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THE COERCIVENESS FOR INTEGRO-DIFFERENTIAL QUADRATIC FORMS AND KORN'S INEQUALITY

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Introduction

Let Ω be a bounded open set of $\boldsymbol{R}^n$ $(n \geqq 2)$ with a C^∞ boundary Γ and assume that $a^{ij}_{\alpha\beta}(x)\,(|\alpha| = s_i, |\beta| = s_j\,; i, j = 1, \cdots, N)$ be functions in $C^\infty(\bar{\Omega})$ such that $\overline{a^{ij}_{\alpha\beta}(x)} = a^{ji}_{\beta\alpha}(x)$, where $s_1, \cdots, s_N$ are integers $\geqq 1$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_n)$ are n-tuples of non-negative integers with $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\beta| = \beta_1 + \cdots + \beta_n$. Then we consider an integro-differential bilinear form

$$Q[u, v] = \sum_{i,j=1}^{N} \int_\Omega \sum_{|\alpha|=s_i} \sum_{|\beta|=s_j} a^{ij}_{\alpha\beta}(x) D^\alpha u_i \overline{D^\beta v_j} dx$$

over a Sobolev space

$$H_{(s)}(\Omega) = \{u = (u_1, \cdots, u_N)\,; u_j \in H_{s_j}(\Omega), j = 1, \cdots, N\},$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with $D_j = -i\partial/\partial x_j$ $(i = \sqrt{-1})$.

We shall say that the quadratic form $Q[u, u]$ is coercive over a subspace V of $H_{(s)}(\Omega)$ if there exist two constants $c_1 > 0$ and c_0 such that

$$Q[u, u] \geqq c_1 \|u\|^2_{(s)} - c_0 \|u\|^2_0\,, \qquad u \in V\,,$$

where

$$\|u\|^2_{(s)} = \sum_{i=1}^{N} \|u_i\|^2_{s_i}\,.$$

It is well-known that $Q[u, u]$ is coercive over $C_0^\infty(\Omega)^N$ (i.e. the Gårding inequality holds) if and only if there exists a constant $c > 0$ such that the inequality

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$$\sum_{i,j=1}^{n}\Big(\sum_{|\alpha|=s_i}\sum_{|\beta|=s_j} a_{\alpha\beta}^{ij}(x)\xi^{\alpha+\beta}\Big)\eta_i\bar{\eta}_j \geqq c\sum_{j=1}^{n}|\xi|^{2s_i}|\eta_j|^2$$

holds for every $x\in\bar{\Omega}$, every $\xi\in\boldsymbol{R}^n$ and every $\eta\in\boldsymbol{C}^N$ (strong ellipticity).

In applying the variational approach to elliptic boundary value problems, the space V is usually given by a system of differential operators defined on Γ. Let $B_{(i,p)}^j(x, D)$ $(p=0,\cdots,s_i-1\,;\, i,j=1,\cdots,N)$ be linear partial defferential operators with coefficients in $C^\infty(\Gamma)$ homogeneous of order s_j-s_i+p and assume that this system is normal, i.e., for any $P\in\Gamma$ and any real, non-zero vector $\dot{\xi}$ tangent to P (the totality of such $\dot{\xi}$, we denote by T_p), let us regard $B_{(i,p)}^j(P, \dot{\xi}+\tau\nu(P))$ as polynomials in τ ($\nu(P)$ denotes the unit inner normal vector to Γ at P), then

$$\sum_{i,p} C_{(i,p)}B_{(i,p)}^j(P,\xi+\tau\nu(P))=0\,,\qquad j=1,\cdots,N\,,$$

only if the constants $C_{(i,p)}$ are all zero, where summation $\sum_{i,p}$ means $\sum_{i=1}^N\sum_{p=0}^{s_i-1}$. Let S_1, S_2 be two subsets of the set

$$M=\bigcup_{i=1}^{N}\{(i,p)\,;\,p=0,\cdots,s_i-1\}$$

and Γ_1,Γ_2 be the disjoint open portions of Γ such that $\bar{\Gamma}_1\cup\bar{\Gamma}_2=\Gamma$ and $\gamma=\bar{\Gamma}_1\cap\bar{\Gamma}_2$ is a C^∞-manifold of dimension $n-2$. Then we introduce the closed subspace of $H_{(s)}(\Omega)$ as follows:

$$V(S_1,S_2)=\bigcap_{\alpha=1,2}\Big\{u\in H_{(s)}(\Omega)\,;\sum_{j=1}^{N}B_{(i,p)}^j(x,D)u_j=0 \text{ on } \Gamma_\alpha \text{ for } (i,p)\in S_\alpha\Big\}\,.$$

The mixed boundary value problems for strongly elliptic differential systems lead naturally to the investigation of coerciveness for quadratic forms Q over $V(S_1,S_2)$. In [4], we have already studied this problem in the case $N=1$ and $s_1=m$. By a slight improvement of the argument used in [4], we can characterize the coerciveness for Q over $V(S_1,S_2)$. We shall state briefly the main result in the following.

For any point P fixed on Γ, denoting by Q^P the form obtained by replacing $a_{\alpha\beta}^{ij}(x)$ with $a_{\alpha\beta}^{ij}(P)$ in the form Q and integrating by part, we have Green's formula

$$\begin{aligned}Q^P[u,v]=&\int_\Omega\sum_{i,j=1}^{N}A^{ij}(P,D)u_i\cdot\bar{v}_j dx\\&+\int_\Gamma\sum_{j,q}\sum_{i=1}^{N}N_{(j,q)}^i(P,D)u_i\overline{D_\nu^q v_j}d\sigma\,,\end{aligned}$$

where

$$A^{ij}(P, D) = \sum_{|\alpha|=s_i} \sum_{|\beta|=s_j} a^{ij}_{\alpha\beta}(P) D^{\alpha+\beta}$$

and $N^i_{(j,q)}(P, D)$ are the differential operators on Γ homogeneous of order $s_i + s_j - q - 1$. For any $P \in \Gamma$, any $\dot{\xi} \in T_P$ and any $(i, p) \in M$, we denote $h^{(i,p)}_j(\dot{\xi}, t)$ $(j = 1, \cdots, N)$ the exponentially decaying solution of the Dirichlet problem

$$\begin{cases} \sum_{j=1}^{N} A^{ij}(P, \dot{\xi} + \nu(P) D_t) u_j = 0 & \text{in } t > 0 \\ D^q_t u_j = \delta^{(i,p)}_{(j,q)} & \text{for } (j, q) \in M \text{ on } t = 0 , \end{cases}$$

where $\delta^{(i,p)}_{(j,q)}$ denotes Kronecker's delta. For any $(i, p), (j, q) \in M$, we set

$$b^{(i,p)}_{(j,q)}(P, \dot{\xi}) = \sum_{k=1}^{N} N^k_{(j,q)}(P, \dot{\xi} + \nu(P) D_t) h^{(i,p)}_k(\dot{\xi}, t)|_{t=0} .$$

The normality of the system $\{B^j_{(i,p)}\}$ guarantees that for each $(i, p), (j, q) \in M$ there exists a number $e^{(j,q)}_{(i,p)}(P, \dot{\xi})$ such that, for any $u(t)$,

$$D^p_t u_i(t) = \sum_{j,q} e^{(j,q)}_{(i,p)}(P, \dot{\xi}) \sum_{k=1}^{N} B^k_{(j,q)}(P, \dot{\xi} + \nu(P) D_t) u_k(t)$$
$$\text{on } t = 0, p = 0, \cdots, s_i - 1 .$$

For a subset S of M and a $\dot{\xi} \in T_P$, we introduce a subspace of $\boldsymbol{C}^s$ $(s = s_1 + \cdots + s_N)$ as follows;

$$L^P_{\dot{\xi}}(S) = \left\{ \left(\sum_{j,q} e^{(j,q)}_{(i,p)}(P, \dot{\xi}) b_{(j,q)} \right)_{(i,p) \in M} ; b \in \boldsymbol{C}^s \text{ such that } b_{(j,q)} = 0 \text{ for } (j, q) \in S \right\} .$$

Then we have

THEOREM. *In order that the $Q[u, u]$ be coercive over $V(S_1, S_2)$, it is necessary and sufficient that for every P on $\bar{\Gamma}_\alpha$ $(\alpha = 1, 2)$ there exists a constant $c > 0$ such that for any $\dot{\xi}, |\dot{\xi}| = 1$, in T_P the inequality*

$$\sum_{i,p} \sum_{j,q} b^{(i,p)}_{(j,q)}(P, \dot{\xi}) a_{(i,p)}(\dot{\xi}) \overline{a_{(j,q)}(\dot{\xi})} \geqq c \sum_{(i,p)} |a_{(i,p)}(\dot{\xi})|^2$$

holds for every $(a_{(i,p)}(\dot{\xi})) \in L^P_{\dot{\xi}}(S_\alpha)$.

