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CHARACTERIZATIONS OF LOCALLY FINITE ACTIONS OF GROUPS ON SETS

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Abstract. We show that an action of a group on a set X is locally finite if and only if X is not equidecomposable with a proper subset of itself. As a consequence, a group is locally finite if and only if its uniform Roe algebra is finite.

1. Introduction. Given a group acting on a set X, a property that has been well-studied is the existence of an invariant mean on X, that is, amenability of the action (see [**1**] for historical remarks). By Tarski's Theorem [**6**, Corollary 9.2], this is equivalent to X not being equidecomposable with two disjoint subsets of itself.

In this note, we address the following question: Given an action of a group G on a set X, when is X not equidecomposable with a proper subset of itself? We show that this holds if and only if the action is locally finite (Definition 2.2), if and only if $\ell^\infty(X) \rtimes_r G$ is a finite C*-algebra (Theorem 2.3). It follows from this that a group is locally finite if and only if its uniform Roe algebra ($\ell^\infty(G) \rtimes_{\mathrm{r}} G$) is finite (Proposition 2.5). In [**3**], it was shown that $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ is finite if G is locally finite and asked if the converse holds.

It was already known that amenability of a group G is equivalent to $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ not being properly infinite, and supramenability is equivalent to $\ell^\infty(G) \rtimes_{\mathrm{r}} G$ not containing any properly infinite projections [**3**, Proposition 5.3]. Therefore, Proposition 2.5 completes the dictionary between equidecomposition properties of groups and the type of projections in the uniform Roe algebra.

2. Characterizations of locally finite actions of groups on sets. We start by recalling some definitions:

Definition 2.1. Let G be a group acting on a set X. Two subsets A and B of X are said to be *equidecomposable* if there are finite partitions $\{A_i\}_{i=i}^n$ and $\{B_i\}_{i=i}^n$ of A and B, respectively, and elements $s_1, \ldots, s_n \in G$ such that $B_i = s_i A_i$ for $1 \leq i \leq n$. When we say that two subsets of G are equidecomposable, it is with respect to the left action of G on itself.

The next definition has already been introduced in [**5**] for actions on semi-lattices.

Definition 2.2. An action of a group G on a set X is said to be *locally finite* if, for every finitely generated subgroup H of G and every $x \in X$, the H-orbit of x is finite.

The left action of a group on itself is locally finite if and only if the group is locally finite.

The following result shows that the notion of locally finite action is a natural strengthening of the notion of amenable action on a set.

THEOREM 2.3. *Let G be a group acting on a set X. The following conditions are equivalent:*

(1) *The action is locally finite.*
(2) *$\ell^\infty(X) \rtimes_r G$ is finite.*
(3) *X is not equidecomposable with a proper subset of itself.*
(4) *No subset of X is equidecomposable with a proper subset of itself.*

Proof. (1) $\Rightarrow$ (2). Since the inductive limit of finite unital C*-algebras with unital connecting maps is finite, it suffices to show that $\ell^\infty(X) \rtimes_\mathrm{r} H$ is finite for every finitely generated subgroup H of G. Let H be such a subgroup and $X = \sqcup_{i \in I} X_i$ be the partition of X into its H-orbits.

For every $i \in I$, the restriction map $\ell^\infty(X) \to \ell^\infty(X_i)$ is H-equivariant. Therefore, there is a homomorphism $\psi \colon \ell^\infty(X) \rtimes_\mathrm{r} H \to \prod(\ell^\infty(X_i) \rtimes_\mathrm{r} H)$. We claim that ψ is injective.

Let $\varphi \colon \ell^\infty(X) \rtimes_\mathrm{r} H \to \ell^\infty(X)$ and, for every $i \in I$, $\varphi_i \colon \ell^\infty(X_i) \rtimes_\mathrm{r} H \to \ell^\infty(X_i)$ be the canonical conditional expectations. Also, let $\varphi_I \colon \prod(\ell^\infty(X_i) \rtimes_\mathrm{r} H) \to \prod \ell^\infty(X_i)$ be the product of the maps φ_i, and $T \colon \ell^\infty(X) \to \prod \ell^\infty(X_i)$ be the isomorphism which arises from the product of the restriction maps. The following diagram commutes:

$$\begin{tikzcd} \ell^\infty(X) \rtimes_\mathrm{r} H \arrow[r, "\psi"] \arrow[d, "\varphi"'] & \prod(\ell^\infty(X_i) \rtimes_\mathrm{r} H) \arrow[d, "\varphi_I"] \\ \ell^\infty(X) \arrow[r, "T"'] & \prod \ell^\infty(X_i). \end{tikzcd}$$

Since φ is faithful, we conclude that ψ is injective. Since the product of finite unital C*-algebras is finite, it suffices to show that $\ell^\infty(X_i) \rtimes_\mathrm{r} H$ is finite for every $i \in I$ in order to conclude that $\ell^\infty(X) \rtimes_\mathrm{r} H$ is finite.

Given $i \in I$, let τ_i be the tracial state on $\ell^\infty(X_i)$ which arises from the uniform probability measure on the finite set X_i. Since τ_i is H-invariant and faithful, it follows that the map $\tau_i \circ \varphi_i \colon \ell^\infty(X_i) \rtimes_\mathrm{r} H \to \mathbb{C}$ is a faithful tracial state. Therefore, $\ell^\infty(X_i) \rtimes_\mathrm{r} H$ is finite.

(2) $\Rightarrow$ (3). This follows from the fact that, if A and B are equidecomposable subsets of X, then the projections 1_A and 1_B are equivalent in $\ell^\infty(X) \rtimes_\mathrm{r} G$.

