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RIGHT INVERSES OF VECTOR FIELDS

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Abstract

D. Przeworska-Rolewicz developed an algebra-based theory around linear, not necessarily continuous, operators $D : X \to X$ which admit a right inverse, the elementary example being $D = d/dt$ or, more generally, $D = \sum_{i=1}^{m} a^i \partial/\partial x^i$ where a_i are constants. We give conditions for the right invertibility of D in the case where a^i are functions, or more generally, where D is the Lie or covariant derivative associated with a vector field on a (Banach) manifold M.

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1. Introduction

Let M be a smooth (that is, C^∞) Banach manifold, and v a smooth vector field on M. Denote by $\mathscr{F}(M)$ the $\mathbb{R}$-algebra of smooth real valued functions on M. The vector field v can be identified with its Lie derivative, an $\mathbb{R}$-linear map $\pounds_v : \mathscr{F}(M) \to \mathscr{F}(M)$. Recall that $\pounds_v f$ is also written as vf which in coordinates — assuming $\dim M = m$ — can be expressed as $(vf)(x) = \sum_{i=1}^{m} v^i(x)\partial_i f(x)$ whenever $v(x) = \sum_{i=1}^{m} v^i(x)\partial_i$.

In general, if $T_s^r(M)$ denotes the bundle of r-contravariant and s-covariant tensors on M, and $\mathscr{T}_s^r(M)$ its $\mathbb{R}$-vector space of smooth sections, that is, (r, s)-tensors on M, then the Lie derivative associated with $v \in \mathscr{T}(M) \equiv \mathscr{T}_0^1(M)$ is an $\mathbb{R}$-linear self-map $\pounds_v : \mathscr{T}_s^r(M) \to \mathscr{T}_s^r(M)$ for any pair (r, s), $r, s = 0, 1, 2 \ldots$.

Finally, if M is the base of a vector bundle $E(M)$ with a connection, then for each $v \in \mathscr{T}(M)$ the covariant derivative is an $\mathbb{R}$-linear self-map $\nabla_v : \mathscr{E}(M) \to \mathscr{E}(M)$ of the space of smooth sections of $E(M) \to M$.

We shall be concerned with the problem of right invertibility of $\pounds_v$ and ∇_v starting with the case of functions: Given a vector field $v \in \mathscr{T}(M)$, does the associated Lie derivative $\pounds_v : \mathscr{F}(M) \to \mathscr{F}(M)$ admit $\mathbb{R}$-linear self-maps $R \equiv R_v : \mathscr{F}(M) \to$

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$\mathcal{F}(M)$ such that $\mathcal{L}_v \circ R$ is the identity on $\mathcal{F}(M)$? The motivation for this problem can be found in [3] from where we recall some basic definitions.

If X is an $\mathbb{R}$-vector space, End(X) the $\mathbb{R}$-vector space of its $\mathbb{R}$-linear self-maps $X \to X$, define for each $D \in \mathrm{End}(X)$ the subspace $\mathrm{Right}_D(X) \subset \mathrm{End}(X)$ of its right inverses, that is, of such $R \in \mathrm{End}(X)$ for which $D \circ R = \mathrm{id}_X$. Note that in the quoted book D does not have to be defined on the whole of X, but for our purposes this simplified situation suffices. If $D \in \mathrm{End}(X)$ is such that $\mathrm{Right}_D(X) \neq \emptyset$, call the subspace $\mathrm{Ker} D \subset X$ its *space of constants*. An *initial operator* for D is a map $F \in \mathrm{End}(X)$ satisfying $F^2 = F$ and $\mathrm{Im} F = \mathrm{Ker} D$. Denote by $\mathrm{Init}_D(X) \subset \mathrm{End}(X)$ the subspace of initial operators for D. Given any $F \in \mathrm{Init}_D(X)$, c is a constant (of D) if and only if $c = F(c)$. The initial operator F for D is said to *correspond to* $R \in \mathrm{Right}_D(X)$ if moreover $F \circ R = 0$. This condition defines an $\mathbb{R}$-isomorphism $\mathcal{I} : \mathrm{Right}_D(X) \to \mathrm{Init}_D(X)$ given explicitly by $\mathcal{I}(R) \equiv F = \mathrm{id}_X - R \circ D$ or $\mathcal{I}^{-1}(F) \equiv R = R_1 - F \circ R_1$, where $R_1 \in \mathrm{Right}_D(X)$ is arbitrary.