In this paper, we shall confine ourself to the proof of the theorem in which $s_1 = \cdots = s_N = 1$, because this special case remains the essential part of the proof and the general case has nothing but a further complication. As its application, we shall give an another proof of Korn's inequality which is fundamental in the boundary value problem of linear elastostatics.

In section 1, we shall reduce the problem to the coerciveness problem for quadratic forms with constant coefficients over some space contained in $H_1(\boldsymbol{R}^n_+)^N$. With this reduced problem we deal in Section 2 (see Theorem 1). The algebraic condition for coerciveness is given in Section 3 (see Theorem 2). Section 4 is devoted to quadratic forms with variable coefficients and the main theorem (Theorem 3) is proved there. In Section 5, we shall apply this theorem to a formally positive quadratic form (see Theorem 4) and, as an example, deduce Korn's inequality.

§ 1. Localization of the problem

We consider the integro-differential quadratic form

$$Q[u, u] = \int_\Omega \sum_{i,j=1}^n \langle a_{ij}(x) D_i u, D_j u\rangle dx \tag{1.1}$$

on the closed subspace V of $H_1(\Omega)^N$ which contains $C_0^\infty(\Omega)$, where $a_{ij}(x)$ $(i, j = 1, \cdots, n)$ are N-square matrices with entries in $C^\infty(\bar{\Omega})$, $u = (u_1, \cdots, u_N) \in V$ and

$$\langle a, b\rangle = \sum_{j=1}^N a_j \bar{b}_j \qquad \text{for } a, b \in \boldsymbol{C}^N .$$

Here we can assume that

$$a_{ij}(x)^* = a_{ji}(x) \;, \qquad i, j = 1, \cdots, n \;, \tag{1.2}$$

hold for every $x \in \Omega$ ($a_{ij}(x)^*$ denotes the adjoint matrix of $a_{ij}(x)$) and that there exists a constant $c > 0$ such that the inequality

$$\left\langle \left(\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \right) a, a \right\rangle \geqq c|\xi|^2|a|^2 \tag{1.3}$$

holds for every $x \in \bar{\Omega}$, every $\xi \in \boldsymbol{R}^n$ and every $a \in \boldsymbol{C}^N$, as far as the coerciveness over V is concerned (see Lecture 14 of [1]). We further assume that V satisfies the property:

$$\zeta u \in V \quad \text{for any} \quad u \in V \quad \text{and} \quad \zeta \in C^\infty(\bar{\Omega}) . \tag{1.4}$$

By freezing any point P on Γ, we introduce a quadratic form with constant matrices $a_{ij}(P)$:

$$Q^P[u, u] = \int_\Omega \sum_{i,j=1}^N \langle a_{ij}(P) D_i u, D_j u \rangle dx .$$

Then we have

PROPOSITION 1.1. *In order that the form* (1.1) *be coercive over* V, *it is necessary and sufficient that*

$$\text{(1.5)} \quad \begin{cases} \textit{for every } P \in \Gamma, \textit{ there exists an } n\textit{-dimensional open neighbour-} \\ \textit{hood } G \textit{ of } P \textit{ such that } Q^P[u, u] \textit{ is coercive over a closed sub-} \\ \textit{space } V_G \textit{ of } V: \\ \qquad\qquad V_G = \{u \in V \,;\, \operatorname{supp}[u] \subset G\} . \end{cases}$$

Let $y = (y_1, \cdots, y_n)$ be a Cartesian coordinate system such that $y' = (y_1, \cdots, y_{n-1})$ represents a coordinate systems of the tangent hyperplane through P and y_n a coordinate of the direction of inner normal to Γ at P, and assume that the generic point $x \in \boldsymbol{R}^n$ can be written as $x = P + Sy$ with an orthogonal matrix S. It then follows that there exists an open neighbourhood G of P and a C^∞-function $f_P(y')$ such that $\Gamma \cap G$ is represented by $y_n - f_P(y') = 0$ and $f_P(y')$ together with $\partial f_P / \partial y_j$ $(j = 1, \cdots, n-1)$ vanishes at $y' = 0$. Therefore, by the coordinate transformation

$$\text{(1.6)} \qquad \chi_P : \begin{cases} z_j = y_j \,, & i = 1, \cdots, n-2 \\ z_{n-1} = y_{n-1} - g_P(y_1, \cdots, y_{n-2}) \\ z_n = y_n - f_P(y_1, \cdots, y_{n-1}) \,, \end{cases}$$

$G \cap \Omega$ is mapped in a one-to-one way onto an open portion of a half space $\boldsymbol{R}_+^n = \{z \in \boldsymbol{R}^n \,;\, z_n > 0\}$ and $G \cap \Gamma$ is transformed onto an open portion of $z_n = 0$, where g_P is a C^∞-function and together with $\partial g_P / \partial y_j$ $(j = 1, \cdots, n-2)$ vanishes at $(y_1, \cdots, y_{n-2}) = 0$.

For $u \in V_G$, we define a function $\tilde{u}(z)$ on a half space $z_n \geqq 0$ by

$$\tilde{u}(z) = \begin{cases} u(P + S\chi_P^{-1}(z)) \,, & z \in \chi_P(G \cap \bar{\Omega}) \\ 0 \,, & z \notin \chi_P(G \cap \bar{\Omega}) \,, \qquad z_n \geqq 0 \end{cases}$$

and set $\tilde{V}_G = \{\tilde{u} \,;\, u \in V_G\}$. Then, by $Q_1^P[\tilde{u}, \tilde{u}]$ we denote the form

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$$(1.7)\qquad Q_1^P[\tilde{u}, \tilde{u}] = \int_{z_n \geqq 0} \sum_{i,j=1}^{N} \langle \tilde{a}_{ij}(P) D_i \tilde{u}, D_j \tilde{u} \rangle dz ,$$

where

$$\tilde{a}_{ij}(P) = \sum_{k,\ell=1}^{n} s_{ki} a_{k\ell}(P) s_{\ell j} , \qquad D_j = -i\partial/\partial z_j ,$$

with $S = (s_{ij})$. Since the coerciveness is invariant under coordinate transformations, we can establish

PROPOSITION 1.2. *In Proposition* 1.1, *we can replace* (1.5) *with*

$$(1.8)\qquad \begin{cases} \text{for every } P \in \Gamma, \text{ there exists an open neighbourhood } G \text{ of } P \text{ such} \\ \text{that the form (1.7) is coercive over } \tilde{V}_G. \end{cases}$$

For the proof of Propositions 1.1 and 1.2, we refer Propositions 3.1 and 3.2 of [4].

Now we introduce the spaces V associated to mixed boundary value problems. Let S_1, S_2 be two subsets of the set $\{1, 2, \cdots, N\}$, Γ_1, Γ_2 be the disjoint open portions of Γ such that $\bar{\Gamma}_1 \cup \bar{\Gamma}_2 = \Gamma$ and $\gamma = \bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is a C^∞-manifold of dimension $n-2$, and $\boldsymbol{b}_j(x)\,(j = 1, \cdots, N)$ be the given N-vectors with components in $C^\infty(\Gamma)$ which are linearly independent at each $x \in \bar{\Omega}$. We define as V the closed subspace of $H_1(\Omega)^N$:

$$(1.9)\qquad V(S_1, S_2) = \bigcap_{\alpha=1,2} \{u \in H_1(\Omega)^N ; \langle \boldsymbol{b}_j(x), u \rangle = 0 \text{ on } \Gamma_\alpha \text{ for } j \in S_\alpha\} .$$

Obviously it satisfies property (1.4).

