A. V. Jayanthan and J. K. Verma
Nagoya Math. J.
Vol. 177 (2005), 155–179

FIBER CONES OF IDEALS WITH ALMOST MINIMAL MULTIPLICITY

A. V. JAYANTHAN* AND J. K. VERMA†

Abstract. Fiber cones of 0-dimensional ideals with almost minimal multiplicity in Cohen-Macaulay local rings are studied. Ratliff-Rush closure of filtration of ideals with respect to another ideal is introduced. This is used to find a bound on the reduction number with respect to an ideal. Rossi's bound on reduction number in terms of Hilbert coefficients is obtained as a consequence. Sufficient conditions are provided for the fiber cone of 0-dimensional ideals to have almost maximal depth. Hilbert series of such fiber cones are also computed.

§1. Introduction

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring having infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R and K an ideal containing I. The fiber cone of I with respect to K is the standard graded algebra $F_K(I) = \bigoplus_{n \geq 0} I^n/KI^n$. The graded algebra $F_K(I)$ for $K = \mathfrak{m}$ is called the fiber cone $F(I)$ of I. For $K = I$, $F_K(I) = G(I)$, the associated graded ring of I. The objective of this paper is to study the depth of the ring $F_K(I)$ subject to certain conditions imposed on the coefficients of the Hilbert polynomial $P(I, n)$ corresponding to the Hilbert function $H(I, n) = \lambda(R/I^n)$, where λ denotes the length function. This theme has been studied for the associated graded rings by Elias [8], Goto [10], Huckaba [15], Huckaba-Marley [17], Jayanthan-Singh-Verma [19], Rossi [25], Rossi-Valla [26], [27], Sally [28], [29], Wang [34], [35] and for the fiber cones by Cortadellas-Zarzuela [5], D'Cruz-Raghavan-Verma [6], D'Cruz-Verma [7], Jayanthan-Verma [20], Shah [30].

The form ring $G(I)$ and the fiber cone $F(I)$ have been studied separately. By studying the ring $F_K(I)$ we hope to unify the results obtained for

Received August 27, 2002.
Revised May 14, 2003.
2000 Mathematics Subject Classification: Primary 13H10, 13H15; Secondary 13C15, 13A02.
*Supported by the National Board for Higher Mathematics, India.
†Corresponding author.

$G(I)$ and $F(I)$. An initial motivation for this paper was to find conditions on Hilbert coefficients which will ensure high depth for $F_K(I)$, thereby obtaining the results known for $G(I)$ and $F(I)$. We have not been completely successful in providing a unified approach. However, techniques developed to study the two rings separately can be unified in the hope of obtaining results for $F_K(I)$.

We now point out a few results in the literature which indicate the importance of a systematic study of fiber cones. We begin with the classic paper [22] of Northcott and Rees. Throughout this section, let $(R, \mathfrak{m})$ be a local ring of dimension d and let I be an R-ideal. Let $k = R/\mathfrak{m}$ be infinite. The analytic spread, $l := \ell(I)$, of I is defined to be the dimension of the fiber cone $F(I)$. By Noether normalization lemma, there exist elements $a_1, a_2, \ldots, a_l \in I$ such that their images $b_1, b_2, \ldots, b_l$ in $I/\mathfrak{m}I$ are algebraically independent over k and $F(I)$ is an integral extension of $k[b_1, b_2, \ldots, b_l]$. It follows that there exists an $n \geq 0$ so that $JI^n = I^{n+1}$ where $J = (a_1, a_2, \ldots, a_l)$. The ideal J is called a minimal reduction of I and the smallest n so that $JI^n = I^{n+1}$ is called the reduction number, $r_J(I)$, of I with respect to J. Minimal reductions have played an important role in the study of many problems in commutative algebra, e.g. Hilbert functions, Rees algebras and associated graded rings, to name a few.

The fiber cone $F(I)$ is the fiber over the closed point of the blowup

$$\operatorname{Spec}\left(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}\right) \longrightarrow \operatorname{Spec}(R).$$

Thus it plays an important role in resolutions of singularities of algebraic varieties. Hironaka, in his paper [16] on resolution of singularities, introduced the concept of permissibility of I as a center of blowing-up. Recall that I is called *permissible* in R, if R/I is regular and the associated graded ring $G(I)$ is R/I-flat. Put

$$H(G(\mathfrak{m}), \lambda) = \sum_{n=0}^{\infty} \dim\left(\frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}}\right)\lambda^n \quad \text{and} \quad H(F(I), \lambda) = \sum_{n=0}^{\infty} \dim\left(\frac{I^n}{\mathfrak{m}I^n}\right)\lambda^n.$$

Let e denote the embedding dimension of R/I. In 1976, B. Singh [31], proved that I is a permissible center of blowing-up if and only if the Hilbert series of $G(\mathfrak{m})$ and that of $F(I)$ are related by the equation:

$$H(G(\mathfrak{m}), \lambda)(1-\lambda)^e = H(F(I), \lambda).$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

In the same year 1976, Cowsik and Nori [4] used fiber cones to characterize complete intersections. They showed that in a regular local ring R, a radical ideal I is a complete intersection if and only if $\ell(I) = ht(I)$.

Let R be a commutative ring. An element $x \in R$ is called integrally dependent on an R-ideal I if it satisfies an equation of the form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

where $a_i \in I^i$ for $i = 1, 2, \ldots, n$. Elements of R that are integrally dependent on I form an ideal denoted by $\overline{I}$. We say that I is complete if $I = \overline{I}$.

Recently, the work on fiber cones has been inspired by their usefulness in detecting evolutionary stability of local algebras. Let A be a ring and let T be a local A-algebra essentially of finite type over A, i.e. T is a localization of a finitely generated A-algebra. An *evolution* of T over A is a local A-algebra essentially of finite type over A, and a surjection $R \to T$ of A-algebras inducing an isomorphism $\Omega_{R/A} \otimes_R T \simeq \Omega_{T/A}$. The evolution R of T is called trivial if $R \to T$ is an isomorphism. We say T is *evolutionarily stable* if all its evolutions are trivial. Eisenbud and Mazur remarked in [9] that they have been unable to find any nontrivial evolution of any reduced k-algebra in equicharacteristic zero or of any reduced algebra which is flat over a discrete valuation ring of mixed characteristic. Hübl [12] observed that if I is a complete ideal in a local domain $(R, \mathfrak{m})$ and $F(I)$ is reduced in degree one then $\mathfrak{m}I$ is complete. He also proved that if $(R, \mathfrak{m})$ is a smooth k-algebra where k is a field of characteristic zero with $[R/\mathfrak{m} : k] < \infty$ and I is a reduced, equidimensional ideal with $\mathfrak{m}I$ complete, then R/I is evolutionarily stable.

Therefore it becomes important to know when $\mathfrak{m}I$ is complete. Huneke and Hübl provided sufficient conditions in [13] for completeness of $\mathfrak{m}I$. An ideal is called *normal* if all its powers are complete. Let I be a normal ideal in a normal local domain $(R, \mathfrak{m})$ with $R/\mathfrak{m}$ infinite and $\ell(I) = \dim R$. If $F(I)$ is equidimensional and has no embedded components, e.g. when $F(I)$ is Cohen-Macaulay, then $\mathfrak{m}I^n$ is complete for all $n \in \mathbb{N}$.

Therefore we observe that it becomes important to know when $F(I)$ is Cohen-Macaulay or what exactly is its depth. Once we know that $F(I)$ is Cohen-Macaulay, it becomes easy to determine its Hilbert series, provided we have access to a minimal reduction of J of I. The images of minimal generators of J in $I/\mathfrak{m}I$ form a homogeneous system of parameters in $F(I)$.

Hence they form a regular sequence, whence

$$H(F(I),t) = \frac{\sum_{n=0}^{r_J(I)} \lambda\big(I^n/(JI^{n-1}+\mathfrak{m}I^n)\big)t^n}{(1-t)^{\ell(I)}}.$$

We now describe the contents of the paper. For a Cohen-Macaulay local ring $(R,\mathfrak{m})$ the 'Abhyankar-Sally' equality gives that

$$\mu(\mathfrak{m}) + \lambda(\mathfrak{m}^2/(y_1,\dots,y_d)\mathfrak{m}) = e_0(\mathfrak{m}) + d - 1,$$

where $(y_1,\dots,y_d)$ is a minimal reduction of $\mathfrak{m}$. J. D. Sally studied the associated graded rings of the maximal ideal with $\mu(\mathfrak{m}) = e_0(\mathfrak{m}) + d - 1$ and showed that $G(\mathfrak{m})$ is Cohen-Macaulay in such cases. She conjectured that if $\mu(\mathfrak{m}) = e_0(\mathfrak{m}) + d - 2$ then $\operatorname{depth} G(\mathfrak{m}) \geq d-1$. This conjecture was settled in affirmative by M. E. Rossi and G. Valla [26] and independently by H.-S. Wang, [34]. Later M. E. Rossi generalized this to the case of $\mathfrak{m}$-primary ideals to prove that if $\lambda(I/I^2) = e_0(I) - (1-d)\lambda(R/I) - 1$, then $\operatorname{depth} G(I) \geq d-1$. She further generalized this to prove that if there exists an integer k such that $I^n \cap J = JI^{n-1}$ for all $n = 1,\dots,k$ and $\lambda(I^{k+1}/JI^k) \leq 1$, then $\operatorname{depth} G(I) \geq d-1$.