2. Conditions for right invertibility

We shall apply this situation to $X = \mathcal{F}(M)$ and $D = \mathcal{L}_v$, where M is a Banach manifold modelled on the Banach space E. Thus let v be a smooth vector field on M, and let $\Phi : \mathbb{R} \times M \rightsquigarrow M$ be the flow associated with v. In other words, Φ is defined on an open subset $\bigcup_{x \in M} [I_x \times \{x\}] \subset \mathbb{R} \times M$, where $I_x = (\alpha_x, \beta_x) \subset \mathbb{R}$ is an open interval for each $x \in M$ and satisfies $\Phi(0, x) = x$ and $\Phi(t + s, x) = \Phi(t, \Phi(s, x))$ for each $x \in M$, and $t, s, t + s \in I_x$. Also $I_{\Phi(t,x)} = I_x - t$. Association with v means that $(\mathcal{L}_v f)(x) = (\partial/\partial t) f(\Phi(t, x))|_{t=0}$ for $f \in \mathcal{F}(M)$. It follows that $(\mathcal{L}_v f)(\Phi(t, x)) = (\partial/\partial t) f(\Phi(t, x))$ for any $t \in I_x$. $\mathrm{Ker} D \subset \mathcal{F}(M)$ consists of functions which are constant along the trajectories $t \longmapsto \Phi(t, x)$, that is, are first integrals of the corresponding dynamical system.

First observe that if D admits a right inverse R, then $f = R(1)$ gives $Df = \mathcal{L}_v(f) = 1$, hence necessarily v is a nowhere zero vector field. Thus if D admits a right inverse then the associated dynamical system must be non-singular, and its trajectories form a [codimension $m - 1$ if $\dim M = m$] foliation of M.

Put $\mathcal{F}_D(M) = \{r \in \mathcal{F}(M) : Dr = -1 \text{ and } r(x) \in I_x \text{ for all } x \in M\}$. Observe that $Dr = -1$ means $(\partial/\partial t) r(\Phi(t, x)) = -1$ and so

$$r(\Phi(t, x)) = r(x) - t \quad \text{for any} \quad r \in \mathcal{F}_D(M), \quad x \in M, \quad t \in I_x. \tag{2.1}$$

THEOREM 2.1. *Let $v \in \mathcal{T}(M)$. Then for each $r \in \mathcal{F}_D(M)$ the formula*

$$(R_r f)(x) = \int_{r(x)}^{0} f(\Phi(\tau, x))\, d\tau \tag{2.2}$$

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defines a right inverse of $D : \mathscr{F}(M) \to \mathscr{F}(M)$, that is, an $\mathbb{R}$-linear $R_r : \mathscr{F}(M) \to \mathscr{F}(M)$ such that $D \circ R_r$ is the identity on $\mathscr{F}(M)$. In particular, $R_r(-1) = r$. Any other right inverse R of D satisfies

$$(Rf)(x) = \int_{r(x)}^{0} f(\Phi(\tau, x))\, d\tau + (Rf)(\pi_r(x)), \tag{2.3}$$

where $\pi_r(x) = \Phi(r(x), x)$.