In the below, we shall localize the boundary conditions. We set, for $P \in \Gamma_\alpha\ (\alpha = 1, 2)$

$$V_R^P = \{u \in H_1(\boldsymbol{R}_+^n)^N \cap \mathscr{E}'_R ; \langle \boldsymbol{b}_j(P), u \rangle = 0 \text{ on } z_n = 0 \text{ for } j \in S_\alpha\} ,$$

and for $P \in \gamma$

$$V_R^P = \bigcap_{\alpha=1,2} \{u \in H_1(\boldsymbol{R}_+^n)^N \cap \mathscr{E}'_R ; \langle \boldsymbol{b}_j(P), u \rangle = 0 \text{ on } H_\alpha \text{ for } j \in S_\alpha\} ,$$

where and in the following we write, for simplicity, $\tilde{u}$ as u and $\mathscr{E}'(\Sigma_R)$ as $\mathscr{E}'_R$ with

$$\Sigma_R = \{z \in \boldsymbol{R}^n ; |z| < R, z_n \geqq 0\}$$

and

$$H_1 \text{ (resp. } H_2) = \{z \in \boldsymbol{R}^n ; z_n = 0, z_{n-1} > 0 \text{ (resp. } z_{n-1} < 0)\} .$$

PROPOSITION 1.3. *In order that the form* (1.1) *satisfying* (1.2) *and* (1.3) *be coercive over* $V(S_1, S_2)$, *it is necessary and sufficient that*

$$(1.10)\quad \begin{cases} \textit{for every } P \in \Gamma, \textit{ there exists a number } R > 0 \textit{ such that the} \\ \textit{form } Q_1^P[u, u] \textit{ defined by (1.7) is coercive over } V_R^P. \end{cases}$$

Proof. Let P be in Γ_α and G be an open neighbourhood of Γ such that $G \cap \Gamma = G \cap \Gamma_\alpha$ and assume that G is contained in the definition domain of the transformation (1.6) with $g_P = 0$. Then we get

$$\tilde{V}_G = \{u \in H_1(\boldsymbol{R}_+^n)^N ; \operatorname{supp}[u] \subset \chi_P(G \cap \bar{\Omega}), \langle \tilde{\boldsymbol{b}}_j(z'), u\rangle = 0 \text{ on } z_n = 0 \text{ for } j \in S_\alpha\},$$

where $\tilde{\boldsymbol{b}}_j(z') = \boldsymbol{b}_j(P + S\chi_P^{-1}(z', 0))$ and $z' = (z'_1, \cdots, z_{n-1})$. If P is in γ, we choose χ_P and g_P so that $\chi_P(G \cap \gamma)$ is represented by $z_n = z_{n-1} = 0$, and $\chi_P(G \cap \Gamma_1)$ (resp. $\chi_P(G \cap \Gamma_2)$) by $z_{n-1} > 0, z_n = 0$ (resp. $z_{n-1} < 0, z_n = 0$). Thus we have

$$\tilde{V}_G = \bigcap_{\alpha=1,2} \{u \in H_1(\boldsymbol{R}_+^n)^N ; \operatorname{supp}[u] \subset \chi_P(G \cap \bar{\Omega}), \langle \boldsymbol{b}_j(z'), u\rangle = 0 \text{ on } H_\alpha \text{ for } j \in S_\alpha\}.$$

Now we shall prove the equivalence of (1.8) and (1.10) for $V = V(S_1, S_2)$ which is defined by (1.9). Let P be any point on Γ.

(i) (1.8) $\Rightarrow$ (1.10). Let R be a positive number such that $\Sigma_R \subset \chi_P(G \cap \bar{\Omega})$. We set

$$\phi_j(z') = \langle \tilde{\boldsymbol{b}}_j(z') - \tilde{\boldsymbol{b}}_j(0), u\rangle \qquad \text{on } z_n = 0 \ (j = 1, \cdots, N)$$

for u in V_R^P. It then easily follows that there exists a $v \in H_1(\boldsymbol{R}_+^n)^N \cap \mathscr{E}'_R$ such that

$$\langle \tilde{\boldsymbol{b}}_j(z'), v\rangle = \begin{cases} \phi_j(z') & \text{on } z_n = 0, \ j \in S_1 \cup S_2 \\ 0 & \text{on } z_n = 0, \ j \notin S_1 \cup S_2 \end{cases}$$

and

$$(1.11)\qquad \|v\|_1 \leqq C \sum_{j \in S_1 \cup S_2} \|\phi_j\|_{1/2},$$

C being a positive constant not depending on u, v. Remembering that $\phi_j(0) = 0$ for $j = 1, \cdots, N$, we can establish that for any $\varepsilon > 0$ there exists a number $R > 0$ such that for every $j \in S_1 \cup S_2$

$$(1.12)\qquad \|\phi_j\|_{1/2} \leqq \varepsilon \|u\|_1, \qquad u \in V_R^P.$$

Obviously $u - v$ is in $\tilde{V}_G$. Accordingly it follows from (1.8) that

$$Q_1^P[u-v, u-v] \geqq c_1 \|u-v\|_1^2 - c_0 \|u-v\|_0^2$$

with some constants $c_1 > 0$ and c_0. So that

$$\begin{aligned} Q_1^P[u,u] &\geqq c_1 \|u-v\|_1^2 - c_0 \|u-v\|_0^2 - K_1(\|u\|_1 \|v\|_1 + \|v\|_1^2) \\ &\geqq c_1 \|u\|_1^2 - c_0 \|u\|_0^2 - K_2(\|u\|_1 \|v\|_1 + \|v\|_1^2) \\ &\geqq \frac{c_1}{2} \|u\|_1^2 - c_0 \|u\|_0^2 - K_3 \|v\|_1^2 , \end{aligned}$$

where $K_1, K_2, \cdots$ denote appropriate constants. Using (1.11) and (1.12), we can obtain

$$Q_1^P[u,u] \geqq \frac{c_1}{4} \|u\|_1^2 - c_0 \|u\|_0^2$$

for every $u \in V_R^P$ if ε is sufficient small. That is, (1.10) holds for $R = R_\varepsilon$.

(ii) (1.10) $\Rightarrow$ (1.8). Let G be a neighbourhood of P such that $\chi_P(G \cap \bar{\Omega}) \subset \Sigma_R$. For $u \in \tilde{V}_G$, we can choose a v in $H_1(\boldsymbol{R}_+^n)^N \cap \mathscr{E}'_R$ so that (1.11) and

$$\langle \tilde{\boldsymbol{b}}_j(0), v \rangle = \begin{cases} -\phi_j(z') & \text{on } z_n = 0,\ j \in S_1 \cup S_2 \\ 0 & \text{on } z_n = 0,\ j \notin S_1 \cup S_2 \end{cases}$$

are valid. Thus by the same argument as in (i), we can conclude (1.8) from (1.10).

§ 2. Coercive forms with constants coefficients in a half space

Let there be given the integro-differential bilinear form in the half space $\boldsymbol{R}_+^n = \{x \in \boldsymbol{R}^n ; x_n > 0\}$:

$$Q_0[u,v] = \int_{t>0} \sum_{i,j=1}^{n} \langle a_{ij} D_i u, D_j v \rangle dx , \tag{2.1}$$

where a_{ij} $(i, j = 1, \cdots, n)$ are N-square matrices with complex constant entries and satisfy (1.2) and (1.3), and $t = x_n$, $D_j = -i\partial/\partial x_j$. Integrating by parts, we obtain

$$Q_0[u,v] = \int_{t>0} \langle A(D)u, v \rangle dx + \int_{t=0} \langle N(D)u, v \rangle dx' , \tag{2.2}$$

where $x' = (x_1, \cdots, x_{n-1})$ and

$$A(D)u = \sum_{i,j=1}^{n} a_{ij} D_i D_j u \,, \qquad N(D)u = -i \sum_{j=1}^{n} a_{jn} D_j u \,.$$

For λ real, we set

$$Q_\lambda[u, v] = Q_0[u, v] + \lambda^2(u, v)$$

with

$$(u, v) = \int_{t>0} \langle u, v \rangle dx \,.$$

Putting $A_\lambda(D) = A(D) + \lambda^2 I$ (I denotes the identity N-matrix), we have by (2.2)

$$Q_\lambda[u, v] = (A_\lambda u, v) + \int_{t=0} \langle N(D)u, v \rangle dx' \,. \tag{2.3}$$

For $\lambda \neq 0$ and $\phi(x')$ in $\mathscr{S}^N$ ($\mathscr{S}$ is the Schwartz space on $\boldsymbol{R}^{n-1}$), we denote by $u_\lambda(x)$ the unique solution of the Dirichlet problem

$$\begin{cases} A_\lambda(D)u = 0 & \text{in } t > 0 \\ u = \phi & \text{on } t = 0 \end{cases} \tag{2.4}$$

in $H_2(\boldsymbol{R}^n_+)^N$. Substituting u_λ for u, v in (2.3), we have

$$Q_\lambda[u_\lambda, u_\lambda] = \int_{t=0} \langle N(D)u_\lambda, u_\lambda \rangle dx' \,.$$

By the Fourier transformation in the variable x' and Parseval's formula, we can obtain

$$Q_\lambda[u_\lambda, u_\lambda] = (2\pi)^{1-n} \int_{t=0} \langle N(\eta, D_t)\hat{u}_\lambda(\eta, t)|_{t=0}, \hat{u}_\lambda(\eta, 0) \rangle d\eta \,, \tag{2.5}$$

where

$$\hat{u}_\lambda(\eta, t) = \int e^{-i\langle x', \eta \rangle} u_\lambda(x', t) dx' \,, \qquad \eta \in \boldsymbol{R}^{n-1}$$

is a solution of the initial value problem

$$\begin{cases} A_\lambda(\eta, D_t)\hat{u} = 0 & \text{in } t > 0 \,. \\ \hat{u} = \hat{\phi}(\eta) & \text{on } t = 0 \end{cases} \tag{2.6}$$

and exponentially decays as $t \to \infty$.