The Abhyankar-Sally equality was generalized by J. Chuai to $\mathfrak{m}$-primary ideals I. It was shown in [3] that for an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring R $\mu(I) + \lambda(\mathfrak{m}I/\mathfrak{m}J) = e_0(I) - \lambda(R/I) + d$. Hence $\mu(I) \leq e_0(I) - \lambda(R/I) + d$ and the equality occurs if and only if $\mathfrak{m}I = \mathfrak{m}J$. S. Goto defined an ideal to have *minimal multiplicity* if $\mathfrak{m}I = \mathfrak{m}J$. He characterized various properties of the associated graded rings, the Rees algebras and fiber cones of such ideals.

In Section 4 we prove the main result of the paper, Theorem 4.4. Let $\gamma(I)$ denote $\operatorname{depth} G(I)$. We prove that if I is an $\mathfrak{m}$-primary ideal such that there exists an integer $k > 0$ such that $\mathfrak{m}I^n \cap J = \mathfrak{m}JI^{n-1}$ for all $n = 1,\dots,k$ and $\lambda(\mathfrak{m}I^{k+1}/\mathfrak{m}JI^k) \leq 1$ for any minimal reduction J of I and $\gamma(I) \geq d-2$, then $F(I)$ has almost maximal depth. The method of the proof is inspired by the methods employed in [25]. The first main ingredient of the proof is a bound on the K-reduction number $r_J^K(I) = \min\{n \mid KI^{n+1} = KJI^n\}$. Such a bound for the usual reduction number $r(I)$ was provided in [25]. This bound played a crucial role in the solution to an analogue of Sally's conjecture for $\mathfrak{m}$-primary ideals by M. E. Rossi. By specializing the bound on the reduction number $r_J^K(I)$ for $K = \mathfrak{m}$, we are able to use it for the fiber cone $F(I)$.

The second main ingredient of the main theorem is the notion of *Ratliff-Rush closure*, $rr_K(I_n) = \bigcup_{n\geq 0} KI_{n+k} : I_1^k$, of a filtration, $\mathcal{F} = \{I_n\}_{n\geq 0}$ of ideals with respect to an ideal K containing I_1. We will develop the basic properties of $rr_K(I_n)$ in Section 2. We shall find an analogue of Huneke's fundamental lemma [18] for the Hilbert function $\lambda(R/rr_K(I_n))$. As a consequence of this generalization, we shall provide formulas, in dimension 2, for the coefficients of the Hilbert polynomial, $P_K(\mathcal{F}, n)$ corresponding to the Hilbert function $H_K(\mathcal{F}, n) = \lambda(R/KI_n)$. These formulas are crucial for obtaining the bound on the reduction number $r_J^K(I)$ in Corollary 3.6. We shall recover Rossi's bound [25] for $r(I)$ as a consequence of our bound for $r_J^K(I)$.

One of the motivations for finding numerical conditions which ensure high depth for $G(I)$ and $F(I)$ is to compute the Hilbert series. Because of high depth one can work in dimension 1 or 2 where computation of Hilbert series is relatively easy. As a result, by imposing conditions on the multiplicity and minimum number of generators, one can predict the Hilbert series. In the final section of this paper, we obtain a formula for the generating function, $\sum_{n\geq 0} H_K(I, n)t^n$, where I is an $\mathfrak{m}$-primary ideal with almost minimal multiplicity with respect to K. This formula generalizes results of Sally and Rossi-Valla.

§2. Ratliff-Rush closure of a filtration of ideals with respect to an ideal

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. A filtration of ideals $\mathcal{F} = \{I_n\}_{n\geq 0}$ is said to be a Hilbert filtration if

(i) $I_n I_m \subseteq I_{n+m}$ for all $n, m \geq 0$,

(ii) there exists $p \geq 0$ such that $I_1 I_n = I_{n+1}$ for all $n \geq p$ and

(iii) I_1 is $\mathfrak{m}$-primary.

For a Hilbert filtration $\mathcal{F}$, let $H(\mathcal{F}, n) = \lambda(R/I_n)$ denote the Hilbert function of $\mathcal{F}$ and

$$P(\mathcal{F}, n) = e_0(\mathcal{F})\binom{n+d-1}{d} - e_1(\mathcal{F})\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F})$$

denote the corresponding polynomial.

Let K be an ideal such that $I_{n+1} \subseteq KI_n$ for all $n \geq 0$. Let $H_K(\mathcal{F}, n) = \lambda(R/KI_n)$ be the Hilbert function of $\mathcal{F}$ with respect to K. Since $H_K(\mathcal{F}, n) =$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

$\lambda(R/I_n) + \lambda(I_n/KI_n)$, $H_K(\mathcal{F}, n)$ coincides with a polynomial for $n \gg 0$. Let the corresponding polynomial be denoted by

$$P_K(\mathcal{F}, n) = g_0\binom{n+d-1}{d} - g_1\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d.$$

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $\mathcal{F}$ an I_1-good filtration of R. Let K be an ideal of R such that $I_1 \subseteq K$. Let $F_K(\mathcal{F}) = \bigoplus_{n\geq 0} I_n/KI_n$ be the fiber cone of $\mathcal{F}$ with respect to K. For $x \in I_1\backslash KI_1$, let x^* denote its initial form in the associated graded ring, $G(\mathcal{F}) = \bigoplus_{n\geq 0} I_n/I_{n+1}$, of $\mathcal{F}$ and x^o denote its initial form in the fiber cone $F_K(\mathcal{F})$.

We begin by recalling some of the properties of superficial elements in $F_K(\mathcal{F})$ proved in [20].

Proposition 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d with $R/\mathfrak{m}$ infinite. Let $\mathcal{F} = \{I_n\}$ be a Hilbert filtration of R, K an ideal such that $I_{n+1} \subseteq KI_n$ for all $n \geq 0$. Then*

1. *There exists an $x \in I_1\backslash KI_1$ such that x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$.*
2. *If, for $x \in I_1\backslash KI_1$, x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$, then there exists a $c > 0$ such that $(KI_n : x) \cap I_c = KI_{n-1}$ for all $n > c$. Moreover if x is regular in R, then $KI_n : x = KI_{n-1}$ for all $n \gg 0$.*
3. *If x^o is regular in $F_K(\mathcal{F})$ and x^* is regular in $G(\mathcal{F})$, then $KI_n : x = KI_{n-1}$ for all $n \geq 1$.*
4. *Let $x \in I_1$ be such that x^* is superficial in $G(\mathcal{F})$ and $x^o \in F_K(\mathcal{F})$ is superficial in $F_K(\mathcal{F})$. Let $\bar{\mathcal{F}} = \{I_n + xR/xR\}_{n\geq 0}$ and $\bar{K} = K/xR$. If* $\operatorname{depth} F_{\bar{K}}(\bar{\mathcal{F}}) > 0$, *then x^o is regular in $F_K(\mathcal{F})$.*
5. *Let $x_1, \dots, x_k \in I_1$. Assume that*
   - *(i) $x_1, \dots, x_k$ is a regular sequence in R.*
   - *(ii) $x_1^*, \dots, x_k^* \in G(\mathcal{F})$ is a regular sequence.*
   - *(iii) $x_1^o, \dots, x_k^o \in F_K(\mathcal{F})$ is a superficial sequence.*

   Then $\operatorname{depth}_{(x_1^o,\dots,x_k^o)} F_K(\mathcal{F}) = k$ *if and only if* $(x_1, \dots, x_k) \cap KI_n = (x_1, \dots, x_k)KI_{n-1}$ *for all $n \geq 1$.*

DEFINITION 2.2. The Ratliff-Rush closure of $\mathcal{F} = \{I_n\}$ with respect to K is the sequence of ideals $rr_K(\mathcal{F}) = \{rr_K(I_n)\}_{n\geq 0}$ given by

$$rr_K(I_n) = \bigcup_{k\geq 1}(KI_{n+k} : I_1^k).$$

The Ratliff-Rush closure of a filtration of ideals with respect to an ideal behaves quite similar to the Ratliff-Rush closure of an ideal. We summarize some of its properties.

PROPOSITION 2.3. 1. *$rr_K(I_n) = \bigcup_{k\geq 1}(KI_{nk+n} : I_n^k)$.*

2. *If* $\operatorname{grade} I_1 > 0$, *then* $rr_K(I_n) = KI_n$ *for* $n \gg 0$.
3. *If* $J = (x_1, \dots, x_s)$ *is a reduction of* I_1, *then* $rr_K(I_n) = \bigcup_{k\geq 1}(KI_{n+k} : (x_1^k, \dots, x_s^k))$.
4. *If* J *is a reduction of* I_1, *then* $rr_K(I_n) = \bigcup_{k\geq 1}(KI_{n+k} : J^k)$.
5. *If* J *is a minimal reduction of* I_1, *then* $rr_K(I_n) : J = rr_K(I_{n-1})$ *for all* $n \geq 1$.