PROOF. Let $f \in \mathscr{F}(M)$. We have $(DR_r f)(x) = (\partial/\partial t)(R_r f)(\Phi(t,x))|_{t=0} =$

$$\frac{\partial}{\partial t}\int_{r(\Phi(t,x))}^{0} f(\Phi(\tau, \Phi(t,x)))\, d\tau\bigg|_{t=0} = \frac{\partial}{\partial t}\int_{r(\Phi(t,x))}^{0} f(\Phi(t+\tau, x))\, d\tau\bigg|_{t=0}.$$

Substituting $s = t + \tau$ and using (2.1) we obtain

$$(DR_r f)(x) = \frac{\partial}{\partial t}\int_{r(x)}^{t} f(\Phi(s,x))\, ds|_{t=0} = f(\phi(t,x))|_{t=0} = f(x).$$

This proves the first part. In general, the condition $DRf = f$ means that we have $(\partial/\partial t)(Rf)(\Phi(t,x)) = f(\Phi(t,x))$ for each fixed $x \in M$ and all $t \in I_x$. Antidifferentiation gives

$$(Rf)(\Phi(t,x)) - (Rf)(x) = \int_0^t f(\Phi(\tau, x))\, d\tau.$$

Substitution $t = r(x)$ gets (2.3).

THEOREM 2.2. *Let $v \in \mathscr{T}(M)$ and $r \in \mathscr{F}_D(M)$. Then $\pi_r(x) = \Phi(r(x), x)$ satisfies*

(a) *$\pi_r : M \to M$ is constant along trajectories, and $r \circ \pi_r = 0$;*
(b) *the subset $N_r = \{x \in M : x = \pi_r(x)\} = \{x \in M : r(x) = 0\}$ of M is a regular submanifold of codimension one transversal to each trajectory;*
(c) *π_r is a projection onto $N_r \subset M$, that is, $\pi_r^2 = \pi_r$.*

PROOF. (a) $\pi_r(\Phi(t,x)) = \Phi(r(\Phi(t,x)), \Phi(t,x))$. By (2.1) this is $\Phi(r(x) - t, \Phi(t,x)) = \Phi(r(x), x) = \pi_r(x)$. Also, $r(\Phi(r(x), x)) = r(x) - r(x)$ again by (2.1). (b) If $r(x) = 0$ then $\pi_r(x) = \Phi(0, x) = x$, and conversely, if $x = \pi_r(x)$ then $r(x) = r(\pi_r(x)) = 0$. To see that $N_r \subset M$ is a regular submanifold it suffices to show that $d_x r \neq 0$ at each $x \in N_r$. This follows from the fact that $-1 = (Dr)(x) = \langle d_x r, v(x)\rangle$ at each $x \in M$. This relation also shows that $v(x)$ is not in the tangent plane to N_r at x, which implies transversality. (c) Follows from (b) and the fact that $r \circ \pi_r = 0$. This completes the proof.

We shall call $N_r \subset M$ the *initial submanifold* corresponding to $r \in \mathscr{F}_D(M)$. Observe that Theorem 2.2 implies that $\Phi(t,x) \in N_r$ if and only if $t = r(x)$. In particular, none of the trajectories $t \longmapsto \Phi(t,x)$ is periodic. It also follows that there is a surjective submersion $p_r : M \to N_r$ such that $\pi_r = i_r \circ p_r$, where $i_r : N_r \to M$ is the natural embedding.

Thus each $r \in \mathscr{F}_D(M)$ defines a submanifold N_r transversal to the trajectories of Φ as described in Theorem 2.2. Also the converse is true.

THEOREM 2.3. *Let $v \in \mathscr{T}(M)$, admit a regular submanifold $N \subset M$ which is transversal to each trajectory of v, and has the property that each trajectory crosses N exactly once, that is,*

(2.4) *for each $x \in M$ there is a unique $r(x) \in I_x$ such that $\Phi(t,x) \in N$ if and only if $t = r(x)$.*

Then $r \in \mathscr{F}_D(M)$, hence D admits a right inverse.