For any $(\eta, \lambda) \neq 0$, let $u^{(h)}(\eta, \lambda, t)$ be the exponentially decaying solution of (2.6) with the initial condition $u^{(h)}(\eta, \lambda, 0) = e_h$ (e_h is the N-vector whose h-th component is 1 and others zero). Then we have

PROPOSITION 2.1. (i) *The* $u^{(h)}(\eta, \lambda, t)$ $(h = 1, \cdots, N)$ *are analytic in* (η, λ) *and have the homogeneity property*:

$$u^{(h)}(\theta^{-1}\eta, \theta^{-1}\lambda, \theta t) = u^{(h)}(\eta, \lambda, t) , \qquad \theta > 0 . \tag{2.7}$$

(ii) *The exponential decaying solution of* (2.6) *is given by*

$$\hat{u}(\eta, \lambda, t) = \sum_{h=1}^{N} u^{(h)}(\eta, \lambda, t)\hat{\phi}_h(\eta) \tag{2.8}$$

and the inequality

$$\int_0^\infty dt \int \left\{(|\eta|^2 + \lambda^2)|\hat{u}|^2 + \left|\frac{d\hat{u}}{dt}\right|^2\right\} d\eta \leqq C \int (|\eta|^2 + \lambda^2)^{1/2} |\hat{\phi}|^2 \, d\eta \tag{2.9}$$

is satisfied with a suitable constant $C > 0$ *independent of* λ.

(iii) *The inverse Fourier transformation of* $\hat{u}(\eta, \lambda, t)$:

$$u_\lambda(x', t) = (2\pi)^{1-n} \int e^{i\langle x', \eta\rangle} \hat{u}(\eta, \lambda, t) d\eta$$

is the solution of (2.4) *in* $H_2(\boldsymbol{R}^n_+)^N$ *if* $\lambda \neq 0$.

Proof. By the theory of ordinary differential equations, (i) is immediately obtained. Differentiating (2.7), we obtain

$$\theta^j (d_t^j u)(\theta^{-1}\eta, \theta^{-1}\lambda, \theta t) = d_t^j u(\eta, \lambda, t) , \qquad \theta > 0 ,$$

where $d_t = d/dt$. For brevity, we put $\theta = \theta(\eta, \lambda) = (|\eta|^2 + \lambda^1)^{1/2}$. It then follows from (2.8) that, for $j = 0, 1$,

$$\begin{aligned}\int_0^\infty dt \int \theta^{2(1-j)} |d_t \hat{u}|^2 \, d\eta &\leqq N \int_0^\infty dt \int \theta^{2(1-j)} \sum_{h=1}^{N} |d_t^j u^{(h)}(\eta, \lambda, t)\hat{\phi}_h(\eta)|^2 \, d\eta \\ &\leqq N \int_0^\infty ds \int \theta \sum_{h=1}^{N} |d_t^j u^{(h)}(\theta^{-1}\eta, \theta^{-1}\lambda, s)|^2 |\hat{\phi}_h(\eta)|^2 \, d\eta \\ &\leqq N \sup_{\theta=1} \int |d_s^j u^{(h)}(\eta, \lambda, s)|^2 \, ds \int \theta |\hat{\phi}_h(\eta)|^2 \, d\eta .\end{aligned}$$

This completes (2.9). By the same way, we can obtain, for $j = 0, 1, 2$,

$$\int_0^\infty dt \int \theta^{2(2-j)} |d_t^j \hat{u}|^2 \, d\eta \leqq C \int \theta^3 |\hat{\phi}(\eta)|^2 \, d\eta ,$$

which guarantees that $u_\lambda(x', t)$ is in $H_2(\boldsymbol{R}^n_+)^N$. Q.E.D.

Now we return (2.5). Since $\hat{u}_\lambda(\eta, t) = \hat{u}(\eta, \lambda, t)$, we have

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(2.10) $$N(\eta, D_t)\hat{u}_\lambda(\eta, t)|_{t=0} = B(\eta, \lambda)\hat{\phi}(\eta) ,$$

provided that $B(\eta, \lambda)$ is the N-square matrix whose (j, h)-entry $b_{jh}(\eta, \lambda)$ is equal to the j-component of the N-vector $N(\eta, D_t)u^{(h)}(\eta, \lambda, t)|_{t=0}$. Consequently, (2.5) becomes

(2.11) $$Q_\lambda[u_\lambda, u_\lambda] = (2\pi)^{1-n} \int \langle B(\eta, \lambda)\hat{\phi}(\eta), \hat{\phi}(\eta)\rangle d\eta .$$

Here note that each entry $b_{jh}(\eta, \lambda)$ is analytic for $(\eta, \lambda) \neq 0$ and homogeneous of degree 1. In fact, this follows immediately from (2.7).

Let V_0 be a closed subspace of $H_1(\boldsymbol{R}^n_+)^N$ and assume that for every $\varepsilon > 0$

(2.12) $$u(x) \in V_0 \Rightarrow u^{(\varepsilon)}(x) = u(x/\varepsilon) \in V_0 .$$

Then we can prove

PROPOSITION 2.2. *If the quadratic form $Q_0[u, u]$ of the form* (2.1) *is coercive over V_0 satisfying* (2.12), *then there exists a constant $c > 0$ independent of λ such that for every λ*

(2.13) $$\int \langle B(\eta, \lambda)U(\eta), U(\eta)\rangle d\eta \geqq c \int |\eta|\langle U(\eta), U(\eta)\rangle d\eta, U \in \sigma(\hat{V}_0) ,$$

where

$$\sigma(\hat{V}_0) = \left\{U(\eta) = \int e^{-i\langle x', \eta\rangle}u(x', t)dx'|_{t=0}, u \in V_0\right\} .$$

Proof. Let $\lambda \neq 0$ and let u_λ be the solution of the Dirichlet problem (2.4) with $\phi \in H_{1/2}(\boldsymbol{R}^{n-1})$ such that $\hat{\phi} \in \sigma(\hat{V}_0)$. Then we have $u_\lambda \in V_0$ and note that, for the u_λ, (2.11) is also valid. Then, the coerciveness of Q_0 and (2.11) guarantee that

(2.14) $$\int \langle B(\eta, \lambda)U(\eta), U(\eta)\rangle d\eta \geqq c_1 \int (1 + |\eta|)|U(\eta)|^2 d\eta - c_0 \|u_\lambda\|_0^2 ,$$

where $c_1 > 0$ and c_0 are constants independent of λ, and we put $\hat{\phi} = U$ and used the well-known inequality

$$\int (1 + |\eta|)|U(\eta)|^2 d\eta \leqq \text{const.} \|u_\lambda\|_1^2 .$$

By (2.12), we have $u_\lambda^{(\varepsilon)} \in V_0$ $(\varepsilon > 0)$ and

$$\begin{cases} A(D)u_\lambda^{(\varepsilon)}(x) = -\left(\dfrac{\lambda}{\varepsilon}\right)^2 u_\lambda^{(\varepsilon)}(x) \\ u_\lambda^{(\varepsilon)}(x', 0) = \phi\left(\dfrac{x'}{\varepsilon}\right). \end{cases}$$

Noting

$$\widehat{\phi\left(\frac{x'}{\varepsilon}\right)} = \varepsilon^{n-1}U(\varepsilon\eta),$$

and applying (2.14) for $u_\lambda^{(\varepsilon)}$, we can immediately obtain

$$\varepsilon^{2n-2}\int\left\langle B\left(\eta, \frac{\lambda}{\varepsilon}\right)U(\varepsilon\eta), U(\varepsilon\eta)\right\rangle d\eta \geqq \varepsilon^{2n-2}c_1\int (1+|\eta|)|U(\varepsilon\eta)|^2\, d\eta - c_0\|u_\lambda^{(\varepsilon)}\|_0^2 .$$

Devising the both side by ε^{n-2} and tending ε to zero, we can conclude (2.13) for $\lambda \neq 0$ and hence also for $\lambda = 0$. Q.E.D.