Proof. 1. Note that $KI_{n+1} : I_1 \subseteq KI_{n+2} : I_1^2 \subseteq \cdots$ is an increasing chain of ideals in R. Hence $rr_K(I_n) = KI_{n+k} : I_1^k$ for $k \gg 0$. Since the chain $KI_{n+n} : I_n \subseteq KI_{n+2n} : I_n^2 \subseteq \cdots$ also terminates, it is enough to show that $rr_K(I_n) = KI_{nk+n} : I_n^k$ for $k \gg 0$. Suppose $x \in KI_{nk+n} : I_n^k$. Since $I_1^{nk} \subseteq I_n^k$, $xI_1^{nk} \subseteq KI_{nk+n}$. Therefore for $k \gg 0$, $x \in KI_{nk+n} : I_1^{nk} = rr_K(I_n)$. Conversely, let $xI_1^k \subseteq KI_{n+k}$ for $k \gg 0$. Since $\mathcal{F}$ is an I_1-good filtration, there exists p_0 such that $I_1I_p = I_{p+1}$ for $p \geq p_0$. Choose $k \gg 0$. Then

$$xI_n^k \subseteq xI_{nk} \subseteq xI_{p_0}I_1^{nk-p_0} \subseteq I_{p_0}KI_{nk-p_0+n} \subseteq KI_{nk+n}.$$

Therefore $x \in KI_{nk+n} : I_n^k$, so that $rr_K(I_n) = KI_{nk+n} : I_n^k$ for $k \gg 0$.

2. Let $x \in I_1$ be such that x is regular in R and x^o is superficial in $F_K(\mathcal{F})$ and x^* is superficial in $G(\mathcal{F})$. Then, by Proposition 2.1(2), $KI_n : x = KI_{n-1}$ for $n \gg 0$. Therefore $KI_n \subseteq KI_{n+1} : I_1 \subseteq KI_{n+1} : x = KI_n$ for $n \gg 0$. Thus $KI_{n+1} : I_1 = KI_n$ for $n \gg 0$. We show that $KI_{n+k} : I_1^k = KI_n$ for all $k \geq 1$. Apply induction on k. The result is proved for $k = 1$. Assume that the result is true for $k - 1$. Then

$$\begin{aligned} KI_{n+k} : I_1^k &= (KI_{n+k} : I_1^{k-1}) : I_1 \\ &= KI_{n+1} : I_1 \quad \text{(by induction)} \\ &= KI_n \quad \text{for } n \gg 0. \end{aligned}$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Therefore $rr_K(I_n) = KI_n$ for $n \gg 0$.

3. Let $(\underline{x}) = (x_1, \ldots, x_s)$ and $(\underline{x})^{[k]} = (x_1^k, \ldots, x_s^k)$. Clearly $KI_{n+k} : I_1^k \subseteq KI_{n+k} : (\underline{x})^{[k]}$. Since $(\underline{x})$ is a reduction of I_1, there exists an integer r such that $(\underline{x})^m I_1^n = I_1^{n+m}$ for all $n \geq r$ and $m \geq 1$. Let $z \in KI_{n+k} : (\underline{x})^{[k]}$ for $k \gg 0$. Then

$$\begin{aligned} zI_1^{r+sk} &= z(\underline{x})^{sk} I_1^r \\ &= \left(\sum_{|\alpha|=sk} z x_1^{\alpha_1} \cdots x_s^{\alpha_s} \right) I_1^r \\ &\subseteq \sum_{|\alpha|=sk} KI_{n+\alpha_i} x_1^{\alpha_1} \cdots \widehat{x_i^{\alpha_i}} \cdots x_s^{\alpha_s} I_1^r \quad \text{where } \alpha_i \geq k \\ &\subseteq KI_{n+r+sk}. \end{aligned}$$

Therefore $z \in KI_{n+r+sk} : I_1^{sk+r} = rr_K(I_n)$.

4. Let $J = (x_1, \ldots, x_s)$. Since $KI_{n+k} : I_1^k \subseteq KI_{n+k} : J^k \subseteq KI_{n+k} : (x_1^k, \ldots, x_s^k)$ for all k, the assertion follows.

5. For $k \gg 0$, we have

$$\begin{aligned} rr_K(I_n) : J &= (KI_{n+k} : J^k) : J \\ &= KI_{n+k} : J^{k+1} = rr_K(I_{n-1}). \end{aligned}$$

$\square$

The next lemma inspired the definition of Ratliff-Rush closure of a filtration of ideals with respect to another ideal.

Proposition 2.4. *Let $F = F_K(\mathcal{F})$ and let $[H^0_{F_+}(F)]_n$ denote the n-th graded component of the local cohomology module $H^0_{F_+}(F)$. Suppose* $\operatorname{grade} I_1 > 0$. *Then for all $n \geq 0$,*

$$[H^0_{F_+}(F)]_n = \frac{rr_K(I_n) \cap I_n}{KI_n}.$$

If $\operatorname{grade}(I_1) > 0$ *and* $\gamma(\mathcal{F}) > 0$, *then* $\operatorname{grade} F_+ > 0$ *if and only if* $rr_K(I_n) = KI_n$ *for all* $n \geq 0$.

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Proof. Let $y \in I_n$ and $y^o \in [H^0_{F_+}(F)]_n = 0 :_{F_n} F_+^k$ for $k \gg 0$. Then $y^o F_+^k = 0$. Therefore $yI_1^k \subseteq KI_{n+k}$. Hence $y \in (KI_{n+k} : I_1^k) \cap I_n = rr_K(I_n) \cap I_n$. Therefore $[H^0_{F_+}(F)]_n \subseteq (rr_K(I_n) \cap I_n)/KI_n$. Suppose $y^o \in (I_n \cap rr_K(I_n))/KI_n$. Then for $k \gg 0$, $yI_1^k \subseteq KI_{n+k}$. Therefore $y^o F_+^k = 0$ so that $y^o \in 0 :_{F_n} F_+^k = [H^0_{F_+}(F)]_n$.

Suppose $rr_K(I_n) = KI_n$ for all $n \geq 0$. Then $H^0_{F_+}(F) = \bigoplus_{n\geq 0} I_n \cap rr_K(I_n)/KI_n = 0$. Therefore $\operatorname{grade} F_+ > 0$. Conversely, suppose $\operatorname{grade} F_+ > 0$. Then $rr_K(I_n) \cap I_n = KI_n$ for all $n \geq 0$. Suppose $y \in rr_K(I_n) = KI_{n+k} : I_1^k$. Choose a regular element $x_1 \in I_1$ such that x_1^o is regular in F and x_1^* is regular in $G(\mathcal{F})$. Then $yx_1^k \in KI_{n+k}$ so that $yx_1^k \in KI_{n+k} \cap (x_1^k) = x_1^k KI_n$. Therefore $y \in KI_n$ and hence $rr_K(I_n) = KI_n$. $\square$

In the next proposition we obtain a generalization of Huneke's fundamental lemma [18] for the function $H_K(\mathcal{F}, n)$. It also shows that once we know a minimal reduction of I_1, we can compute the coefficients g_1 and g_2 and hence the Hilbert polynomial of $F_K(\mathcal{F})$ can be completely determined.

PROPOSITION 2.5. *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring. Let $\mathcal{F} = \{I_n\}$ be a Hilbert filtration of R. Let $J = (x, y)$ be a minimal reduction of I_1. Then $\lambda(R/rr_K(I_n))$ coincides with the polynomial $P_K(\mathcal{F}, n)$, for $n \gg 0$ and the following are true:*
1. *For $n \geq 2$,*

$$\Delta^2[P_K(\mathcal{F}, n) - \lambda(R/rr_K(I_n))] = \lambda\left(\frac{rr_K(I_n)}{Jrr_K(I_{n-1})}\right).$$

2. *Set*

$$v_n = \begin{cases} e_0(\mathcal{F}) - \lambda(R/rr_K(I_0)) & \text{if } n = 0 \\ \lambda\left(\frac{rr_K(I_n)}{Jrr_K(I_{n-1})}\right) & \text{if } n \geq 1. \end{cases}$$

Then, $g_1 = \sum_{n\geq 1} v_n - \lambda(R/rr_K(I_0))$ and $g_2 = \sum_{n\geq 1}(n-1)v_n - \lambda(R/rr_K(I_0))$.

