7

The large N limit of two-dimensional models

7.1 Introduction

The number of approximation techniques in quantum field theory is very limited. Perturbation expansion in small interaction coupling, like $\alpha_{\rm em} = \frac{1}{137}$ of QED, is obviously the most important one. Other methods include dimensional expansion, high temperature expansion and large radius expansion. Quite surprisingly, one of the most useful approximation techniques is expansion in the number of degrees of freedom. A priori one would tend to think that the larger the number of degrees of freedom, the more complex the system. However, it turns out that theories with infinitely many degrees of freedom are much easier to solve than those with a finite number of degrees of freedom. Once the system with $N \to \infty$ is known, a systematic expansion in $\frac{1}{N}$ provides an approximation procedure for computing quantities that describe systems of finite N.

Large N methods have been applied in a very wide range of physical systems. Starting from non-critical phenomena in spin systems like the Heisenberg ferromagnet (discussed in Section 5.14), then $SU(N)$ QCD theories in various dimensions, and later matrix models associated with either string models or two-dimensional models.

The large N approximation in field theory or correspondingly the planar expansion of Feynman diagrams was introduced by 't Hooft in his seminal paper [122].

In this book we will focus on four arenas where large N approximations are being used:

(i) Two-dimensional quantum field theory models, which include the Gross–Neveu model and the CP^N models, will be addressed in this chapter.
(ii) Quantum chromodynamics with large N $SU(N)$ gauge theory in two dimensions. In Chapter 10 of Part 2 the solution of two-dimensional QCD, following 't Hooft [124], will be described,[1] together with a certain generalization of it.
(iii) The approach to four-dimensional QCD based on the $\frac{1}{N}$ expansion.
(iv) Baryons in large N QCD.

[1] Known as the 't Hooft model.

The last two topics will be described in the third part of the book in Chapters 19 and 20.

In Nature the number of colors is three, and thus one may wonder whether it makes sense to in expand a not-so-small parameter 1/3. Even though there is no general proof that this expansion is indeed reliable, a vast literature on the subject brings out a large amount of evidence that indeed this is the case. It turns out that for certain quantities the $\frac{1}{N}$ term vanishes and the correction starts as $\frac{1}{N^2}$, and hence puts the approximation on a more solid base.

As an example of the accuracy of the large N limit consider the Stirling formula for N, where the leading term is $\sqrt{2\pi N}(N/e)^N$ for large N. But the correction is actually $\frac{1}{12N}$ as compared to 1, making it only an 8 percent correction even for $N = 1$.

In the next sections we describe the $O(N)$ model, the Gross–Neveu model and the CP^N models.

There are several review articles on large N expansions, for instance [46], [160], [165]. In this chapter we make use of [160].

7.2 The Gross–Neveu model

The Gross–Neveu (GN) model, proposed in [117], describes a set of N Dirac fermions interacting via a four-fermi interaction. It turns out that one can solve the model, and prove that it is asymptotically free and admits a dynamical symmetry breaking, using $1/N$ expansion. The Lagrangian of the system can be written in the form,

$$\mathcal{L}_{\mathrm{GN}} = i\bar{\psi}^a \not{\partial}\psi^a + \frac{\lambda_0^2}{2}(\bar{\psi}^a\psi^a)^2, \tag{7.1}$$

where $a = 1...N$ and λ_0 is a bare coupling of dimension zero. The corresponding action is invariant under a continuous $SU(N)$ global transformation and a discrete chiral transformation,

$$\psi^a \to g^a_b\psi^b \quad g\in SU(N), \qquad \psi^a \to \gamma_5\psi^a \quad \bar{\psi}^a \to \gamma_5\bar{\psi}^a. \tag{7.2}$$

The discrete chiral symmetry forbids a mass term. In fact this is the most general action invariant under these symmetry transformations, with terms of dimension two or less, and hence it is a renormalizable action.