PROPOSITION 2.3. *In Proposition* 2.2, *if* (2.13) *is valid for* $\lambda = 0$, *then* $Q_0[u, u]$ *is coercive over* V_0.

Proof. Let $u \in V_0$. We can choose a sequence u_ε, $\varepsilon > 0$, in $C_0^\infty(\bar{\boldsymbol{R}}_+^n)^N$ such that $u_\varepsilon \to u$ in $H_1(\boldsymbol{R}_+^n)^N$ as $\varepsilon \to 0$. Let w_λ be in $H_2(\boldsymbol{R}_+^n)^N$ and be a solution of (2.4) with $\phi = u_\varepsilon|_{t=0}$ and with $\lambda \neq 0$, and set $v_\varepsilon = u_\varepsilon - w_\lambda$.

By (2.9) in Proposition 2.1, we have

$$\int_0^\infty dt\int\left\{(|\eta|^2+\lambda^2)|\hat{w}_\lambda|^2 + \left|\frac{d\hat{w}_\lambda}{dt}\right|^2\right\}d\eta \leqq C\int (|\eta|^2+\lambda^2)^{1/2}|U_\varepsilon(\eta)|^2\, d\eta ,$$

where $U_\varepsilon(\eta) = \hat{u}_\varepsilon|_{t=0}$ and C is a constant > 0 independent of λ and ε. Writing here the left hand side as $|w_\lambda|_{1,\lambda}^2$, we have

$$|v_\varepsilon|_{1,\lambda}^2 \geqq \tfrac{1}{2}|u_\varepsilon|_{1,\lambda}^2 - C\int (|\eta|^2+\lambda^2)^{1/2}|U_\varepsilon(\eta)|^2\, d\eta . \tag{2.15}$$

Substituting w_λ in place of u_λ in (2.11), we obtain

$$Q_\lambda[u_\varepsilon - v_\varepsilon, u_\varepsilon - v_\varepsilon] = (2\pi)^{1-n}\int \langle B(\eta,\lambda)U_\varepsilon(\eta), U_\varepsilon(\eta)\rangle d\eta . \tag{2.16}$$

Since $v_\varepsilon = 0$ on $t = 0$ and $A_\lambda u_\varepsilon = A_\lambda v_\varepsilon$ in $t > 0$, we can calculate as follows:

$$\begin{aligned} Q_\lambda[u_\varepsilon - v_\varepsilon, u_\varepsilon - v_\varepsilon] &= Q_\lambda[u_\varepsilon, u_\varepsilon] + Q_\lambda[v_\varepsilon, v_\varepsilon] - 2\,\mathrm{Re}\,(A_\lambda u_\varepsilon, v_\varepsilon) \\ &= Q_\lambda[u_\varepsilon, u_\varepsilon] + Q_\lambda[v_\varepsilon, v_\varepsilon] - 2\,\mathrm{Re}\,(A_\lambda v_\varepsilon, v_\varepsilon) \end{aligned}$$

$$= Q_\lambda[u_\varepsilon, u_\varepsilon] - Q_\lambda[v_\varepsilon, v_\varepsilon] .$$

Therefore, it follows from (2.15), (2.16) and the ellipticity condition (1.3) that for any δ, $0 < \delta \leqq 1$,

$$\begin{aligned} Q_\lambda[u_\varepsilon, u_\varepsilon] &\geqq \delta Q_\lambda[v_\varepsilon, v_\varepsilon] + Q_\lambda[u_\varepsilon - v_\varepsilon, u_\varepsilon - v_\varepsilon] \\ &\geqq \delta C_1 \int_0^\infty dt \int \left\{(|\eta|^2 + \lambda^2)|\hat{u}_\varepsilon|^2 + \left|\frac{du_\varepsilon}{dt}\right|^2\right\} d\eta \\ &\quad - \delta C_2 \int (|\eta|^2 + \lambda^2)^{1/2} |U_\varepsilon(\eta)|^2 d\eta \\ &\quad + (2\pi)^{1-n} \int \langle B(\eta, \lambda) U_\varepsilon(\eta), U_\varepsilon(\eta) \rangle d\eta , \end{aligned}$$

where C_1 and C_2 are constants > 0 independent of ε and λ. Tending $\varepsilon \to 0$ and using (2.13) with $\lambda = 0$, we obtain

$$Q_0[u, u] \geqq \delta C_1 \int_0^\infty dt \int \left(|\eta|^2 |\hat{u}|^2 + \left|\frac{d\hat{u}}{dt}\right|^2\right) d\eta + ((2\pi)^{1-n} c - \delta C_2) \int |\eta| |U(\eta)|^2 d\eta .$$

If we choose δ so that $(2\pi)^{1-n} c - \delta C_2 \geqq 0$, then we can conclude that $Q_0[u, u]$ is coercive over V_0. Q.E.D.

Thus, combining Propositions 2.2 and 2.3, we have

THEOREM 1. *An integro-differential quadratic form $Q_0[u, u]$ of the form* (2.1) *with properties* (1.2) *and* (1.3) *is coercive over a closed subspace V_0 of $H_1(\boldsymbol{R}_+^n)^N$ satisfying* (2.12), *if and only if*

$$(2.17)\quad \begin{cases} \textit{there exists a constant } c > 0 \textit{ such that} \\ \displaystyle\int \langle B(\eta) U(\eta), U(\eta) \rangle d\eta \geqq c \int |\eta| \langle U(\eta), U(\eta) \rangle d\eta , \qquad U \in \hat{V}_0 , \\ \textit{where } B(\eta) = B(\eta, 0) \textit{ and } B(\eta, \lambda) \textit{ is the } N\textit{-square matrix defined} \\ \textit{by } (2.10). \end{cases}$$

§ 3. Algebraic characterization

Let $\boldsymbol{b}_j$, $j = 1, \cdots, N$, be N-vectors with complex components (i.e., $\boldsymbol{b}_j \in \boldsymbol{C}^N$) which are linearly independent, and let S_+, S_- be two subsets of the set $\{1, \cdots, N\}$. In this section we shall consider the problem of coerciveness for the quadratic form (2.1) over

$$V_0(S_+, S_-) = \{u \in H_1(\boldsymbol{R}_+^n)^N ; \langle \boldsymbol{b}_j, u \rangle = 0 \text{ on } \Gamma_\pm \text{ for } j \in S_\pm\} ,$$

which clearly satisfies (2.12) and where

$$\Gamma_\pm = \{(x_1, \cdots, x_{n-1}, 0) \in \boldsymbol{R}^n\,;\, x_{n-1} \gtrless 0\}\,.$$

Here and in the following, the signs $\pm$ and $\gtrless$ are taken in the same order. For a subset S of the set $\{1, \cdots, N\}$, we introduce the subspace $L(S)$ of $\boldsymbol{C}^N$ as follows:

$$L(S) = \{a \in \boldsymbol{C}^N\,;\, \langle \boldsymbol{b}_j, a\rangle = 0 \text{ for } j \in S\}$$

THEOREM 2. *An integro-differential quadratic form $Q_0[u, u]$ of the form (2.1) with properties (1.2) and (1.3) is coercive over $V_0(S_+, S_-)$ if and only if*

$$(3.1)\quad \begin{cases} \textit{there exists a constant } c > 0 \textit{ such that for every } \eta \in \boldsymbol{R}^{n-1}, \eta \neq 0, \\ \qquad \langle B(\eta)a, a\rangle \geqq c|\eta|\langle a, a\rangle\,, \qquad a \in L(S_+) \cup L(S_-) \\ \textit{holds, where } B(\eta) \textit{ is the same matrix as in Theorem 1.} \end{cases}$$

Proof. For the proof we have only to establish the equivalence of (3.1) and (2.17) with $V_0 = V_0(S_+, S_-)$.

For $u \in V_0(S_+, S_-)$, we set $w_j(x') = \langle \boldsymbol{b}_j, u\rangle|_{t=0}$ for $j = 1, \cdots, N$. Clearly we have $w_j \in H_{1/2}(\boldsymbol{R}^{n-1})$ and, in particular, $w_j \in H^{\mp}_{1/2}(\boldsymbol{R}^{n-1})$ for $j \in S_\pm$, where

$$H^\pm_s(\boldsymbol{R}^{n-1}) = \{\phi \in H_s(\boldsymbol{R}^{n-1})\,;\, \phi = 0 \ \ \Gamma_\mp\}\,.$$

Conversely, for any $w_j \in H_{1/2}(\boldsymbol{R}^{n-1})$ $(j = 1, \cdots, N)$ such that $w_j \in H^{\mp}_{1/2}(\boldsymbol{R}^{n-1})$ for $j \in S_\pm$, we can find a u in $V_0(S_+, S_-)$ satisfying $w_j(x') = \langle \boldsymbol{b}_j, u\rangle|_{t=0}$ for $j = 1, \cdots, N$. Then we denote by E a non-singular N-square matrix with constant entries such that $u(x', 0) = Ew(x')$, where $w(x') = (w_1(x'), \cdots, w_N(x'))$.