Proof. 1. Since $rr_K(I_n) = KI_n$ for $n \gg 0$, $\lambda(R/rr_K(I_n)) = H_K(\mathcal{F}, n) = P_K(\mathcal{F}, n)$ for $n \gg 0$. Consider the exact sequence:

$$0 \longrightarrow \frac{R}{rr_K(I_{n-1}):J} \xrightarrow{\beta} \left(\frac{R}{rr_K(I_{n-1})}\right)^2 \xrightarrow{\alpha} \frac{J}{Jrr_K(I_{n-1})} \longrightarrow 0,$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

where the maps α and β are defined as, $\alpha(\bar{r},\bar{s}) = \overline{xr+ys}$ and $\beta(\bar{r}) = (\bar{y}\bar{r}, -\bar{x}\bar{r})$. It follows that for all $n \geq 2$,

$$\begin{aligned}
2\lambda(R/rr_K(I_{n-1})) &= \lambda(R/(rr_K(I_{n-1}):J)) + \lambda(J/Jrr_K(I_{n-1}))\\
&= \lambda(R/(rr_K(I_{n-1}):J)) + \lambda(R/Jrr_K(I_{n-1})) - \lambda(R/J).
\end{aligned}$$

Therefore $e_0(\mathcal{F})+2\lambda(R/rr_K(I_{n-1})) = \lambda(R/Jrr_K(I_{n-1}))+\lambda(R/(rr_K(I_{n-1}):J))$. Hence

$$\begin{aligned}
&e_0(\mathcal{F}) - \lambda(R/rr_K(I_n)) + 2\lambda(R/rr_K(I_{n-1})) - \lambda(R/rr_K(I_{n-2}))\\
&\quad = \lambda(R/Jrr_K(I_{n-1})) - \lambda(R/rr_K(I_n)) + \lambda(R/(rr_K(I_{n-1}):J))\\
&\qquad - \lambda(R/rr_K(I_{n-2}))\\
&\quad = \lambda(rr_K(I_n)/Jrr_K(I_{n-1})) - \lambda(rr_K(I_{n-1}):J/rr_K(I_{n-2})).
\end{aligned}$$

Since $\Delta^2 P_K(\mathcal{F},n) = e_0(\mathcal{F})$,

$$\Delta^2[P_K(\mathcal{F},n) - H_K(\mathcal{F},n)] = \lambda\left(\frac{rr_K(I_n)}{Jrr_K(I_{n-1})}\right) - \lambda\left(\frac{rr_K(I_{n-1}):J}{rr_K(I_{n-2})}\right).$$

By Proposition 2.3(5), we have $rr_K(I_n) : J = rr_K(I_{n-1})$ for all $n \geq 1$. Therefore for all $n \geq 2$,

$$\Delta^2[P_K(\mathcal{F},n) - \lambda(R/rr_K(I_n))] = \lambda\left(\frac{rr_K(I_n)}{Jrr_K(I_{n-1})}\right).$$

2. Define a filtration of ideals $F = \{F_n\}$ in R as follows:

$$F_n = \begin{cases} R & \text{if } n = 0\\ rr_K(I_{n-1}) & \text{if } n \geq 1. \end{cases}$$

Then it can be seen that F is a Hilbert Fitration. Since R is Cohen-Macaulay, grade $I_1 \geq 0$ and hence by Proposition 2.3(2), the Hilbert polynomial corresponding to the functions $H_K(\mathcal{F},n)$ and $\lambda(R/rr_K(I_n))$ are equal. Also, Proposition 2.3(5) shows that $\gamma(F) \geq 1$. Let $H(F,n) = \lambda(R/F_n)$ and $P(F,n)$ denote the Hilbert function and Hilbert polynomial corresponding to the filtration F. Then it can be seen that

$$\begin{aligned}
P(F,n) = P_K(\mathcal{F},n-1) &= g_0\binom{n}{d} - g_1\binom{n-1}{d-1} + g_2\\
&= g_0\binom{n+1}{d} - (g_0+g_1)\binom{n}{d-1} + (g_1+g_2).
\end{aligned}$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Applying Proposition 1.9 of [11], we get that

$$g_0 + g_1 = \sum_{j\geq 0} \lambda(F_{j+1}/JF_j) \quad \text{and} \quad g_1 + g_2 = \sum_{j\geq 1} \binom{n}{1} \lambda(F_{j+1}/JF_j).$$

Therefore

$$\begin{aligned} g_1 &= \lambda(rr_K(I_0)/J) + \sum_{n\geq 1} \lambda(rr_K(I_n)/Jrr_K(I_{n-1})) - g_0 \\ &= \sum_{n\geq 1} v_n - \lambda(R/rr_K(I_0)). \end{aligned}$$

Substituting the value of g_1, we obtain $g_2 = \sum_{n\geq 1}(n-1)v_n - \lambda(R/rr_K(I_0))$. ∎

As a consequence we derive formulas for the Hilbert coefficients obtained by Huneke for the I-adic filtration in [18].

Corollary 2.6. *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring and $\mathcal{F}$ be a Hilbert filtration. Then $e_1(\mathcal{F}) = \sum_{n\geq 1} v_n$ and $e_2(\mathcal{F}) = \sum_{n\geq 1}(n-1)v_n$.*

Proof. Put $K = R$ in Proposition 2.5(2) and note that, for $K = R$, $rr_K(I_0) = R$. ∎

§3. Bounds on reduction numbers

In this section we obtain a bound on the K-reduction number of an $\mathfrak{m}$-primary ideal (see Definition 3.3) from which we derive Rossi's bound ([25, Corollary 1.5]) for the reduction number. We use this bound to prove the almost maximal depth condition for the fiber cone. We set the following notation for the rest of the section. Let $(R, \mathfrak{m})$ denote a Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal of R and let J be its minimal reduction. Let K be an ideal containing I and let $rr_K(I^n)$ denote the Ratliff-Rush closure of I^n with respect to K. For $n \geq 0$, set

$$\rho_n^K = \lambda(rr_K(I^{n+1})/Jrr_K(I^n)) \quad \text{and} \quad \nu_n^K = \lambda(KI^{n+1}/KJI^n).$$

Lemma 3.1. *If $KI^{n+1} \cap J = KJI^n$ for some n, then $\rho_n^K - \nu_n^K = \lambda(rr_K(I^{n+1})/Jrr_K(I^n) + KI^{n+1})$.*

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Proof. Since $Jrr_K(I^n) \subseteq Jrr_K(I^n) + KI^{n+1} \subseteq rr_K(I^{n+1})$, we have

$$\lambda\left(\frac{rr_K(I^{n+1})}{Jrr_K(I^n) + KI^{n+1}}\right) = \rho_n^K - \lambda\left(\frac{Jrr_K(I^n) + KI^{n+1}}{Jrr_K(I^n)}\right)$$
$$= \rho_n^K - \lambda\left(\frac{KI^{n+1}}{KI^{n+1} \cap Jrr_K(I^n)}\right).$$

Since $KJI^n \subseteq Jrr_K(I^n) \cap KI^{n+1} \subseteq J \cap KI^{n+1} \subseteq KJI^n$, $KJI^n = Jrr_K(I^n) \cap KI^{n+1}$. ∎

The next proposition, due to M. E. Rossi, played a crucial role in solving the conjecture of Sally [26] and its generalization to case of $\mathfrak{m}$-primary ideals [25]. For an ideal I in R, let $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ denote the Rees algebra of I. For an $\mathcal{R}(I)$-module M, put $\operatorname{Ann}_{I^\nu}(M) = \{x \in I^\nu \mid xt^\nu M = 0\}$.

Proposition 3.2. *Let I be an ideal of a Noetherian local ring R and let J be a minimal reduction of I. Let M be an $\mathcal{R}(I)$-module of finite length as R-module. Let ν be the minimum number of generators of $M/\mathcal{R}(J)_+M$ as an R-module. Then*

$$I^\nu = JI^{\nu-1} + \operatorname{Ann}_{I^\nu}(M).$$

Definition 3.3. Let J be a minimal reduction of an ideal I. Put

$$r_J^K(I) := \min\{n \mid KI^{n+1} = KJI^n\}.$$

The integer $r_J^K(I)$ is called the K-reduction number of I with respect to J.

We now give a bound for the K-reduction number of an $\mathfrak{m}$-primary ideal. Put

$$S_J^K(I) := \{n \in \mathbb{N} \mid KI^{j+1} \cap J = KJI^j \text{ for all } j \leq n\}.$$

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R and let J be a minimal reduction of I. Let K be an ideal containing I and let $n \in S_J^K(I)$. Then*

$$r_J^K(I) \leq \sum_{i\geq 0} \rho_i^K + n + 1 - \sum_{i=0}^{n} \nu_i^K.$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Proof. Let $M := \bigoplus_{n\geq 1} rr_K(I^n)/KI^n$. Then M is a finitely generated $\mathcal{R}(I)$-module and $\lambda_R(M) < \infty$, by 2.3(2). For $j \geq 0$, $(M/\mathcal{R}(J)_+M)_{j+1} = M_{j+1}/(J^{j+1}M_0 + J^jM_1 + \cdots + JM_j)$. For $1 \leq i \leq j+1$ and $k \gg 0$, we have

$$\begin{aligned} J^iM_{j-i+1} &= JJ^{i-1}M_{j-i+1} \subseteq JI^{i-1}M_{j-i+1} \\ &= \frac{JI^{i-1}(KI^{j+1-i+k} : I^k) + KI^{j+1}}{KI^{j+1}} \\ &\subseteq \frac{Jrr_K(I^j) + KI^{j+1}}{KI^{j+1}} \\ &= JM_j. \end{aligned}$$

Therefore $[M/\mathcal{R}(J)_+M]_{j+1} \cong rr_K(I^{j+1})/Jrr_K(I^j) + KI^{j+1}$. We have

$$\lambda(rr_K(I^{j+1})/Jrr_K(I^j) + KI^{j+1}) \leq \lambda(rr_K(I^{j+1})/Jrr_K(I^j))$$

and equality occurs if and only if $KI^{j+1} \subseteq Jrr_K(I^j)$. Since J is a reduction of I and $rr_K(I^n) = KI^n$ for $n \gg 0$, there exists a j such that $KI^{j+1} \subseteq Jrr_K(I^j)$. Let $k = \min\{j \mid KI^{j+1} \subseteq Jrr_K(I^j)\}$. Let μ_j be the minimal number of generators of $rr_K(I^{j+1})/Jrr_K(I^j) + KI^{j+1}$ as an R-module. Then $\mu_j \leq \lambda(rr_K(I^{j+1})/Jrr_K(I^j) + KI^{j+1})$. Let $\mu = \sum_{j\geq 0}\mu_j$. Then by the previous proposition $I^\mu = JI^{\mu-1} + \operatorname{Ann}_{I^\mu}(M)$. Therefore

$$\begin{aligned} KI^{\mu+k+1} &= KI^{k+1}(JI^{\mu-1} + \operatorname{Ann}_{I^\mu}(M)) \\ &= KJI^{\mu+k} + KI^{k+1}\operatorname{Ann}_{I^\mu}(M) \\ &\subseteq KJI^{\mu+k} + Jrr_K(I^k)\operatorname{Ann}_{I^\mu}(M). \end{aligned}$$