Let us now check whether in this formulation of the Lagrangian, where λ_0 is fixed, one can make sense of a large N limit. Consider the scattering process of two fermions with flavor index a that turn into a pair of fermions with a different flavor index b. The leading Feynman diagrams that contribute to this process are given in Fig. 7.1. The first diagram which is the basic interaction vertex is of order λ_0, the second one is of order λ_0^2, the third is of order $\lambda_0^2 N$ due to the N different flavors of the fermions that can run in the loop, and the last two diagrams are of order $\lambda_0^3 N^2$. It is thus clear that the perturbation expansion

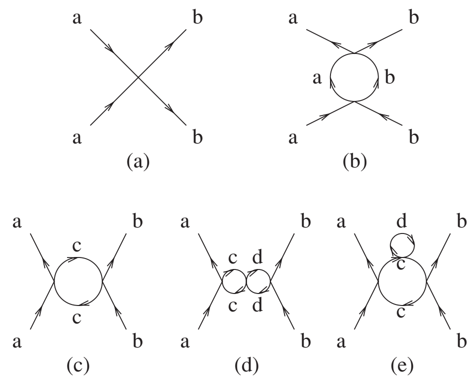

Fig. 7.1. Leading order diagrams for the $a + \bar{a} \rightarrow b + \bar{b}$ scattering.

expressed in terms of the fixed coupling λ_0 does not have a sensible large N expansion. However, one can easily cure this problem by defining the coupling $\lambda \equiv \lambda_0 N$, which is taken to be fixed when $N \rightarrow \infty$. The Lagrangian now reads,

$$\mathcal{L}_{GN} = i\bar{\psi}^a \not{\partial}\psi^a + \frac{\lambda}{2N}(\bar{\psi}^a\psi^a)^2. \tag{7.3}$$

It is now straightforward to see that the $\frac{1}{N}$ in front of the interaction term enables a well-defined large N limit. Consider again the diagrams in Fig. 7.1. The leading contribution is of order $\frac{\lambda}{N}$ and the loop corrections are now of order $\frac{\lambda^2}{N^2}$, $\frac{\lambda^2}{N}$ and $\frac{\lambda^3}{N}$, respectively. Hence it is obvious that the form of any scattering amplitude in perturbation theory is $\frac{1}{N}A(\lambda, 1/N)$, which becomes $\frac{1}{N}A(\lambda, 0)$ at the large N limit.

To further analyze the system it is convenient to introduce the auxiliary field σ, in terms of which the Lagrangian (7.3) can be written as,

$$\mathcal{L}_{\mathrm{GN}} = i\bar{\psi}^a \not{\partial}\psi^a + \sigma\bar{\psi}^a\psi^a - \frac{N}{2\lambda}\sigma^2. \tag{7.4}$$

Integrating over σ or alternatively solving the classical equation of motion for σ and substituting it into the action yields the Lagrangian (7.1). The introduction of the auxiliary field, which will soon acquire a physical interpretation, is a standard step in the large N procedure which enables a simplified counting of powers of $\frac{1}{N}$.

The Feynman diagrams of the theory for this alternative formulation are then given in Fig. 7.2, with the full line representing the fermion propagator, and the dotted line that of σ.

Note that the only non-trivial interaction is the $\sigma\bar{\psi}^a\psi^a$ term and that each σ propagator contributes $\frac{i\lambda}{N}$. The diagrams Fig. 7.1 that contribute to the scattering process are converted to those in Fig. 7.3.

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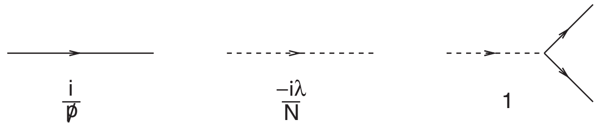

Fig. 7.2. Feynman rules of the Gross–Neveu model.