Set

$$\hat{H}_s(\boldsymbol{R}^{n-1}) \text{ (resp. } \hat{H}^\pm_s(\boldsymbol{R}^{n-1})) = \{\hat{\phi}\,;\, \phi \in H_s(\boldsymbol{R}^{n-1}) \text{ (resp. } \phi \in H^\pm_s(\boldsymbol{R}^{n-1}))\}$$

and

$$\mathscr{W} = \{W = (\hat{w}_1, \cdots, \hat{w}_N) \in \hat{H}_{1/2}(\boldsymbol{R}^{n-1})^N\,;\, w_j \in H^{\mp}_{1/2}(\boldsymbol{R}^{n-1}) \qquad \text{for } j \in S_\pm\}\,.$$

Then (2.17) may be rewritten by

$$(3.2)\qquad \int \langle E^*\mathscr{B}(\eta)EW(\eta), W(\eta)\rangle d\eta \geqq 0\,, \qquad W \in \mathscr{W}\,,$$

where $\mathscr{B}(\eta) = B(\eta) - c|\eta|I$ and E^* is the adjoint matrix of E. If we set

$$\Psi = \{\psi = (\psi_1, \cdots, \psi_N) \in \hat{H}_0(\boldsymbol{R}^{n-1})^N ; \psi_j \in \hat{H}_0^{\mp}(\boldsymbol{R}^{n-1}) \text{ for } j \in S_\pm\}$$

and, for $\varepsilon \geqq 0$,

$$\rho_j^{(\varepsilon)}(\eta) = \begin{cases} (\eta_\pm \pm i\varepsilon)^{-1/2} , & j \in S_\pm - (S_+ \cap S_-) , \\ (\varepsilon + |\eta|)^{-1/2} , & j \notin S_+ \cup S_- \text{ or } j \in S_+ \cap S_- . \end{cases}$$

it then follows from the theorem of the Hilbert transformation that for any $\psi \in \Psi$ and any $\varepsilon > 0$

$$(\rho_1^{(\varepsilon)}(\eta)\psi_1(\eta), \cdots, \rho_N^{(\varepsilon)}(\eta)\psi_N(\eta)) \in \mathscr{W} .$$

Substituting this in (3.2) and letting $\varepsilon \to 0$, we obtain

$$\int \langle \mathscr{B}_0(\eta)\psi(\eta), \psi(\eta)\rangle d\eta \geqq 0 , \qquad \psi \in \Psi , \tag{3.3}$$

where $\mathscr{B}_0(\eta)$ is the matrix defined by

$$\mathscr{B}_0(\eta) = P(\eta)^* E^* \mathscr{B}(\eta) E P(\eta) ,$$

with

$$P(\eta) = [\rho_j(\eta)\delta_{jk}]_{j,k=1}^N (\rho_j(\eta) = \rho_j^{(0)}(\eta)) .$$

Note that every entry of $\mathscr{B}_0(\eta)$ is homogeneous of degree zero in η. Thus we can assert that (3.2) implies (3.3). Conversely (3.3) implies (3.2), for if $W \in \mathscr{W}$ then we have

$$(\rho_1(\eta)^{-1}W_1(\eta), \cdots, \rho_N(\eta)^{-1}W_N(\eta)) \in \Psi .$$

Now it remains to prove the equivalence of (3.1) and (3.3). But this can be really done by the same argument as in [4, pp. 131–133]. Q.E.D.

Set, for $F(t) \in H_1(\boldsymbol{R}_+)^N$,

$$\begin{aligned} Q_\eta[F, F] = \int_0^\infty \Big\{ &\langle a_{nn} D_t F, D_t F\rangle + \sum_{i=1}^{n-1} \langle a_{in}\eta_i F, D_t F\rangle \\ &+ \sum_{i=1}^{n-1} \langle a_{ni} D_t F, \eta_i F\rangle + \sum_{i,j=1}^{n-1} \langle a_{ij}\eta_i\eta_j F, F\rangle \Big\} dt . \end{aligned} \tag{3.4}$$

Then we have

COROLLARY 1. *The form Q_0 is coercive over $V_0(S_+, S_-)$ if and only if the following is satisfied for every $\eta \in \boldsymbol{R}^{n-1}$, $\eta \neq 0$: Any function $F_+(t)$*

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(resp. $F_-(s)) \in H_1(\boldsymbol{R}_+)$ *satisfying*

$$\begin{cases} A(\eta, D_t)F(t) = 0 & \textit{in } t > 0 \\ \langle \boldsymbol{b}_j, F(t)\rangle = 0 & \textit{on } t = 0,\ j \in S_+ \ (\textit{resp. } S_-) \end{cases} \tag{3.5}$$

vanishes in $t > 0$ *if*

$$Q_\eta[F_\pm, F_\pm] \leqq 0\ . \tag{3.6}$$

Proof. Let $F_\pm(t) \in H_1(\boldsymbol{R}_+)$ and assume (3.5) and (3.6). Integrating by parts we find by (3.4)

$$Q_\eta[F_\pm, F_\pm] = \langle N(\eta, D_t)F_\pm(t), F_\pm(t)\rangle_{t=0}\ .$$

Now suppose (3.1) to be valid. By setting $a^\pm = F^\pm(0)$, it then follows that

$$(2\pi)^{n-1}Q_\eta[F_\pm, F_\pm] = \langle B(\eta)a^\pm, a^\pm\rangle \geqq c|\eta|\langle a^\pm, a^\pm\rangle\ ,$$

since $a^\pm \in L(S_\pm)$. Here, the signs $+, -$ are taken in the same order. Accordingly we have $a^\pm = 0$ by (3.6). Hence $F_\pm(t) = 0$ for $t > 0$.

Conversely, we suppose that (3.1) is not valid. Then, we can assume without loss of generality that there exist $\eta_0 \in \boldsymbol{R}^{n-1}, |\eta_0| = 1$, and $a^{(0)} \in L(S_+)$ such that $|a^0| = 1$ and

$$\langle B(\eta_0)a^{(0)}, a^{(0)}\rangle \leqq 0 \tag{3.7}$$

Let $F(t)$ be in $H_1(\boldsymbol{R}_+)^N$ and satisfy

$$\begin{cases} A(\eta_0, D_t)F = 0 & \text{in } t > 0\ , \\ F(0) = a^{(0)} & \text{on } t = 0\ . \end{cases}$$

It then follows from (3.7) that $Q_{\eta_0}[F, F] \leqq 0$. But it is easily seen that $F(t)$ does not vanish identically. Q.E.D.

Before ending this section, we shall state another corollary which will be used in the proof of the main theorem (Theorem 3 in § 4).

COROLLARY 2. *The form* Q_0 *is coercive over* $V_0(S_+, S_-)$ *if it is coercive over* $V_0(S_+, S_-) \cap \mathscr{E}'_R$ *for some positive number* R.

Proof. Suppose that Q_0 is coercive over $V_0(S_+, S_-) \cap \mathscr{E}'_R$ for a positive number R. Then we would like to prove that Q_0 is coercive over $V_0(S_+, S_-)$. We assume for the moment that Q_0 is not coercive over $V_0(S_+, S_-)$. Using Corollary 1, we can assume without loss of generality

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that there exists a $\eta_0 \in \boldsymbol{R}^{n-1}$, $\eta_0 \neq 0$ and a function $F_+(t)$ which fulfils (3.5) and (3.6) for $\eta = \eta_0$ but does not identically vanish in $t > 0$.

For $\mu > 0$, we put

$$u_\mu(x', t) = e^{i\langle x', \mu\eta_0\rangle} F_+(\mu t) .$$

Let $\zeta(x) \in C_0^\infty(\Sigma_R)$ and further assume that $\zeta(x)$ is written in the form $\zeta_1(t)\zeta_2(x')$ where $\zeta_1(t) = 1$ in a neighbourhood of $t = 0$ and $\zeta_2(x') = 0$ in $x_{n-1} < 0$. Then we have $\zeta(x)u_\mu(x', t) \in V_0(S_+, S_-) \cap \mathscr{E}'_R$. Accordingly

$$Q_0[\zeta u_\mu, \zeta u_\mu] \geqq c_1 \|\zeta u_\mu\|_1^2 - c_0 \|\zeta u_\mu\|_0^2 \tag{3.8}$$

holds for all $\mu > 0$. But we can immediately show that $Q_0[\zeta u_\mu, \zeta u_\mu]$ and $\|\zeta u_\mu\|_0^2$ are bounded when $\mu \to \infty$ but $\|\zeta u_\mu\|_1^2$ tends to infinity as $\mu \to \infty$. This contradicts (3.8). Thus we can assert that Q_0 is coercive over $V_0(S_+, S_-)$.