PROOF. The function $r : M \to \mathbb{R}$ is well defined and satisfies $r(\Phi(t,x)) = r(x) - t$ because $\Phi(r(x),x) = \Phi(r(x)-t, \Phi(t,x))$ for $t \in I_x$. It follows that $(\partial/\partial t)r(\Phi(t,x)) = -1$ which means $Dr = -1$, and so it remains to show that r is smooth. Since for each $x_0 \in M$, $x \longmapsto \Phi_0(x) \equiv \Phi(r(x_0),x)$ is a C^∞-diffeomorphism from a neighbourhood of x_0 onto a neighbourhood of $\Phi(r(x_0),x_0)$ and $r \circ \Phi_0 = r - r(x_0)$, it suffices to show that r is smooth in a neighbourhood of any $y \in N$. Since $\Phi : \mathbb{R} \times M \rightsquigarrow M$ is smooth and $N \subset M$ is a regular submanifold, then also $\Phi_N : \mathbb{R} \times N \rightsquigarrow M$ is smooth in a neighbourhood of $(0,y) \in \mathbb{R} \times N$. Because N is transversal to the trajectory through $y \in N$ we have $T(M)_y = T(N)_y + \operatorname{Im}T(\Phi_y)_0 = \operatorname{Im}T(\Phi_N)_{(0,y)}$, which shows that $T(\Phi_N)_{(0,y)}$ is an isomorphism, toplinear in case of Banach manifolds, (cf. [2, p. 29]). By the inverse function theorem $\Phi_N : \mathbb{R} \times N \rightsquigarrow M$ is therefore a local diffemorphism from a neighbourhood of $(0,y)$ onto a neighbourhood of y and so it suffices to verify that $r \circ \Phi_N : \mathbb{R} \times N \rightsquigarrow \mathbb{R}$ is smooth in a neighbourhood of $(0,y)$. The last statement is obvious, since for $x \in N$ and $t \in I_x$ we have $(r \circ \Phi_N)(t,x) = -t$. This completes the proof.

In this sense there is a one-to-one correspondence between elements of $\mathscr{F}_D(M)$ and regular submanifolds $N \subset M$ satisfying (2.4), further referred to as *initial submanifolds* for D. We have therefore

COROLLARY 2.4. *Let $N \subset M$ be an initial submanifold for D. If R_1 and R_2 are two right inverses of D which coincide on N—that is, $(R_1 f)(x) = (R_2 f)(x)$ for any $f \in \mathscr{F}(M)$, $x \in N$—then $R_1 = R_2$.*

COROLLARY 2.5. *Let $r \in \mathscr{F}_D(M)$. The general form of a right inverse R of D is given by $Rf = R_r f + (Hf) \circ p_r$ for some $\mathbb{R}$-linear $H : \mathscr{F}(M) \to \mathscr{F}(N_r)$.*

PROOF. It follows from Theorem 2.1 that the general form of R is $R = R_r + K$, where $K \in \mathrm{End}(\mathscr{F}(M))$ takes values in $\mathrm{Ker}D$, that is, Kf is constant along trajectories, that is, $Kf = Kf \circ \pi_r$ for all $f \in \mathscr{F}(M)$. Clearly, $f \longmapsto f|_{N_r}$ defines an isomorphism $\mathrm{Ker}D \to \mathscr{F}(N_r)$ whose inverse is $g \longmapsto g \circ p_r$ and so Kf can also be written as $(Hf) \circ p_r$ for some $\mathbb{R}$-linear $H : \mathscr{F}(M) \to \mathscr{F}(N_r)$.

