\otimes^{t-1} L) \otimes K\}$. This is the similar situation to $\sigma(L \otimes M_1) = v_1 \otimes (\otimes^t L) \otimes K$. Hence we have inductively $\mu_{t+1} \cdots \mu_2\sigma (L \otimes M_{1,\dots,1}) = L \otimes v_1 \otimes v'_1 \otimes \cdots \otimes v_1'^{\cdots\prime} \otimes K$, where μ_j is an isometry defined by $\mu_j(x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_{t+1} \otimes y) = x_j \otimes \cdots \otimes x_1 \otimes \cdots \otimes x_{t+1} \otimes y$ $(x_i \in L, y \in K)$. Since $L \otimes K$ is indecomposable, $M_{1,\dots,1}$ is also indecomposable. Moreover there are no non-zero elements $m \in M_{1,\dots,1} \subset M$, $m' \in v_1 \otimes v'_1 \otimes \cdots \otimes v'_1 \cdots \otimes K \subset M$ such that $\mu_{t+1} \cdots \mu_2\sigma(L \otimes m) = L \otimes m'$. Lemma 2 implies $v_1 \otimes v'_1 \otimes \cdots \otimes v_1'^{\cdots\prime} \otimes K \cong L \otimes K'$ for some positive lattice K'. This contradicts the assumption on K. Thus the part (ii) for $m = t+1$ has been proved. Now we must prove the part (i) for $m = t + 1$. The part (ii) implies that $O(\otimes^{t+1} L \otimes K) = O(\otimes^{t+1} L) \otimes O(K)$, and $O(\otimes^{t+1} L)$ is generated by $O(L)$ and interchanges of L's. From the part (i) for $m = t$ follows that $\otimes^t L$ is indecomposable. Hence Lemma 5 implies that $\otimes^{t+1} L$ is also indecomposable; then from Lemma 3 follows that $\otimes^{t+1} L \otimes K$ is indecomposable. This completes the proof.

2.

In this section we prove the converse.

Let L be an indecomposable positive lattice which satisfies the condition (**B**).

2.1. Let M, N be indecomposable positive lattices and let σ be an isometry from $L \otimes M$ on $L \otimes N$ such that $\sigma(L \otimes m) = L \otimes n$ $(m \in M, n \in N)$ implies $m = 0, n = 0$. Fix any basis $\{v_i\}$ of L. Assume that $M \cong \otimes^p L \otimes M', N \cong \otimes^q L \otimes N'$ where M', N' are not isometric to any lattice of the form $L \otimes K$. Since M, N are indecomposable, M', N' are also indecomposable. Then the part (ii) in (**B**) implies $p = q$ and $\alpha : M' \cong N'$. Identifying M (resp. N) and $\otimes^p L \otimes M'$ (resp. $\otimes^p L \otimes N'$), we have $\sigma = (\sigma_0 \otimes \cdots \otimes \sigma_p \otimes \beta)\eta$ by virtue of (iii) in (**B**) where $\sigma_i \in O(L), \beta \in O(N')$ and η is an isometry defined by $\eta(x_0 \otimes \cdots \otimes x_p \otimes m) = x_{s(0)} \otimes \cdots \otimes x_{s(p)} \otimes \alpha(m)$ $(x_0, \cdots, x_p \in L, m \in M', s$: a permutation). $s(0) = 0$ implies $\sigma(L \otimes x_1$

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$\otimes \cdots \otimes x_p \otimes m) = L \otimes \sigma_1(x_{s(1)}) \otimes \cdots \otimes \sigma_p(x_{s(p)}) \otimes \beta\alpha(m)$. This contradicts our assumption on σ. Thus we have $s(0) \geq 1$. It is easy to see that $\sigma(v_i \otimes L \otimes \cdots \otimes L \otimes M') = L \otimes \cdots \otimes L \otimes \sigma_{s^{-1}(0)}(v_i) \otimes L \otimes \cdots \otimes L \otimes N'$, $\sigma^{-1}(v_i \otimes L \otimes \cdots \otimes L \otimes N') = L \otimes \cdots \otimes L \otimes \sigma_0^{-1}(v_i) \otimes L \otimes \cdots \otimes L \otimes M'$ where $\sigma_{s^{-1}(0)}(v_i)$ (resp. $\sigma_0^{-1}(v_i)$) is on the $s^{-1}(0)+1$-th (resp. $s(0)+1$-th) component. Put $N_i = L \otimes \cdots \otimes L \otimes \sigma_{s^{-1}(0)}(v_i) \otimes L \otimes \cdots \otimes L \otimes N'$, $M_i = L \otimes \cdots \otimes L \otimes \sigma_0^{-1}(v_i) \otimes L \otimes \cdots \otimes L \otimes M'$ where $\sigma_{s^{-1}(0)}(v_i)$ (resp. $\sigma_0^{-1}(v_i)$) is on the $s^{-1}(0)$-th (resp. $s(0)$-th) component. Then we have $M_i = \{m \in M; \sigma(L \otimes m) \subset v_i \otimes N\}$, $N_i = \{n \in N; \sigma^{-1}(L \otimes n) \subset v_i \otimes M\}$, $M = \oplus M_i$, $N = \oplus N_i$, and $\sigma(v_i \otimes M_i) = v_i \otimes N_i$.