§ 4. Main theorem

Now we can state the main theorem. Let $Q[u, u]$ be a quadratic form of the form (1.1) satisfying (1.2) and (1.3). If $u \in H_2(\Omega)^N$ and $v \in H_1(\Omega)^N$, integrating by parts we find

$$Q[u, v] = \int_\Omega \langle A(x, D)u, v\rangle dx + \int_\Gamma \langle N(x, D)u, v\rangle d\sigma , \tag{4.1}$$

where $d\sigma$ denotes the Lebesgue measure on Γ and

$$A(x, D)u = \sum_{i,j=1}^n D_j(a_{ij}(x)D_i u) , \qquad x \in \Omega ,$$

$$N(x, D)u = -i \sum_{i,j=1}^n \nu_j(x)a_{ij}(x)D_i u , \qquad x \in \Gamma ,$$

$\nu(x) = (\nu_1(x), \cdots, \nu_n(x))$ being the unit inner normal vector to Γ at $x \in \Gamma$.

Let P be arbitrarily fixed on Γ. We denote by T_P the totality of real vectors $\dot{\xi} \neq 0$ parallel to Γ at P. Let $\dot{\xi} \in T_P$. For any $\phi \in \mathscr{S}(\boldsymbol{R}^{n-1})$, we denote by $u_\phi(\dot{\xi}, t)$ an exponentially decaying solution of the Dirichlet problem

$$\begin{cases} A^0(P, \dot{\xi} + \nu D_t)u = 0 & \text{in } t > 0 , \\ u = \hat{\phi}(\dot{\xi}) & \text{on } t = 0 , \end{cases}$$

where $\nu = \nu(P)$ and A^0 is the leading part of A. Following the same process as in (2.10), we can define the N-square matrix $B(P, \dot{\xi})$ such that

$$N(P, \dot{\xi} + \nu D_t) u_\phi(\dot{\xi}, t)|_{t=0} = B(P, \dot{\xi}) \hat{\phi}(\dot{\xi}) . \tag{4.2}$$

Let $V(S_1, S_2)$ be the closed subspace of $H_1(\Omega)^N$ which is defined by (1.9). For a subset S of the set $\{1, \cdots, N\}$, we introduce a subspace of $\boldsymbol{C}^N$ as follows:

$$L^P(S) = \{a \in \boldsymbol{C}^N ; \langle \boldsymbol{b}_j(P), a \rangle = 0 \text{ for } j \in S\} .$$

Then we can obtain

THEOREM 3. *Let $Q[u, u]$ be an integro-differential quadratic form of the form* (1.1) *which satisfies* (1.2) *and* (1.3), *and whose coefficients $a_{ij}(x), i, j = 1, \cdots, n$, are N-square matrices with entries in $C^\infty(\bar{\Omega})$. Let $\boldsymbol{b}_j(x), j = 1, \cdots, N$, be the given N-vectors with components in $C^\infty(\Gamma)$ which are linearly independent at each $x \in \Gamma$. By $V(S_1, S_2)$ we denote the closed subspace of $H_1(\Omega)^N$ which is defined by* (1.9). *Then, in order that the $Q[u, u]$ be coercive over $V(S_1, S_2)$, it is necessary and sufficient that*

$$\begin{cases} \text{for every } P \text{ on } \bar{\Gamma}_\alpha (\alpha = 1, 2), \text{ there exists a constant } c > 0 \text{ such} \\ \text{that the inequality} \\ \qquad \langle B(P, \dot{\xi}) a, a \rangle \geqq c |\dot{\xi}| \langle a, a \rangle , \qquad a \in L^P(S_\alpha) \\ \text{holds for every } \dot{\xi} \in T_P, B(P, \dot{\xi}) \text{ being defined by (4.2).} \end{cases} \tag{4.3}$$

Proof. The proof easily follows from Proposition 1.3, Theorem 2 and its Corollary 2, where we should note that

$$V_R^P = \begin{cases} V_0(S_\alpha, S_\alpha) \cap \mathscr{E}'_R \text{ with } \boldsymbol{b}_j = \boldsymbol{b}_j(P) , & \text{if } P \in \Gamma_\alpha \\ V_0(S_1, S_2) \cap \mathscr{E}'_R \text{ with } \boldsymbol{b}_j = \boldsymbol{b}_j(P) , & \text{if } P \in \gamma = \bar{\Gamma}_1 \cap \bar{\Gamma}_2 . \end{cases}$$

Q.E.D.

Corresponding to Corollary 1 of Theorem 2, we can replace (4.3) with

$$\begin{cases} \text{for every } P \text{ on } \bar{\Gamma}_\alpha \ (\alpha = 1, 2) \text{ and every } \dot{\xi} \in T_P, \text{ any function } F(t) \\ \in H_1(\boldsymbol{R}_+) \text{ satisfying} \\ \qquad \begin{cases} A^0(P, \dot{\xi} + \nu(P) D_t) F(t) = 0 & \text{in } t > 0 , \\ \langle \boldsymbol{b}_j(P), F(t) \rangle = 0 \text{ on } t = 0 & \text{for } j \in S_\alpha \end{cases} \\ \text{vanishes in } t > 0 \text{ if} \end{cases} \tag{4.4}$$

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$$Q_{\dot\xi}^P[F,F] = \int_0^\infty \left\langle \sum_{i,j=1}^n a_{ij}(P)(\dot\xi_i + \nu_i D_t)F, (\dot\xi_j + \nu_j D_t)F \right\rangle dt \leqq 0 .$$

§5. Formally positive quadratic forms and Korn's inequality

Let $\Omega, \Gamma, \Gamma_\alpha, S_\alpha$ $(\alpha = 1, 2)$ and $V(S_1, S_2)$ be the same as in Section 1 and let $A_j(x)$, $j = 1, \cdots, n$, be $M \times N$-matrices $(M \geqq N)$ with entries in $C^\infty(\bar\Omega)$. For $u = (u_1(x), \cdots, u_N(x))$, we write as

$$L(x, D)u = \sum_{j=1}^n A_j(x) D_j u \tag{5.1}$$

and introduce a formally positive quadratic form

$$Q[u, u] = \int_\Omega \langle L(x, D)u, L(x, D)u \rangle dx , \tag{5.2}$$

which is nothing but the form (1.1) with $a_{ij}(x) = A_j(x)^* A_i(x)$. The condition (1.3) leads to

$$\operatorname{rank} L(x, \xi) = N \tag{5.3}$$

for every $x \in \bar\Omega$ and every $\xi \in \boldsymbol{R}^n$, $\xi \neq 0$. On the other hand, it is obvious that the condition (1.2) is automatically satisfied. Then we shall study the coerciveness of the form (5.2) over $V(S_1, S_2)$. To this purpose, we are going to characterize (4.4), where A^0 and Q_ξ^P are given by

$$\begin{cases} A^0(P, \dot\xi + \nu D_t) = L(P, \dot\xi + \nu D_t)^* L(P, \dot\xi + \nu D_t) , \\ Q_{\dot\xi}^P[F, F] = \int_0^\infty \langle L(P, \dot\xi + \nu D_t)F, L(P, \dot\xi + \nu D_t)F \rangle dt \end{cases} \tag{5.4}$$

with $\nu = \nu(P)$ and

$$L(x, D)^* u = \sum_{j=1}^n D_j (A_j(x)^* u) .$$

That is, we would like to get the algebraic conditions under which any function $F(t) \in H_1(\boldsymbol{R}_+)$ satisfying, for any $P \in \bar\Gamma_\alpha$ $(\alpha = 1, 2)$ and any $\dot\xi \in T_P$,

$$\begin{cases} L(P, \dot\xi + \nu D_t)F(t) = 0 & \text{in } t > 0 \\ \langle \boldsymbol{b}_j(P), F(t) \rangle = 0 & \text{on } t = 0 , \quad \text{for } j \in S_\alpha \end{cases} \tag{5.5}$$

identically vanishes in $t > 0$.