If D admits a right inverse $R \in \mathrm{Right}_D(\mathscr{F}(M))$ then $r = R(-1)$ satisfies $Dr = -1$, but we cannot conclude that $r \in \mathscr{F}_D(M)$, that is, that $r(x) \in I_x$ unless v is complete, that is, $I_x = \mathbb{R}$ for all $x \in M$. However, if M is paracompact, (and the Banach space E on which M is modelled admits smooth partitions of unity subordinate to any locally finite cover), there is a nowhere zero function $\rho \in \mathscr{F}(M)$ such that ρv is a global vector field on M, whose flow Φ^* is equivalent to that of v, that is, is a reparametrisation of Φ (cf. [4]). The last statement means that there is a smooth $t^* : \bigcup_{x \in M}[I_x \times \{x\}] \to \mathbb{R}$ such that for each $x \in M$ the map $t_x^* \equiv t^*(.,x) : I_x \to \mathbb{R}$ is a smooth diffeomorphism, and $\Phi(t,x) = \Phi^*(t_x^*(t), x)$. Writing D^* for $\mathcal{L}_{\rho v} = \rho D$ we see that $r^* = R(-1/\rho)$ satisfies $D^* r^* = -1$ which implies $r^* \in \mathscr{F}_{D^*}(M)$ because ρv was a global vector field. Therefore by Theorems 2.1 and 2.2, the flow Φ^* must admit a transversal submanifold $N = N_{r^*}$ such that for each $x \in M$, $\Phi^*(t', x) = \Phi(t_x^{*\,-1}(t'), x) \in N$ if and only if $t' = r^*(x)$. Thus Φ has the property described in (2.4) with $r(x) = t_x^{*\,-1}(r^*(x))$. We have proved

THEOREM 2.6. *Let $v \in \mathscr{T}(M)$. Then D admits a right inverse if and only if it admits an initial submanifold, that is, a regular submanifold $N \subset M$ which has the property that each trajectory crosses N transversally and exactly once as described in* (2.4).

The initial operator corresponding to $R \in \mathrm{Right}_D(X)$ is defined in [3] as $\mathscr{I}(R) \equiv F = \mathrm{id}_X - R \circ D$. If R is referred to some $r \in F_D(M)$ as in (2.3), then this gives $(Ff)(x) = f(x) - \int_{r(x)}^0 (\partial/\partial\tau) f(\Phi(\tau,x)\, d\tau - (RDf)(\pi_r(x)) = (f - RDf)(\pi_r(x))$ or $[f|_{N_r} - (H \circ D)f] \circ p_r$. In particular, the initial operator corresponding to $R = R_r$ is given by $F_r f = f \circ \pi_r$. In other words, $(F_r f)(x)$ 'is the value of f at the point where the trajectory through x intersects N_r'. Note that $F_r r = 0$ and that (2.3) can be written as $R = R_r + F_r \circ R$, which is in fact the formula for $\mathscr{I}^{-1}(F_r)$ from [3].

EXAMPLE 1. If $M = (a,b) \subset \mathbb{R}$, $Df = f'$ then $\Phi(t,x) = x+t$, $I_x = (a-x, b-x)$ and $Dr = -1$ means $r(x) = -x + c$ and so $r \in \mathscr{F}_D(M)$ if and only if $a < c < b$, in which case $N_r = \{c\}$, $\pi_r : x \longmapsto c$, $(R_r f)(x) = \int_c^x f(t)\, dt$, and $(F_r f)(x) = f(c)$. The general right inverse must satisfy $(Rf)(x) = \int_{c-x}^0 f(x+\tau)\, d\tau + (Rf)(c)$, or, by

Corollary 2.5, it must be given by $(Rf)(x) = \int_{c-x}^{0} f(x+\tau)\,d\tau + Hf$, where H is an arbitrary $\mathbb{R}$-linear map from $\mathscr{F}(M)$ into $\mathbb{R}$. In particular, $Hf = f(x_0)$ for some $x_0 \in (a, b)$.