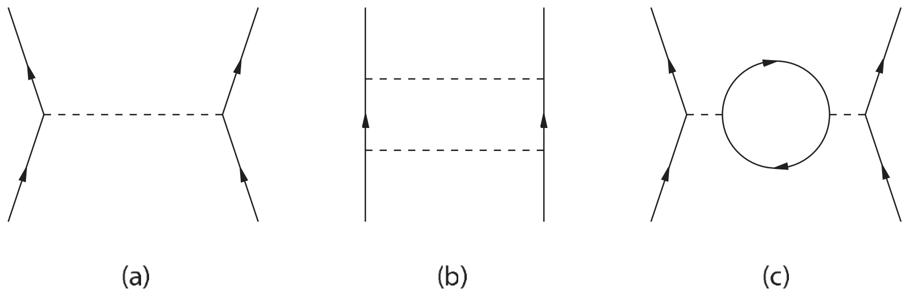

Fig. 7.3. Leading diagrams that contribute to the two-to-two scattering.

We would now like to integrate over the fermions and derive an effective Lagrangian, $\mathcal{L}_{\text{effective}}(\sigma)$. Since the Lagrangian is quadratic in the fermion fields, $\mathcal{L}_{\text{effective}}(\sigma)$ is given by the sum of terms (Fig. 7.4).

Note that all diagrams are with an even number of σ only, since σ is odd under the transformation (7.2). The first term is the tree level contribution $-\frac{N}{2\lambda}\sigma^2$ and the rest are the one loop contributions. Both are of order N, the latter due to the N fermions that can run in the loop. The N dependence of $\mathcal{L}_{\text{effective}}$ therefore has the form,

$$\mathcal{L}_{\text{effective}}(\sigma, \lambda, N) = N\hat{\mathcal{L}}_{\text{effective}}(\sigma, \lambda). \tag{7.5}$$

This makes the counting of powers of $\frac{1}{N}$ very easy. Consider a graph with E external σ lines, I internal σ lines, V vertices and L independent loops. The parameters (E, I, V, L) are not independent. For each internal line there is a momentum integration and hence a loop. However each vertex introduces a delta function in momenta that cancels one momentum apart from an overall delta function associated with the momentum conservation.

Thus one has,

$$L = I - V + 1. \tag{7.6}$$

Recall that each σ external or internal line carries a $\frac{1}{N}$ factor, while each vertex contributes a factor of N.

Thus the net power N of each graph is,

$$N^{-I+V-E} = N^{-E-L+1}. \tag{7.7}$$

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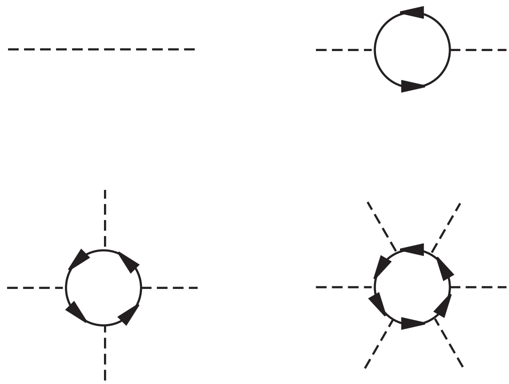

Fig. 7.4. Diagrams of $\mathcal{L}_{\text{effective}}(\sigma)$.

It is obvious from this expression that adding loops and external σ lines suppresses the corresponding contribution due to additional powers of $\frac{1}{N}$. Since the minimal number of σ external lines is two the leading behavior is of order $\frac{1}{N}$.

For the purpose of investigating the possibility of spontaneous breaking of the discrete symmetry of (7.2), it is enough to compute the effective potential $V(\sigma)$ rather than the effective action, namely, the limit where all the external lines carry zero momentum. The effective potential is given by the sum of the diagrams in Fig. 7.4:

$$-iV = -i\frac{N\sigma^2}{2\lambda} - N\sum_{n=1}^{\infty}\frac{1}{2n}\text{Tr}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\left[\frac{-\not{p}\sigma}{p^2+i\epsilon}\right]^{2n}, \tag{7.8}$$

where $\frac{1}{2n}$ is the symmetry factor of the graph, N comes from summing over all possible flavors, –1 from the fermion loop and the expression in the bracket is the product of the propagator and $(i\frac{\not{p}}{p^2+i\epsilon})$ and the vertex $(i\sigma)$. Using the identity,