Since $Jrr_K(I^k)\operatorname{Ann}_{I^\mu}(M) \subseteq KJI^{\mu+k}$, $KI^{\mu+k+1} = KJI^{\mu+k}$. Therefore

$$r_J^K(I) \leq \mu + k = \sum_{j\geq 0}\mu_j + k \leq \sum_{j\geq 0}\lambda\left(\frac{rr_K(I^{j+1})}{Jrr_K(I^j) + KI^{j+1}}\right) + k.$$

Since $n \in S_J^K(I)$, by Lemma 3.1, $\lambda(rr_K(I^{j+1})/Jrr_K(I^j) + KI^{j+1}) = \rho_j^K - \nu_j^K$ for $j \leq n$. Hence

$$r_J^K(I) \leq \sum_{j=0}^{n}(\rho_j^K - \nu_j^K) + \sum_{j\geq n+1}\lambda\left(\frac{rr_K(I^{j+1})}{Jrr_K(I^j) + KI^{j+1}}\right) + k.$$

If $k \leq n+1$, then $\lambda(rr_K(I^{j+1})/Jrr_K(I^j) + KI^{j+1}) = \rho_j^K$ for all $j \geq n+1$ so that

$$r_j^K(I) \leq \sum_{j\geq 0}\rho_j^K + k - \sum_{j=0}^{n}\nu_j^K \leq \sum_{j\geq 0}\rho_j^K + n + 1 - \sum_{j=0}^{n}\nu_j^K.$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Suppose $k \geq n+2$. Then for $n+1 \leq j \leq k-1$,

$$\begin{aligned}\lambda(rr_K(I^{j+1})/Jrr_K(I^j)+KI^{j+1}) &= \rho_j^K - \lambda(Jrr_K(I^j)+KI^{j+1}/Jrr_K(I^j)) \\ &\leq \rho_j^K - 1.\end{aligned}$$

Therefore

$$\begin{aligned} r_J^K(I) &\leq \sum_{j=0}^{n}(\rho_j^K - \nu_j^K) + \sum_{j=n+1}^{k-1}(\rho_j^K - 1) + \sum_{j \geq k}\rho_j^K + k \\ &= \sum_{j\geq 0}\rho_j^K - \sum_{j=0}^{n}\nu_j^K + n + 1. \end{aligned}$$

∎

The following lemma is quite well-known. We include it for the sake of completeness.

Lemma 3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $J = (x_1, \dots, x_s)$ be an ideal generated by a regular sequence in R. Then for any ideal K containing J, $J/KJ \cong (R/K)^s$.*

Proof. Consider the map $\phi : (R/K)^s \to J/KJ$, defined as

$$\phi(\bar{r}_1, \dots, \bar{r}_s) = \overline{r_1x_1 + \cdots + r_sx_s}.$$

The map ϕ is clearly surjective. Suppose for some $r_1, \dots, r_s \in R$, $r_1x_1 + \cdots + r_sx_s \in KJ$. Write $r_1x_1 + \cdots + r_sx_s = t_1x_1 + \cdots + t_sx_s$ for some $t_1, \dots, t_s \in K$. Then $(r_1 - t_1)x_1 = (t_2 - r_2)x_2 + \cdots + (t_s - r_s)x_s$. Since $x_1, \dots, x_s$ is a regular sequence, $r_1 - t_1 \in (x_2, \dots, x_s) \subseteq K$ and hence $r_1 \in K$. Similarly $r_i \in K$ for all $i = 1, \dots, s$. Therefore ϕ is an isomorphism. ∎

We obtain a bound on the reduction number $r_J^K(I)$ in terms of the Hilbert coefficient g_1.

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal, K an ideal containing I and J a minimal reduction of I. If $n \in S_J^K(I)$, then*

$$r_J^K(I) \leq g_1 - \sum_{j=0}^{n}\nu_j^K + n + 1 + \lambda(R/K).$$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Proof. By Theorem 3.4, we get

$$\begin{aligned}
r_J^K(I) &\le \lambda(rr_K(I)/Jrr_K(I^0)) + \sum_{j\ge 1}\rho_j^K + n + 1 - \sum_{j=0}^{n}\nu_j^K \\
&= \lambda(R/Jrr_K(I^0)) - \lambda(R/rr_K(I)) + \sum_{j\ge 1}\rho_j^K + n + 1 - \sum_{j=0}^{n}\nu_j^K \\
&= \lambda(R/J) + \lambda(J/Jrr_K(I^0)) - \lambda(R/rr_K(I)) \\
&\quad + \sum_{j\ge 1}\rho_j^K + n + 1 - \sum_{j=0}^{n}\nu_j^K \\
&= e_0(I) + 2\lambda(R/rr_K(I^0)) - \lambda(R/rr_K(I)) + \sum_{j\ge 1}\rho_j^K + n + 1 - \sum_{j=0}^{n}\nu_j^K \\
&\le e_0(I) + \lambda(R/rr_K(I^0)) + \lambda(R/K) - \lambda(R/rr_K(I)) \\
&\quad + \sum_{j\ge 1}\rho_j^K + n + 1 - \sum_{j=0}^{n}\nu_j^K.
\end{aligned}$$

The last equality follows from Lemma 3.5 and the inequality follows since $K \subseteq rr_K(I^0)$. By Lemma 2.5, $g_1 = e_0(I) + \lambda(R/rr_K(I^0)) - \lambda(R/rr_K(I)) + \sum_{j\ge 1}\rho_j^K$. Therefore

$$r_J^K(I) \le g_1 - \sum_{j=0}^{n}\nu_j^K + n + 1 + \lambda(R/K).$$

∎

COROLLARY 3.7. (Rossi's bound) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension 2. Let I be an $\mathfrak{m}$-primary ideal of R and J be a minimal reduction of I. Then*

$$r_J(I) \le e_1(I) - e_0(I) + \lambda(R/I) + 1.$$

Proof. For $K = R$, we have $0 \in S_J^K(I)$ and note that $g_i = e_i$ for all $i = 0, \dots, d$. Hence from Corollary 3.6 it follows that $r(I) \le e_1(I) - \lambda(I/J) + 1 = e_1(I) - e_0(I) + \lambda(R/I) + 1$. ∎

Our objective in introducing $r_J^K(I)$ is to obtain bounds for $r_J^{\mathfrak{m}}(I)$ which in turn is used to study the depth of fiber cones of ideals with almost minimal multiplicity.

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Corollary 3.8. *Let $(R,\mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal and J a minimal reduction of I. If $n \in S_J^{\mathfrak{m}}(I)$, then*

$$r_J^{\mathfrak{m}}(I) \leq g_1 + n + 2 - \sum_{j=0}^{n} \nu_j^{\mathfrak{m}}.$$

Proof. The assertion follows directly from Corollary 3.6 by puttting $K = \mathfrak{m}$. ∎

If I is an $\mathfrak{m}$-primary ideal, J is a minimal reduction of I and x^* is regular in $G(I)$, then $r_J(I) = r_{\bar{J}}(\bar{I})$, [14]. In the following lemma we prove that a similar result holds for the K-reduction number also.

Lemma 3.9. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R, K an ideal containing I and J a minimal reduction of I. Let $x \in I \setminus KI$ be such that x^* is regular in $G(I)$ and x^o is regular in $F_K(I)$. Then $r_{\bar{J}}^{\bar{K}}(\bar{I}) = r_J^K(I)$, where "$-$" denote images* modulo$(x)$.

Proof. Clearly $r_{\bar{J}}^{\bar{K}}(\bar{I}) \leq r_J^K(I)$. Suppose for some n, $\bar{K}\bar{I}^n = \bar{K}\bar{J}\bar{I}^{n-1}$. Then $KI^n + xR = KJI^{n-1} + xR$ and hence $KI^n = KI^n \cap (KJI^{n-1} + xR) = KJI^{n-1} + (xR \cap KI^n)$. Since x^* is regular in $G(I)$ and x^o is regular in $F_K(I)$, by Proposition 2.1(5), $xR \cap KI^n = xKI^{n-1}$. Hence $KI^n = KJI^{n-1}$. Therefore $r_{\bar{J}}^{\bar{K}}(\bar{I}) = r_J^K(I)$. ∎

§4. Ideals with almost minimal multiplicity

Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal and J a minimal reduction of I.

Definition 4.1. An ideal I is said to have almost minimal multiplicity with respect to an ideal $K \supseteq I$ if for any minimal reduction J of I, $\lambda(KI/KJ) = 1$. We say that I has almost minimal multiplicity if $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$.