EXAMPLE 2. If $M = \mathbb{R}^m$ and $v = a \neq 0$ is a constant vector, that is, $D = \sum_{i=1}^{m} a^i \partial/\partial x^i$ is a directional derivative on $\mathbb{R}^m$, where the coefficients a^i are constants, then $\Phi(t, x) = x + ta$ and $r_0(x) = -\sum_{i=1}^{m} a^i x^i / \sum_{i=1}^{m} (a^i)^2$ is one element of $\mathscr{F}_D(M)$. Any other element $r \in \mathscr{F}_D(M)$ must be of the form $r_0 + s$, where $Ds = 0$, that is, s is constant along the trajectories $t \longmapsto x + ta$. The intial operator F_r is given by $(F_r f)(x) = f(x + r(x)a)$. Observe that $N_0 = \{x \in M : x = x + r_0(x)a\} = \{x \in \mathbb{R}^m : r_0(x) = 0\}$ is the hyperplane through origin perpendicular to a, hence the corresponding π_0 is the perpendicular projection of $\mathbb{R}^m$ onto this N_0. For a general $r \in \mathscr{F}_D(M)$ the submanifold $N_r \subset \mathbb{R}^m$ is a hypersurface intersecting transversally each trajectory $t \longmapsto x + ta$. Formula (2.3) gives then the general form of a right inverse of D as

$$(Rf)(x) = \int_{r_0(x)}^{0} f(x+\tau)\,d\tau + (Rf)(\Psi_0(x)) = \int_{r_0(x)}^{0} f(x+\tau)\,d\tau + (Hf)(p_{r_0}(x)), \tag{2.5}$$

where H is an $\mathbb{R}$-linear map $\mathscr{F}(\mathbb{R}^m) \to \mathscr{F}(N_0)$.

In particular, if a is the first coordinate vector, that is, $D = \partial/\partial x^1$, then $r_0(x) = -x^1$ and $\mathscr{F}_D(M) = \{r \in \mathscr{F}(\mathbb{R}^m) : r(x) = -x^1 + s(x^2, \dots, x^m), s \in \mathscr{F}(\mathbb{R}^{m-1})\}$.

Przeworska-Rolewicz (cf. [3]) defines the definite integral $I_\alpha^\beta : X \to \operatorname{Ker} D$ determined by the initial operators F_α and F_β by $I_\alpha^\beta = F_\beta \circ R - F_\alpha \circ R$ and shows that this is independent of the choice of $R \in \operatorname{Right}_D(X)$, hence can also be expressed as $F_\beta \circ R_\alpha$. It is not hard to see that in our case—where the initial operators F_i are determined by $r_i \in \mathscr{F}_D(M)$, $i = 1, 2$—the definite integral 'from F_1 to F_2' is simply

$$R_{r_1} \circ \pi_{r_2} : x \longmapsto \int_{r_1(\Phi(r_2(x), x))}^{0} f(\Phi(\tau, \Phi(r_2(x), x)))\,d\tau = \int_{r_1(x)}^{r_2(x)} f(\Phi(\tau, x))\,d\tau.$$

Exponentials, defined as solutions of $Dy = \lambda y$ for $\lambda \in \mathbb{R}$, are functions $y \in \mathscr{F}(M)$ satisfying $y(\Phi(t, x)) = y(x)e^{\lambda t}$ for $x \in M, t \in I_x$. In particular, any such exponential is uniquely determined by its values on an initial submanifold $N \subset M$.

The result of Example 2.2.2 in [3] can also be generalized.