(3) $\Rightarrow$ (4). If $A \subset X$ is equidecomposable with $B \subsetneq A$, then $X = A \sqcup A^c$ is equidecomposable with $B \sqcup A^c \subsetneq X$.

(4) $\Rightarrow$ (1). Suppose that there is $H < G$ generated by a finite and symmetric set S and $x \in X$ such that Hx is infinite. Then there exists a sequence $(s_n)_{n \in \mathbb{N}} \subset S$ such that

$$\forall n, m \in \mathbb{N} \colon n \neq m \Rightarrow s_n \cdots s_1 x \neq s_m \cdots s_1 x.$$

The sequence $(s_n \cdots s_1 x)_{n \in \mathbb{N}}$ can be seen as an infinite simple path in the graph whose vertex set is Hx and whose edges come from S.

We claim that $\gamma := \{s_n \cdots s_1 x \colon n \in \mathbb{N}\}$ is equidecomposable with $\gamma \setminus \{s_1 x\}$.

Given $s \in S$, let $A_s := \{s_n \cdots s_1 x \colon s_{n+1} = s\}$. It is easy to check that $\{A_s\}_{s \in S}$ partitions γ and $\{sA_s\}_{s \in S}$ partitions $\gamma \setminus \{s_1 x\}$. Hence, γ is equidecomposable with its proper subset $\gamma \setminus \{s_1 x\}$. $\square$

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We now proceed to give a characterization of locally finite groups which can be seen as an analogy to parts of [**3**, Theorem 1.1].

The next definition is from [**4**].

DEFINITION 2.4. Let H and G be groups. A map $f: H \to G$ is said to be a *uniform embedding* if, for every finite set $S \subset H$, there is a finite set $T \subset G$ such that

$$\forall x, y \in H: xy^{-1} \in S \implies f(x)f(y)^{-1} \in T,$$

and, for every finite set $T \subset G$, there is $S \subset H$ finite such that

$$\forall x, y \in H: f(x)f(y)^{-1} \in T \implies xy^{-1} \in S.$$

The implication (1) $\Rightarrow$ (2) in the next result had already been observed in [**3**, Remark 5.4], and (5) $\Rightarrow$ (1) is an immediate consequence of [**8**, Lemma 1].

PROPOSITION 2.5. *Let G be a group. The following conditions are equivalent:*

(1) *G is locally finite.*
(2) *The uniform Roe algebra $\ell^\infty(G) \rtimes_r G$ is finite.*
(3) *G is not equidecomposable with a proper subset of itself.*
(4) *No subset $A \subset G$ is equidecomposable with a proper subset of itself.*
(5) *There is no injective uniform embedding from $\mathbb{Z}$ into G.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) (and (4) $\Rightarrow$ (1)) are a consequence of Theorem 2.3.

(4) $\Rightarrow$ (5). This follows from the fact that $\mathbb{N} \subset \mathbb{Z}$ is equidecomposable with a proper subset of itself and [**3**, Lemma 3.2].

(5) $\Rightarrow$ (1). This is a consequence of [**8**, Lemma 1]. □

REMARK 2.6. After this note was made available on arXiv, we became aware of [**7**], where it is shown that if a group is infinite and finitely generated, then its uniform Roe algebra is infinite.

Any locally finite group acts on itself in a transitive, faithful and locally finite way. If a finitely generated group admits a faithful, transitive, locally finite action on a set, then the group is finite. This is in stark contrast to the fact that there are finitely generated, non-amenable groups which admit faithful, transitive, amenable actions on sets (see [**1**] for various examples).

A finitely generated group admits a faithful, locally finite action if and only if it is residually finite.

PROPOSITION 2.7. *If a group admits a faithful, locally finite action, then it embeds into a group which admits a faithful, locally finite and transitive action.*

Proof. Let G be a group which acts on a set X in a faithful and locally finite way.

Take a set $Y \subset X$ of representatives of all G-orbits, and let S_Y be the group of finitely supported permutations of Y. Consider the natural action of S_Y on X and the associated action of $H := G * S_Y$ on X. This action is transitive and locally finite. By taking the quotient of H by the kernel of this action, we get a faithful, transitive, locally finite action on X by a group which contains G. □

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In analogy to what is known for amenable actions [**2**, Lemma 3.2], the following holds for locally finite actions:

PROPOSITION 2.8. *Let G be a group acting on a set X in a locally finite way. If, for every $x \in X$, the stabilizer subgroup G_x is locally finite, then G is locally finite.*

Proof. Take $H < G$ finitely generated and $x \in X$. Since the action is locally finite, it follows that Hx is finite. Hence, there is H_0 a subgroup of finite index in H such that $H_0 < G_x$. In particular, H_0 is locally finite. Therefore, H is also locally finite. Since H is finitely generated, we conclude that it is finite. □

REMARK 2.9. One can define in a natural way an action of a group on a set X to be supramenable if no subset of X is equidecomposable with two disjoint proper subsets of itself. It is not true that if the action of a group G is supramenable, and all the stabilizer subgroups are supramenable, then G is supramenable.

Indeed, it is well-known that the class of supramenable groups is not closed by taking extensions (the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ is such an example). Let then G be a non-supramenable group which contains a supramenable normal subgroup N such that $\frac{G}{N}$ is also supramenable.

Consider the left action of G on $\frac{G}{N}$. Since $\frac{G}{N}$ is supramenable, it follows easily that this action is supramenable. The stabilizer subgroups of the action are all equal to N, hence are supramenable.

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