Hence we have proved that the condition (**A**) holds for indecomposable positive lattices M, N and for any fixed basis $\{v_i\}$ of L.

2.2. Let M, N be positive lattices and let σ be an isometry from $L \otimes M$ on $L \otimes N$ such that $\sigma(L \otimes m) = L \otimes n$ $(m \in M, n \in N)$ implies $m = 0$, $n = 0$. Put $M = \perp M_i$, $N = \perp N_i$ where M_i, N_i are indecomposable; then the part (i) in (**B**) implies that $L \otimes M_i, L \otimes N_i$ are indecomposable. By virtue of 105:1 in [8] we may assume $\sigma(L \otimes M_i) = L \otimes N_i$. Hence 2.1 implies the condition (**A**) for decomposable lattices M, N.

3. Miscellaneous remarks

3.1. Let k be a totally real algebraic number field with maximal order O_k. We considered the following question in [3], [4] (see also [1], [2], [9]).

If σ is an isometry from $O_kL \cong O_kM$, where L, M are positive lattices, then does $\sigma(L) = M$ hold?

This is equivalent to the following if $k/\boldsymbol{Q}$ is a Galois extension.

Assume that k is a totally real Galois extension over $\boldsymbol{Q}$. Let G be a finite group in $GL(n, O_k)$ such that $g(G) = \{g(A); A \in G\} = G$ for any g in Gal$(k/\boldsymbol{Q})$. Then does $G \subset GL(n, \boldsymbol{Z})$ hold?

Sketch of the proof of the equivalence. Suppose that $G \subset GL(n, O_k)$ is given. Put $P = \sum_{A \in G} {}^tAA$. Then P is a positive definite symmetric matrix with rational numbers as entries since $g(G) = G$ for any g in Gal$(K/\boldsymbol{Q})$. Let L be a positive lattice corresponding to P. Then $O(O_kL)$ contains G. If $O(O_kL) = O(L)$ holds, then $G \subset GL(n, \boldsymbol{Z})$ holds. Conversely, suppose that $\sigma: O_kL \cong O_kM$ is given. Define an isometry $\tilde{\sigma}$ of $O(O_k(L \perp M))$ by $\tilde{\sigma} = \sigma$ on O_kL, $\tilde{\sigma} = \sigma^{-1}$ on O_kM. Taking G as

$O(O_k(L \perp M))$, we have $\tilde{\sigma} \in O(L \perp M)$ and $\sigma(L) = M$ if $G = O(L \perp M)$.

3.2. Let F be a totally real algebraic number field. Suppose that there is an unramified totally real Galois extension E of F. Denote the Galois group $G(E/F)$ by G. Put $V = F[G]$ (group ring) and introduce an inner product by $(g, g') = \delta_{g,g'}$ (= Kronecker's delta) for $g, g' \in G$. This makes V a positive definite quadratic space over F. We define the operation G to $EV = E[G]$ by $g'(\sum_{g\in G} a_i g) = \sum_{g\in G} g'(a_i)g'g$ for $g' \in G$, $a_i \in E$. Put $\tilde{L} = \perp_{g\in G} O_E g, L = \{\sum_{g\in G} g(a1_G)\,; a \in O_E\}$. Then $\tilde{L} = O_E L$ and L is an indecomposable quadratic lattice over O_F [3]. Put $M = \perp_{g\in G} O_F g$; then $\tilde{L} = O_E M$. Hence we have

(a) L, M are not isometric positive lattices over O_F, but $O_E L, O_E M$ are isometric.

Defining an inner product in O_E by $(x, y) = \mathrm{tr}_{E/F}\, xy (x, y \in O_E)$, we have a positive lattice $\tilde{O}_E$. Taking traces, we have $\tilde{O}_E \otimes L \cong \tilde{O}_E \otimes M$. Here $\tilde{O}_E \otimes L$ is decomposable since $O_E L$ is decomposable. $\tilde{O}_E$ is indecomposable because it is isometric to L. Hence we have, putting $N = \tilde{O}_E$,

(b) L, N are indecomposable positive lattices over O_F but $L \otimes N$ is decomposable.

(c) $N \otimes L \cong N \otimes M$ but $L \not\cong M$.

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