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{2n} = -\frac{1}{2}\log(1-x^2), \tag{7.9}$$

and analytically continuing to the Euclidean space gives,

$$V = N\left[\frac{\sigma^2}{2\lambda} - \int\frac{\mathrm{d}^2p_E}{(2\pi)^2}\log\left(1+\frac{\sigma^2}{p_E^2}\right)\right]. \tag{7.10}$$

The momentum integral is logarithmically divergent so by introducing a cutoff on the Euclidean momentum $p_E^2 \leq \Lambda^2$ we find,

$$V = N\left[\frac{\sigma^2}{2\lambda} - \frac{1}{4\pi}\sigma^2[\log\left(\frac{\sigma^2}{\Lambda^2}\right)+1\right]. \tag{7.11}$$

The effective potential can be rewritten in terms of the coupling λ_r, renormalized at a scale μ, defined as,

$$\frac{1}{\lambda_r} \equiv \frac{1}{N}\frac{\mathrm{d}^2V}{\mathrm{d}\sigma^2}|_{\sigma=\mu} = \frac{1}{\lambda} + \frac{1}{2\pi}\log\left(\frac{\mu^2}{\Lambda^2}\right) + \frac{1}{\pi}, \tag{7.12}$$

and substitute it into the $V(\sigma)$ to find,

$$V = N\left[\frac{\sigma^2}{2\lambda_r} + \frac{1}{4\pi}\sigma^2\left[\log\left(\frac{\sigma^2}{\mu^2}\right) - 3\right]\right] \tag{7.13}$$

The fact that the cutoff disappears obviously implies that the theory is indeed renormalizable. The β function and the anomalous dimension can be determined by substituting the effective potential into the renormalization group equation (see Section 17.6),

$$[\mu\partial_\mu + \beta(\lambda_r)\partial_{\lambda_r} - \gamma_\sigma(\lambda_r)\partial_\sigma]\, V(\sigma) = 0. \tag{7.14}$$

We thus find the exact (to all orders of λ_r) expression of $\beta(\lambda_r)$ and $\gamma_\sigma(\lambda_r)$, which take the form,

$$\beta(\lambda_r) = -\frac{{\lambda_r}^3}{2\pi} \quad \gamma_\sigma(\lambda_r) = 0. \tag{7.15}$$

Thus we have deduced that the Gross–Neveu model is asymptotically free, namely that the effective coupling goes to zero at high momenta. It turns out that the minus sign ensures this.

Let us now examine whether the chiral symmetry of this theory is spontaneously broken. For that we determine the extremum points of the potential. The vanishing points of the derivative,

$$\frac{\mathrm{d}V}{\mathrm{d}\sigma} = N\left[\frac{\sigma}{\lambda_r} + \frac{\sigma}{2\pi}\left(\log\frac{\sigma^2}{\mu^2} - 2\right)\right], \tag{7.16}$$

are at

$$\sigma = 0 \quad \text{and} \quad \sigma = \pm\sigma_0 = \pm\mu e^{1-\frac{\pi}{\lambda_r}}, \tag{7.17}$$

where

$$V(0) = 0 \quad \text{and} \quad V(\sigma_0) = -N\frac{\sigma_0^2}{4\pi} < 0. \tag{7.18}$$

Now since the potential vanishes at $\sigma = 0$ and it is negative at $\sigma = \pm\sigma_0$, its global minima are at $\pm\sigma_0$. Therefore the discrete chiral symmetry is broken and the massless fermions acquire mass which to the leading order is σ_0.

A further interesting property of the model is the dimensional transmutation. The bare theory depends on one continuous dimensionless parameter and the effective theory depends on one continuous parameter with dimensions, σ_0. Whereas one may anticipate that observables will depend on the dimensionless parameter in a complicated way, one finds a simple dependence on the parameter with dimensions which follows a dimensional analysis.