Remark. For any $\mathfrak{m}$-primary ideal I, an ideal $K \supseteq I$ and a minimal reduction J of I, $\lambda(KI/KJ) = 1$ if and only if $\lambda(I/KI) = e_0(I) - \lambda(R/I) + \lambda(J/KJ) - 1 = e_0(I) - \lambda(R/I) + d\lambda(R/K) - 1$, by Lemma 3.5. Hence the definition of almost minimal multiplicity with respect to K is independent of the minimal reduction J chosen for I.

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

For $K = I$, the almost minimal multiplicity condition is equivalent to $\lambda(I/I^2) = e_0(I) - (d-1)\lambda(R/I) - 1$, which was the condition imposed on the ideal in [25] to obtain the almost maximal depth for the associated graded ring.

In this section we actually consider a more general class of ideals, described as follows. Let I be an $\mathfrak{m}$-primary ideal, J a minimal reduction of I and K an ideal containing I. Suppose that there exists an integer k such that $\mathfrak{m}I^n \cap J = \mathfrak{m}JI^{n-1}$ for all $n = 1, \dots, k$ and $\lambda(\mathfrak{m}I^{k+1}/\mathfrak{m}JI^k) \leq 1$. We prove that the fiber cones of such ideals have almost maximal depth provided the associated graded rings have high depth. The method of the proof is analogous to the method employed by M. E. Rossi in [25] to prove the almost maximal depth condition for the associated graded ring. We begin with the following lemma.

LEMMA 4.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal, J a minimal reduction of I and K an ideal containing I. If $\lambda(KI^r/KJI^{r-1}) = 1$, for some $r \geq 1$, then $\lambda(KI^n/KJI^{n-1}) \leq 1$ for all $n \geq r$.*

Proof. Since $\lambda(KI^r/KJI^{r-1}) = 1$, there exists $a \in K$, $b \in I$ such that $KI^r = KJI^{r-1} + (ab^r)$ and $\mathfrak{m}ab^r \subseteq KJI^{r-1}$. Then it can easily be seen, by induction, that $KI^n = KJI^{n-1} + (ab^n)$ with $\mathfrak{m}ab^n \subseteq KJI^{n-1}$ for all $n \geq r$. Hence $\lambda(KI^n/KJI^{n-1}) \leq 1$ for all $n = 1, \dots, r_J^K(I)$. $\square$

LEMMA 4.3. 1. *Let $(R, \mathfrak{m})$ be a 2-dimensional Cohen-Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal and let J be a minimal reduction of I. If for some k, $\mathfrak{m}I^k \cap J = \mathfrak{m}JI^{k-1}$ for all $n = 1, \dots, k$ and $\lambda(\mathfrak{m}I^{k+1}/\mathfrak{m}JI^k) = 1$, then $r_J^{\mathfrak{m}}(I) \leq g_1 + k + 1 - \sum_{j=0}^{k-1} \nu_j^{\mathfrak{m}}$.*

2. *Let $x \in I \backslash \mathfrak{m}I$ be such that x^o is superficial in $F(I)$ and x^* is superficial in $G(I)$. Let "$-$" denote images* modulo(x). *If there exists k such that $\bar{\mathfrak{m}}\bar{I}^n \cap \bar{J} = \bar{\mathfrak{m}}\bar{J}\bar{I}^{n-1}$ for all $n = 1, \dots, k$ and $\lambda(\bar{\mathfrak{m}}\bar{I}^{k+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^k) = 1$, then $r_{\bar{J}}^{\bar{\mathfrak{m}}}(\bar{I}) = r_J^{\mathfrak{m}}(I) = g_1 + k + 1 - \sum_{j=0}^{k-1} \nu_j^{\mathfrak{m}}$.*

Proof. 1. The inequality directly follows from Corollary 3.8.

2. Set $s = r_{\bar{J}}^{\bar{\mathfrak{m}}}(\bar{I})$. Clearly $s \leq r_J^{\mathfrak{m}}(I)$. As x^o is superficial in $F(I)$ and x^* is superficial in $G(I)$, $g_1 = \bar{g}_1$, where $\bar{g}_i$ denote coefficients of the polynomial corresponding to $\lambda(\bar{R}/\bar{\mathfrak{m}}\bar{I}^n)$. Since $\dim \bar{R} = 1$, by Theorem 5.3 of [20], $\bar{g}_1 = \sum_{n \geq 1} \lambda(\bar{\mathfrak{m}}\bar{I}^n/\bar{\mathfrak{m}}\bar{J}\bar{I}^{n-1}) - \lambda(\bar{R}/\bar{\mathfrak{m}})$. From the hypothesis it follows that $\lambda(\bar{\mathfrak{m}}\bar{I}^{j+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^j) = 1$ for all $j = k, \dots, s-1$. Therefore, $\bar{g}_1 =$

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

$\sum_{j=0}^{k}\lambda(\bar{\mathfrak{m}}\bar{I}^{j+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^{j})+s-k-1$. Hence $r_J^{\mathfrak{m}}(I)\leq g_1+k+1-\sum_{j=0}^{k-1}\nu_j^{\mathfrak{m}}=s$. ∎

We now prove the main result of this section.

Theorem 4.4. *Let $(R,\mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring, $d\geq 2$, I an $\mathfrak{m}$-primary ideal such that $\gamma(I)\geq d-2$ and J a minimal reduction of I. If there exists an integer k, such that*

$$\mathfrak{m}I^n\cap J=\mathfrak{m}JI^{n-1}\quad \textit{for all}\quad n=1,\ldots,k\quad\textit{and}\quad \lambda(\mathfrak{m}I^{k+1}/\mathfrak{m}JI^k)\leq 1,$$

then depth $F(I)\geq d-1$.

Proof. Induct on d. Let $d=2$ and $J=(x,y)$ such that (x^o,y^o), (x^*,y^*) are superficial sequences in $F(I)$ and $G(I)$ respectively. Let "–" denote images modulo(x).

Case I: $\mathfrak{m}I^{k+1}=\mathfrak{m}JI^k$.
Then we have, $\bar{\mathfrak{m}}\bar{I}^n\cap(\bar{y})=\bar{\mathfrak{m}}(\bar{y})\bar{I}^{n-1}$ for all $n\geq 1$. Since $\bar{y}$ is regular in $\bar{R}$, the above condition is equivalent to saying that $\bar{\mathfrak{m}}\bar{I}^n:(\bar{y})=\bar{\mathfrak{m}}\bar{I}^{n-1}$ for all $n\geq 1$. Therefore, $\bar{y}^o$ is regular in $F(\bar{I})$. Hence Sally machine for fiber cone, [20, Lemma 2.7], yields that $x^o\in F(I)$ is regular and hence depth $F(I)\geq 1$.

Case II: $\lambda(\mathfrak{m}I^{k+1}/\mathfrak{m}JI^k)=1$.
Then for all $n=k+1,\ldots,r_J^{\mathfrak{m}}(I)=r$, we have $\lambda(\mathfrak{m}I^n/\mathfrak{m}JI^{n-1})=1$. If $\bar{\mathfrak{m}}\bar{I}^{k+1}=\bar{\mathfrak{m}}\bar{J}\bar{I}^k$, then proceeding as in Case I, we get that x^o is regular in $F(I)$ and hence depth $F(I)\geq 1$. Therefore assume that $\lambda\big(\bar{\mathfrak{m}}\bar{I}^{k+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^k\big)=1$. Set $s=r_{\bar{J}}^{\bar{\mathfrak{m}}}(\bar{I})$. Then by Lemma 4.3(2), we have $r=s=g_1+k+1-\sum_{j=0}^{k-1}\nu_j^{\mathfrak{m}}$.

Since $\mathfrak{m}I^n\cap J=\mathfrak{m}JI^{n-1}$ for all $n=1,\ldots,k$, from [20, Lemma 5.2], we get $\mathfrak{m}I^n\cap(x)=\mathfrak{m}(x)I^{n-1}$ for all $n=1,\ldots,k$.

For $j\geq k$, consider the following exact sequence:

$$0\longrightarrow\frac{\mathfrak{m}I^j:x}{\mathfrak{m}I^j:J}\xrightarrow{\cdot y}\frac{\mathfrak{m}I^{j+1}:x}{\mathfrak{m}I^j}\xrightarrow{\cdot x}\frac{\mathfrak{m}I^{j+1}}{\mathfrak{m}JI^j}\longrightarrow\frac{\bar{\mathfrak{m}}\bar{I}^{j+1}}{\bar{\mathfrak{m}}\bar{J}\bar{I}^j}\longrightarrow 0.$$

For $j=k,\ldots,r-1$, $\lambda(\mathfrak{m}I^{k+1}/\mathfrak{m}JI^k)=1=\lambda\big(\bar{\mathfrak{m}}\bar{I}^{k+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^k\big)$ and for $j\geq r$, these two modules are zero. Therefore, for $j\geq k$, the last two modules in the above exact sequence have equal length. For $j=k$ we know that $\mathfrak{m}I^j:x=\mathfrak{m}I^{j-1}=\mathfrak{m}I^j:J$, hence by induction, $\mathfrak{m}I^{j+1}:x=\mathfrak{m}I^j$ for all $j\geq k$. Therefore $\mathfrak{m}I^{j+1}:x=\mathfrak{m}I^j$ for all $j\geq 0$ and hence x^o is regular in $F(I)$.