Let P be fixed in $\bar\Gamma_\alpha$ $(\alpha = 1, 2)$ and let $\dot\xi$ be fixed in T_P. Since we have

$$L(P, \dot\xi + \nu D_t) = L(P, \nu) D_t + L(P, \dot\xi)$$

and rank $L(P, \nu) = N$ by (5.3), it follows that the space $\mathcal{N}(P, \dot{\xi})$ of vector-valued functions which satisfy the equation

$$L(P, \dot{\xi} + \nu D_t)F(t) = 0 \qquad \text{in } t > 0$$

and belong to $H_1(\boldsymbol{R}_+)^N$ is isomorphic as a vector space to the space of initial values:

$$\mathcal{N}_0(P, \dot{\xi}) = \{F(0) \in \boldsymbol{C}^N ; F(t) \in \mathcal{N}(P, \dot{\xi})\} .$$

Let $\mathcal{N}_0^{\perp}(P, \dot{\xi})$ be the orthogonal complement of the space $\mathcal{N}_0(P, \dot{\xi})$ with respect to $\boldsymbol{C}^N$ and put

$$\boldsymbol{r}(P, \dot{\xi}) = \dim \mathcal{N}_0(P, \dot{\xi}) .$$

Then we can show

THEOREM 4. *The formally positive quadratic form* $Q[u, u]$ *of the form* (5.2) *satisfying* (5.3) *is coercive over* $V(S_1, S_2)$ *if and only if*

$$\text{(5.6)} \quad \begin{cases} \textit{for every } P \textit{ on } \bar{\Gamma}_\alpha \, (\alpha = 1, 2) \textit{ and every } \dot{\xi} \in T_P, \textit{ there exist } \mathrm{r}\,(P, \dot{\xi}) \\ \textit{vectors among } \{\boldsymbol{b}_j(P) ; j \in S_\alpha\} \textit{ which are linearly independent} \\ \bmod \mathcal{N}_0^{\perp}(P, \dot{\xi}) . \end{cases}$$

Proof. We have only to show that any function $F \in H_1(\boldsymbol{R}_+)^N$ satisfying (5.5) identically vanishes in $t > 0$ if and only if (5.6) is valid, that is, to show that every $a \in \mathcal{N}_0(P, \dot{\xi})$ satisfying $\langle \boldsymbol{b}_j(P), a \rangle = 0$ for all $j \in S_\alpha$ vanishes if and only if (5.6) is valid. But this fact is immediately established. Q.E.D.

Noting that $V(S_1, S_2) = H_1(\Omega)^N$ if $S_1 = S_2 = \phi$, we can prove

COROLLARY. *The quadratic form* $Q[u, u]$ *in Theorem* 4 *is coercive over* $H_1(\Omega)^N$ *if and only if*

$$\text{(5.7)} \quad \begin{cases} \textit{for every } P \in \Gamma \textit{ and every } \dot{\xi} \in T_P, \\ \qquad\qquad \operatorname{rank} L(P, \dot{\xi} + \tau\nu(P)) = N \\ \textit{is valid for every } \tau \textit{ such that } \operatorname{Im} \tau > 0. \end{cases}$$

Proof. It easily follows from (4.4) that the quadratic form $Q[u, u]$ is coercive over $H_1(\Omega)^N$ if and only if $\mathcal{N}(P, \dot{\xi}) = \{0\}$ or $\boldsymbol{r}(P, \dot{\xi}) = 0$ for every $P \in \Gamma$ and every $\dot{\xi} \in T_P$.

Suppose that (5.7) does not hold for some $P \in \Gamma$, $\dot{\xi} \in T_P$ and τ such that $\operatorname{Im} \tau > 0$. Then we can find a vector $a \neq 0$ in $\boldsymbol{C}^N$ such that $L(P, \dot{\xi} + \tau\nu) = 0$. Clearly we have $e^{i\tau t}a \in \mathscr{N}(P, \dot{\xi})$, and hence $\mathscr{N}(P, \dot{\xi}) \neq \{0\}$.

Now we assume that (5.7) holds. It is well-known that any solution $F(t)$ in $H_1(\boldsymbol{R}_+)$ of the equation $L(P, \dot{\xi} + \nu D_t)F(t) = 0$ can be written as a sum of

$$\phi_{jk}(t) = \frac{1}{k!}(it)^k e^{it\tau_j} a_{jk} \qquad (j = 1, \cdots, r;\, k = 0, \cdots, s)$$

where τ_j are the distinct complex numbers such that $\operatorname{Im} \tau_j > 0$ and a_{jk} are some N-vectors. Since

$$L(D_t)\phi_{jk}(t) = \left\{\frac{(it)^k}{k!} L(\tau_j)a_{jk} + \frac{(it)^{k-1}}{(k-1)!} L'(\tau_j)a_{jk}\right\} e^{it\tau_j}$$

and $L(D_t)F(t) = 0$, we have, for $j = 1, \cdots, r$,

$$\begin{cases} L(\tau_j)a_{js} = 0 \\ L(\tau_j)a_{js-1} + L'(\tau_j)a_{js} = 0 \\ \quad \vdots \\ L(\tau_j)a_{j0} + L'(\tau_j)a_{j1} = 0 \,, \end{cases}$$

where we put, for brevity, $L(D_t) = L(P, \dot{\xi} + \nu D_t)$, $L(\tau) = L(P, \dot{\xi} + \tau\nu)$, and $L'(\tau) = dL(\tau)/d\tau$. It then follows from (5.7) that $a_{js} = a_{js-1} = \cdots = a_{j0} = 0$ for all j. Accordingly we have $F(t) = 0$ in $t > 0$. Q.E.D.

By applying this corollary, we can finally give the simple proof of what is called the second Korn inequality:

$$\int_\Omega \sum_{i,j=1}^n |D_j u_i + D_i u_j|^2 \, dx \geqq c\,\|u\|_1^2 - \int_\Omega |u|^2 \, dx \tag{5.8}$$

for any $u = (u_1, \cdots, u_n) \in H_1(\Omega)^n$ with a constant $c > 0$. The proof is anything but trivial. In fact, various proofs have been published by Friedrichs [3] and others (see [2] and Bibliography of Lecture 12 of [1]).

For n functions $u = (u_1, \cdots, u_n)$ defined in Ω, we define a system of n^2 differential equations:

$$L_{ij}(D)u = D_j u_i + D_i u_j \,, \qquad i, j = 1, \cdot\cdot, , n$$

and

$$L(D)u = {}^t(L_{11}(D)u, \cdots, L_{1n}(D)u, \cdots, L_{n1}(D)u, \cdots, L_{nn}(D)u) \,,$$

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which is one of operators of the type (5.1) with $(n^2 \times n)$-matrices $A_j(x)$. Then it is obvious that the quadratic form (5.2) with $L(x, D) = L(D)$ becomes the left hand side of (5.8). Therefore, the inequality (5.8) means that the quadratic form (5.2) with $L(x, D) = L(D)$ is coercive over $H_1(\Omega)^n$, Consequently it is sufficient to verify (5.3) and (5.7), in order to show that the inequality (5.8) is valid for all $u \in H_1(\Omega)^n$.

To do so, let $a \in \boldsymbol{C}^n$ and $\zeta \in \boldsymbol{C}^n, \zeta \neq 0$, and assume that $L(\zeta)a = 0$, i.e.,

$$L_{ij}(\zeta)a = \zeta_j a_i + \zeta_i a_j = 0$$

for all $i, j = 1, \cdots, n$. For i such that $\zeta_i \neq 0$, we have $a_i = 0$, for $L_{ii}(\zeta)a = 2\zeta_i a_i = 0$. Now let $\zeta_i = 0$, for some i. There then exists an integer $j\ (\neq i)$ such that $\zeta_j \neq 0$. For such i, j, we have $L_{ij}(\zeta)a = \zeta_j a_i = 0$, which implies $a_i = 0$. Thus we get $a = 0$. This means $\operatorname{rank} L(\zeta) = n$ for all $\zeta \in \boldsymbol{C}^n, \zeta \neq 0$. In particular, we have $\operatorname{rank} L(\xi) = n$ for $\xi \in \boldsymbol{R}^n, \xi \neq 0$.

Let P be an arbitrary point on Γ. For every $\dot{\xi} \in T_P$ and every τ such that $\operatorname{Im} \tau > 0$, it is easily proved that $\dot{\xi} + \tau\nu \in \boldsymbol{C}^n\ (\nu = \nu(P))$ and $\dot{\xi} + \tau\nu \neq 0$. Accordingly $\operatorname{rank} L(\dot{\xi} + \tau\nu) = n$. Hence it follows from Corollary of Theorem 4 that Korn's inequality (5.8) is valid for all $u \in H_1(\Omega)^n$.

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