7.3 The CP^{N-1} model

Another model that can be solved using the large N expansion is the CP^{N-1} model. The model is an example of a non-linear sigma model where the fields live in a complex projective $N-1$ space,

$$CP^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)}. \tag{7.19}$$

The Lagrangian of the model can be written as,

$$\mathcal{L}_{CPN} = \partial_\mu Z^\dagger \partial^\mu Z - \frac{\lambda}{N} J_\mu J^\mu, \tag{7.20}$$

where $Z^\dagger \equiv (z_1, \ldots, z_N)$, namely, an N-dimensional "unit" vector of complex fields that obey the constraint,

$$Z^\dagger Z = \frac{N}{\lambda}, \tag{7.21}$$

and J_μ is given by,

$$J_\mu = -\frac{i}{2}\left[Z^\dagger \partial_\mu Z - (\partial_\mu Z^\dagger)Z\right]. \tag{7.22}$$

The Lagrangian describes a theory of massless particles with short range interaction which originates from both the explicit JJ interaction as well as from the constraint. The number of degrees of freedom of the CP^{N-1} model is $2N-2$. This is the dimension of the CP^{N-1} coset space,

$$\begin{aligned}\dim\left[CP^{N-1}\right] &= \dim\left[\frac{SU(N)}{SU(N-1)\times U(1)}\right]\\ &= (N^2-1) - ((N-1)^2 - 1 + 1) = 2N-2.\end{aligned} \tag{7.23}$$

Differently, we count N complex numbers Z_i, namely $2N$ real degrees of freedom, minus one degree of freedom due to the constraint (7.21), minus one degree of freedom due to the $U(1)$ local symmetry,

$$Z \to \mathrm{e}^{i\alpha} Z. \tag{7.24}$$

It is easy to verify that (7.20) is indeed invariant under this transformation upon the use of the constraint. In fact one can also write the Lagrangian in the form,

$$\mathcal{L}_{CPN} = \partial_\mu \Phi^\dagger \partial^\mu \Phi, \tag{7.25}$$

where Φ is a traceless hermitian matrix built from Z and $Z^\dagger$ according to,

$$\Phi = \sqrt{\frac{\lambda}{N}}\left[ZZ^\dagger - \frac{1}{\lambda}\right], \tag{7.26}$$

It is clear that the local transformation of the above does not change Φ and hence the Lagrangian is invariant under this transformation.

The first step in the large N program is to eliminate the quartic interaction term, by introducing an auxiliary field in a similar manner to what was done in the Gross–Neveu model. However, since the interaction has the form of a vector times a vector, the auxiliary field should also be a vector. This shifts the Lagrangian according to,

$$\begin{aligned}\mathcal{L}_{CPN} \to \mathcal{L}_{CPN} + \frac{\lambda}{N}\left(J_\mu + \frac{N}{\lambda}A_\mu\right)^2 \\ = \partial_\mu Z^\dagger \partial^\mu Z + 2J_\mu A^\mu + \frac{N}{\lambda}A_\mu A^\mu \\ = \left[(\partial_\mu - iA_\mu)Z^\dagger\right]\left[(\partial^\mu + iA_\mu)Z\right],\end{aligned} \tag{7.27}$$

where in the third line we have used the constraint. It is clear from its last form that the Lagrangian is invariant under the $U(1)$ local transformation,

$$Z \to \mathrm{e}^{i\alpha} Z \quad A_\mu \to A_\mu - \partial_\mu \alpha. \tag{7.28}$$

We now incorporate the fact that the Z are constraint variables by introducing another Lagrange multiplier into the Lagrangian,

$$\mathcal{L}_{CPN} = \left[(\partial_\mu - iA_\mu)Z^\dagger\right]\left[(\partial^\mu + iA_\mu)Z\right] + \sigma\left[Z^\dagger Z - \frac{N}{\lambda}\right]. \tag{7.29}$$

Obviously the path integral over σ, or equivalently using its equation of motion, implies that $Z^\dagger Z = \frac{N}{\lambda}$.

The action is now quadratic in Z, so we integrate out the Z fields similarly to what was done in the Gross–Neveu model. The Feynman diagrams that constitute the leading contributions to the effective action, which is now a functional of σ and A_μ, are drawn in Figure 7.5.