Now assume that $d > 2$. Let $J = (x_1, \dots, x_d)$ be such that $(x_1^o, \dots, x_{d-2}^o)$ is a superficial sequence in $F(I)$ and $(x_1^*, \dots, x_{d-2}^*)$ is a regular sequence in $G(I)$. Let "$-$" denote images modulo$(x_1, \dots, x_{d-2})$. Then we have,

$$\bar{\mathfrak{m}}\bar{I}^n \cap \bar{J} = \bar{\mathfrak{m}}\bar{J}\bar{I}^{n-1} \quad \text{for all} \quad n = 1, \dots, k \quad \text{and} \quad \lambda\big(\bar{\mathfrak{m}}\bar{I}^{k+1}/\bar{\mathfrak{m}}\bar{J}\bar{I}^k\big) \leq 1.$$

Therefore, by the first part, $\operatorname{depth} F(\bar{I}) \geq 1$. Since $(x_1^*, \dots, x_{d-2}^*)$ is a regular sequence in $G(I)$, $F(\bar{I}) \cong F(I)/(x_1^o, \dots, x_{d-2}^o)$ and hence by Sally machine, $\operatorname{depth} F(I) \geq d-1$. ∎

COROLLARY 4.5. *Let I be an $\mathfrak{m}$-primary ideal in a Cohen-Macaulay local ring $(R, \mathfrak{m})$ such that $\gamma(I) \geq d-2$. If I has almost minimal multiplicity, then* $\operatorname{depth} F(I) \geq d-1$.

Proof. By Lemma 4.2 we have $\lambda(\mathfrak{m}I^2/\mathfrak{m}JI) \leq 1$. We also have $\mathfrak{m}I \cap J = \mathfrak{m}J$. Now the assertion directly follows from Theorem 4.4. ∎

We end this section with an example to show that the depth assumption on the associated graded ring in the above theorem is necessary. This example was provided to us by M. E. Rossi.

EXAMPLE 4.6. Let $R = k[\![x, y, z]\!]$, where k is any field. Let $I = (-x^2 + y^2, -y^2 + z^2, xy, yz, zx)$ and $J = (-x^2 + y^2, -y^2 + z^2, xy)$. Then $I^3 = JI^2$. Hence J is a minimal reduction of I. Let $\mathfrak{m} = (x, y, z)$. Then it can be seen that $\mathfrak{m}I = \mathfrak{m}J + (z^3)$ and $\mathfrak{m}(z^3) \subset \mathfrak{m}J$. Hence $\lambda(\mathfrak{m}I/\mathfrak{m}J) = 1$. Therefore I has almost minimal multiplicity. It can be easily seen that $x^2 I \subset I^2$, but $x^2 \notin I$. This shows that the Ratliff-Rush closure $\tilde{I}$ is not equal to I. Hence, $\gamma(I) = 0$.

Now we show that $\operatorname{depth} F(I) = 1$. Since I is generated by homogeneous elements of same degree (equal to 2), $F(I) \cong k[-x^2 + y^2, -y^2 + z^2, xy, yz, zx]$. Therefore $\operatorname{depth} F(I) \geq 1$. Set $F = k[-x^2 + y^2, -y^2 + z^2, xy, yz, zx]$. Consider $\overline{F} = F/(-x^2 + y^2)F$. Let $\mathfrak{N}$ denote the graded maximal ideal of F and let $\overline{\mathfrak{N}}$ be the graded maximal ideal of $\overline{F}$. Then, it can be easily checked that $\overline{\mathfrak{N}}(\overline{-x^2z^2 + y^2z^2}) = 0$. Note that, since $z^2 \notin F$, $\overline{-x^2z^2 + y^2z^2} \neq 0 \in \overline{F}$. Therefore we have produced a nonzero element in $\overline{F}$ which is killed by the maximal ideal of $\overline{F}$ and hence $\operatorname{depth} \overline{F} = 0$. This shows that $\operatorname{depth} F = \operatorname{depth} F(I) = 1$.

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

§5. Cohen-Macaulay $F_K(I)$ when I has almost minimal multiplicity

In this section, we characterize Cohen-Macaulay property of $F_K(I)$ when I has almost minimal multiplicity. For this purpose, we find the generating function of the function $H_K(I,n)$, first in dimension 1 and then in arbitrary dimension. A formula of Rossi and Valla for the Hilbert series of $G(\mathfrak{m})$ when $\lambda(\mathfrak{m}^2/J\mathfrak{m}) = 1$ is generalized for $\mathfrak{m}$-primary ideals with almost minimal multiplicity.

LEMMA 5.1. *Let $(R,\mathfrak{m})$ be a 1-dimensional Cohen-Macaulay local ring and let I be an $\mathfrak{m}$-primary ideal of R with almost minimal multiplicity with respect to $K \supseteq I$. Let $s = r_J^K(I)$. Then*

1. $P_K(I,n) = e_0(I)n - (s - \lambda(R/K))$.
2. $\sum_{n\geq 0} H_K(I,n)t^n = \left[\lambda(R/K) + (e_0(I) - 1 - \lambda(R/K))t + t^{s+1}\right]/(1-t)^2$.

Proof. 1. From the following diagram,

$$\begin{tikzcd} R & KI_n \arrow[l, "h"'] & KI_{n+1} \arrow[l, "l_n"'] \\ (x) \arrow[u, "e_0(I)"] & xKI_n \arrow[l, "h"] \arrow[u, "e_0(I)"'] \arrow[ur, "c"'] & \end{tikzcd}$$

it follows that

$$\lambda(KI^n/KI^{n+1}) = \begin{cases} e_0(I) - 1 & \text{for } n = 0, \ldots, s-1 \\ e_0(I) & \text{for } n \geq s. \end{cases}$$

Therefore,

$$\begin{aligned} \lambda(R/KI^n) &= \lambda(R/K) + \sum_{i=0}^{n-1} \lambda(KI^n/KI^{n+1}) \\ &= \begin{cases} n(e_0(I) - 1) + \lambda(R/K) & \text{for } 1 \leq n \leq s \\ ne_0(I) - (s - \lambda(R/K)) & \text{for } n > s. \end{cases} \end{aligned}$$

Therefore $P_K(I,n) = ne_0(I) - (s - \lambda(R/K))$.

2. Substituting the values of $H_K(I,n)$ from (1) we get,

$$
\begin{aligned}
&\sum_{n\geq 0} H_K(I,n)t^n \\
&\quad = \sum_{n=0}^{s}[n(e_0(I)-1)+\lambda(R/K)]t^n + \sum_{n=s+1}^{\infty}[e_0(I)n-(s-\lambda(R/K))]t^n \\
&\quad = e_0(I)\sum_{n=0}^{\infty} nt^n - \sum_{n=0}^{s} nt^n + \lambda(R/K)\sum_{n=0}^{\infty} t^n - s\sum_{n=s+1}^{\infty} t^n \\
&\quad = \frac{e_0(I)}{(1-t)^2} - \frac{e_0(I)}{(1-t)} + \frac{\lambda(R/K)-st^{s+1}}{(1-t)} - \sum_{n=0}^{s} nt^n \\
&\quad = \frac{e_0(I)}{(1-t)^2} - \frac{e_0(I)-\lambda(R/K)+st^{s+1}+\left(\sum_{n=0}^{s} nt^n\right)(1-t)}{(1-t)} \\
&\quad = \frac{e_0(I)}{(1-t)^2} - \frac{e_0(I)-\lambda(R/K)+st^{s+1}+t(1+t+\cdots+t^{s-1})-st^{s+1}}{(1-t)} \\
&\quad = \frac{e_0(I)-e_0(I)(1-t)+\lambda(R/K)(1-t)-t(1-t^s)}{(1-t)^2} \\
&\quad = \frac{\lambda(R/K)+(e_0(I)-1-\lambda(R/K))t+t^{s+1}}{(1-t)^2}.
\end{aligned}
$$

$\square$

Proposition 5.2. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal with almost minimal multiplicity with respect to K such that $\gamma(I)\geq d-1$. Let $s=r_J^K(I)$. Then*

$$\sum_{n\geq 0} H_K(I,n)t^n = \frac{\lambda(R/K)+(e_0(I)-1-\lambda(R/K))t+t^{s+1}}{(1-t)^{d+1}}.$$

Proof. We induct on d. The case $d=1$ is proved in Lemma 5.1(2). Let $d>1$. Let $x\in I\backslash KI$, such that x^* is a regular element in $G(I)$ and x^o is a regular element in $F_K(I)$. Let "$-$" denote images modulo (x). Then $\bar{I}$ is an $\bar{\mathfrak{m}}$-primary ideal with almost minimal multiplicity with respect to $\bar{K}$ in $\bar{R}$. For $n\geq 1$, consider the exact sequence

$$0\longrightarrow KI^{n+1}:x/KI^n \longrightarrow R/KI^n \xrightarrow{x} R/KI^{n+1} \longrightarrow R/(KI^{n+1}+xR)\longrightarrow 0.$$

Since x^* is regular in $G(I)$ and x^o is regular in $F_K(I)$, $KI^{n+1}:x=KI^n$ for all $n\geq 0$, by Proposition 2.1(3). Therefore $H_{\bar{K}}(\bar{I},n)=\Delta H_K(I,n)$ for

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

all $n \geq 1$. By induction

$$\sum_{n\geq 0} H_{\bar{K}}(\bar{I}, n)t^n = \frac{\lambda(\bar{R}/\bar{K}) + (e_0(\bar{I}) - 1 - \lambda(\bar{R}/\bar{K}))t + t^{s+1}}{(1-t)^d},$$

where $s = r_{\bar{J}}^{\bar{K}}(\bar{I}) = r_J^K(I)$. Therefore

$$\sum_{n\geq 0} H_K(I, n)t^n = \frac{\lambda(R/K) + (e_0(I) - 1 - \lambda(R/K))t + t^{s+1}}{(1-t)^{d+1}}.$$

∎

As a corollary, we recover a result of Rossi, [24, Corollary 3.8(2)].