These diagrams, which include pure σ, pure A_μ and mixed diagrams, are all proportional to N. The computation of $V(\sigma)$ is similar to that in the GN model, leading to,

$$V(\sigma) = -N\left[\frac{\sigma}{\lambda} + \frac{\sigma}{4\pi}\left(\log\frac{\sigma}{\Lambda^2} - 1\right)\right], \tag{7.30}$$

where Λ is the cutoff. Again similar to the GN model the cutoff can be eliminated by performing a renormalization at a scale μ,

$$\frac{1}{\lambda_r} = -\frac{1}{N}\frac{\mathrm{d}V}{\mathrm{d}\sigma}|_{\mu^2}\frac{1}{\lambda} + \frac{1}{4\pi}\left(\log\frac{\mu^2}{\Lambda^2}\right), \tag{7.31}$$

so that the potential takes the form,

$$V(\sigma) = -N\left[\frac{\sigma}{\lambda_r} + \frac{\sigma}{4\pi}\left(\log\frac{\sigma}{\mu^2} - 1\right)\right]. \tag{7.32}$$

It is also evident that the model has a negative β function, or differently stated, for fixed λ_r and μ, when $\Lambda \to \infty$, λ vanishes, namely, the model is asymptotically free.

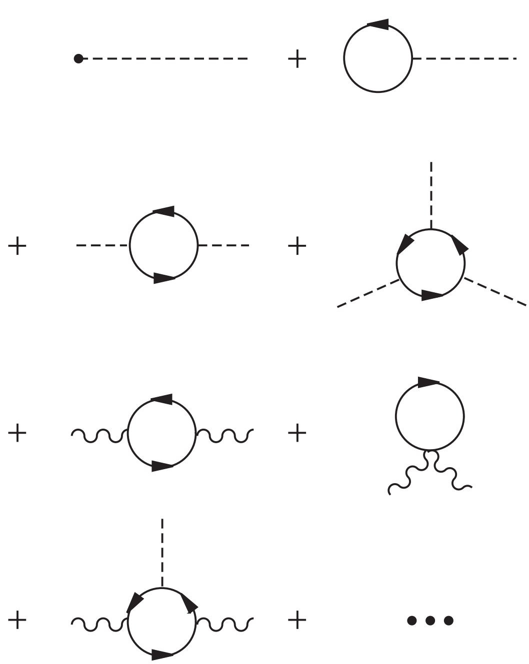

Fig. 7.5. Leading order contributions to the effective action.

Again the model admits a dimensional transmutation. The original dimensionless coupling is traded with a parameter σ_0 with dimensions, at which the potential has a minimum,

$$\frac{\mathrm{d}V}{\mathrm{d}\sigma} = -N\left[\frac{1}{\lambda_r} + \frac{1}{4\pi}\left(\log\frac{\sigma}{\mu^2}\right)\right] = 0, \quad \sigma_0 = \mu^2 \mathrm{e}^{-\frac{4\pi}{\lambda_r}}. \tag{7.33}$$

The following remarks about the model are relevant:

(i) The model admits a dynamical generation of abelian gauge fields. From the diagrams on the third line of Fig. 7.5 we see that the contribution to $S_{\text{effective}}$ quadratic in A_μ is,

$$-\frac{iN}{4\pi}\left[g_{\mu\nu}p^2 - p_\mu p_\nu\right]\int_0^1 \mathrm{d}x \frac{(1-2x)^2}{\sigma_0^2 - p^2 x(1-x) - i\epsilon}.$$

Now for long range interaction, namely, for small momenta we ignore the p^2 term to obtain,

$$-\frac{iN}{12\pi\sigma_0}[g_{\mu\nu}p^2 - p_\mu p_\nu],$$

which corresponds to the following term in the effective action,

$$S_{\text{effective}} = -\frac{N}{48\pi\sigma_0}\int \mathrm{d}^2 x F_{\mu\nu}F^{\mu\nu}, \tag{7.34}$$

namely an action of an abelian gauge field.

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(ii) The Z fields can be interpreted as bosonic "quarks" in the fundamental representation of the group, though transforming in a non-linear way. These Z quarks are confined due to the dynamically generated abelian gauge interaction. As will be shown in Chapter 8, in two dimensions the abelian force between a quark anti-quark pair is linear in separation distance.