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal with $\lambda(I^2/JI) = 1$ for some minimal reduction J of I with reduction number r. Then*

$$H(G(I), t) := \sum_{n\geq 0} \lambda(I^n/I^{n+1})t^n = \frac{\lambda(R/I) + (e_0(I) - 1 - \lambda(R/I))t + t^r}{(1-t)^d}.$$

Proof. By Corollary 1.7 of [25], $\gamma(I) \geq d - 1$. Put $K = I$ in Proposition 5.2. Then we get

$$\sum_{n\geq 0} \lambda(R/I^{n+1})t^n = \frac{\lambda(R/I) + (e_0(I) - 1 - \lambda(R/I))t + t^r}{(1-t)^{d+1}}.$$

Multiplying both sides by $(1-t)$, we get

$$\sum_{n\geq 0} \lambda(I^n/I^{n+1})t^n = \frac{\lambda(R/I) + (e_0(I) - 1 - \lambda(R/I))t + t^r}{(1-t)^d}.$$

∎

We end this paper by characterizing the Cohen-Macaulay fiber cones of ideals with almost minimal multiplicity in the following proposition.

Proposition 5.4. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal with almost minimal multiplicity with respect to $K \supseteq I$ and $\gamma(I) \geq d - 1$. Let $s = r_J^K(I)$. Then $F_K(I)$ is Cohen-Macaulay if and only if $\lambda(KI^n + JI^{n-1}/JI^{n-1}) = 1$ for all $n = 1, \ldots, s$.*

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

Proof. Since $\gamma(I) \geq d-1$, by Theorem 4.2 of [20], we know that $F_K(I)$ is Cohen-Macaulay if and only if $g_1 = \sum_{n\geq 1} \lambda(KI^n + JI^{n-1}/JI^{n-1}) - \lambda(R/K)$. From Proposition 4.1.9 of [2] and Proposition 5.2, we get that $g_1 = r_J^K(I) - \lambda(R/K)$. Therefore, $F_K(I)$ is Cohen-Macaulay if and only if

$$r_J^K(I) - \lambda(R/K) = \sum_{n\geq 1} \lambda(KI^n + JI^{n-1}/JI^{n-1}) - \lambda(R/K).$$

From Lemma 4.2, it follows that $\lambda(KI^n + JI^{n-1}/JI^{n-1}) \leq 1$ for all $n \geq 1$. Therefore $F_K(I)$ is Cohen-Macaulay if and only if $r_J^K(I) = \sum_{n\geq 1} \lambda(KI^n + JI^{n-1}/JI^{n-1})$ if and only if $\lambda(KI^n + JI^{n-1}/JI^{n-1}) = 1$ for all $n = 1, \ldots, r_J^K(I)$. ∎

References

[1] S. Abhyankar, *Local rings of high embedding dimension*, Amer. J. Math., **89** (1967), 1073–1077.

[2] W. Bruns and J. Herzog, Cohen-Macaulay rings, Revised Edition, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, 1998.

[3] J. Chuai, *Generalised parameter ideals in local C-M rings*, Algebra Colloq., **3** (1996), no. 3, 213–216.

[4] R. C. Cowsik and M. V. Nori, *Fibers of blowing up*, J. Indian Math. Soc., **40** (1976), 217–222.

[5] T. Cortadellas and S. Zarzuela, *On the depth of the fiber cone of filtrations*, J. Algebra, **198** (1997), no. 2, 428–445.

[6] C. D'Cruz, K. N. Raghavan and J. K. Verma, *Cohen-Macaulay fiber cones*, Commutative Algebra, Algebraic Geometry and Computational Methods (Hanoi, 1996), Springer, Singapore (1999), pp. 233–246.

[7] C. D'Cruz, J. K. Verma, *Hilbert series of fiber cones of ideals with almost minimal mixed multiplicity*, J. Algebra, **251** (2002), 98–109.

[8] J. Elias, *On the depth of the tangent cone and the growth of the Hilbert function*, Trans. Amer. Math. Soc., **351** (1999), 4027–4042.

[9] D. Eisenbud and B. Mazur, *Evolutions, symbolic squares, and Fitting ideals*, J. Reine Angew. Math., **488** (1997), 189–201.

[10] S. Goto, *Cohen-Macaulayness and negativity of A-invariants in Rees algebras associated to $\mathfrak{m}$-primary ideals of minimal multiplicity*, Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998), J. Pure Appl. Algebra, **152** (2000), no. 1-3, 93–107.

[11] A. Guerrieri and M. E. Rossi, *Hilbert coefficients of Hilbert filtrations*, J. Algebra, **199** (1998), 40–61.

[12] R. Hübl, *Evolutions and valuations associated to an ideal*, Jour. Reine angew. Math., **517** (1999), 81–101.

[13] R. Hübl and C. Huneke, *Fiber cones and the integral closure of ideals*, Collect. Math., **52** (2001), no. 1, 85–100.

[14] S. Huckaba, *Reduction numbers for ideals of higher analytic spread*, Math. Proc. Camb. Phil. Soc., **102** (1987), 49–57.

[15] S. Huckaba, *A d-dimensional extension of a lemma of Huneke's and formulas for the Hilbert coefficients*, Proc. Amer. Math. Soc., **124** (1996), no. 5, 1393–1401.

[16] H. Hironaka, *Resolution of singularities*, Ann. of Math., **79** (1964), 109–326.

[17] S. Huckaba and T. Marley, *Hilbert coefficients and depth of associated graded rings*, J. London Math. Soc. (2), **56** (1997), 64–76.

[18] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J., **34** (1987), no. 2, 293–318.

[19] A. V. Jayanthan, B. Singh and J. K. Verma, *Hilbert coefficients and depth of form rings*, to appear, Comm. Algebra.

[20] A. V. Jayanthan and J. K. Verma, *Hilbert coefficients and depth of fiber cones*, to appear, J. Pure and Applied Algebra.

[21] B. Johnston and J. K. Verma, *On the length formula of Hoskin and Deligne and associated graded rings of two-dimensional regular local rings*, Math. Proc. Cambridge Philos. Soc., **111** (1992), 423–432.

[22] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge Philos. Soc., **50** (1954), 145–158.

[23] A. Ooishi, *On the Gorenstein property of the associated graded ring and the Rees algebra of an ideal*, J. Algebra, **155** (1993), 397–414.

[24] M. E. Rossi, *Primary ideals with good associated graded ring*, J. of Pure and Appl. Algebra, **145** (2000), 75–90.

[25] M. E. Rossi, *A bound on the reduction number of a primary ideal*, Proc. Amer. Math. Soc., **128** (2000), no. 5, 1325–1332.

[26] M. E. Rossi and G. Valla, *A conjecture of J. Sally*, Comm. Algebra, **24** (1996), no. 13, 4249–4261.

[27] M. E. Rossi and G. Valla, *Cohen-Macaulay local rings of embedding dimension* $e+d-3$, Proc. London Math. Soc., **80** (2000), 107–126.

[28] J. D. Sally, *Cohen-Macaulay local rings of embedding dimension* $e+d-2$, J. Algebra, **83** (1983), 393–408.

[29] J. D. Sally, *Tangent cones at Gorenstein singularities*, Comp. Math., **40** (1980), 167–175.

[30] K. Shah, *On the Cohen-Macaulayness of the Fiber Cone of an Ideal*, J. Algebra, **143** (1991), 156–172.

[31] B. Singh, *A numerical criterion for the permissibility of a blowing-up*, Compositio Math., **33** (1976), 15–28.

[32] G. Valla, *On form rings which are Cohen-Macaulay*, J. Algebra, **58** (1979), 475–481.

[33] P. Valabrega and G. Valla, *Form rings and regular sequences*, Nagoya Math. J., **72** (1978), 93–101.

[34] H.-J. Wang, *On Cohen-Macaulay local rings with embedding dimension* $e+d-2$, J. Algebra, **190** (1997), no. 1, 226–240.

https://doi.org/10.1017/S0027763000009089 Published online by Cambridge University Press

[35] H.-J. Wang, *Hilbert coefficients and the associated graded rings*, Proc. Amer. Math. Soc., **128** (1999), 963–973.

[36] J. Watanabe, *The Dilworth number of Artin Gorenstein rings*, Adv. Math., **76** (1989), no. 2, 194–199.

A. V. Jayanthan
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad - 211019
India
jayan@mri.ernet.in

J. K. Verma
Department of Mathematics
Indian Institute of Technology
Bombay, Powai, Mumbai - 400076
India
jkv@math.iitb.ac.in