THEOREM 2.7. *Let $v \in \mathscr{T}(M)$, $r \in \mathscr{F}_D(M)$. Then R_r given by* (2.2) *is a Volterra right inverse, that is, the operator* $\mathrm{id}_X - \lambda R_r$ *is invertible for any $\lambda \in \mathbb{R}$, its inverse being* $\mathrm{id}_X + \lambda B_r$, *where the operator B_r is given by*

$$(B_r f)(x) = \int_{r(x)}^{0} e^{-\lambda\sigma} f(\Phi(\sigma, x))\,d\sigma. \tag{2.6}$$

PROOF. We shall only verify $(\mathrm{id}_X + \lambda B_r)(\mathrm{id}_X - \lambda R_r) = \mathrm{id}_X$, the other equality following similarly. We have

$$\begin{aligned}(\mathrm{id}_X + \lambda B_r)(\mathrm{id}_X - \lambda R_r) f(x) &= f(x) + \lambda[(B_r f)(x) - (R_r f)(x)] - \lambda^2 (B_r R_r f)(x) \\ &= f(x) + \lambda \int_{r(x)}^{0} (e^{-\lambda\sigma} - 1) f(\Phi(\sigma, x))\, d\sigma \\ &\quad -\lambda^2 \int_{r(x)}^{0} e^{-\lambda\sigma} \left(\int_{r(\Phi(\sigma,x))}^{0} f(\Phi(\tau + \sigma, x))\, d\tau \right) d\sigma \\ &= f(x) + \lambda \int_{r(x)}^{0} (e^{-\lambda\sigma} - 1) f(\Phi(\sigma, x))\, d\sigma \\ &\quad -\lambda^2 \int_{r(x)}^{0} e^{-\lambda\sigma} \left(\int_{r(x)}^{\sigma} f(\Phi(s, x))\, ds \right) d\sigma \\ &= f(x) + \lambda \int_{r(x)}^{0} (e^{-\lambda\sigma} - 1) f(\Phi(\sigma, x))\, d\sigma \\ &\quad -\lambda^2 \int_{r(x)}^{0} f(\Phi(s, x)) e^{-\lambda\sigma} \left(\int_{s}^{0} e^{-\lambda\sigma}\, d\sigma \right) ds \\ &= f(x).\end{aligned}$$

This completes the proof.

3. The Lie derivative

Turning to the more general case of a Lie derivative, let λ be a differentiable functor on Banach spaces ([2, p. 54]), in particular that of r-contravariant and s-covariant tensors. Denote by $T_\lambda(M) = \lambda(T(M))$ the vector bundle of tensors of type λ ([2, p. 109]), in particular $T_\lambda(M) = T_s^r(M)$, and by $\mathscr{T}_\lambda(M)$ the $\mathbb{R}$-vector space of its smooth sections. For each smooth diffeomorphism $F : M \to M$, and each $x \in M$, denote by $T_\lambda(F)_x : T_\lambda(M)_x \to T_\lambda(M)_{F(x)}$ the corresponding linear isomorphism. If η is a tensor field of type λ, that is, $\eta \in \mathscr{T}_\lambda(M)$, and $v \in \mathscr{T}(M)$ as before, then

$$(3.1)\qquad (D\eta)(x) \equiv (\mathcal{L}_v \eta)(x) = \frac{\partial}{\partial t} \left[T_\lambda(\Phi_{-t})_{\Phi(t,x)} \eta(\Phi(t, x)) \right] \Big|_{t=0}$$

defines the Lie derivative of η with respect to v as an $\mathbb{R}$-linear self-map $\mathcal{L}_v : T_\lambda(M) \to T_\lambda(M)$ (cf. [2, p. 109]).

THEOREM 3.1. *Let $v \in \mathscr{T}(M)$. Then for each $r \in \mathscr{F}_D(M)$ the formula*

$$(3.2)\qquad (R_{\lambda;r}\eta)(x) = \int_{r(x)}^{0} T_\lambda(\Phi_{-\tau})_{\Phi(\tau,x)} \eta(\Phi(\tau, x))\, d\tau$$

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defines a right inverse of $D \equiv \mathcal{L}_v : \mathcal{T}_\lambda(M) \to \mathcal{T}_\lambda(M)$, *that is, an* $\mathbb{R}$*-linear* $R_{\lambda;r} : \mathcal{T}_\lambda(M) \to \mathcal{T}_\lambda(M)$ *such that* $D \circ R_{\lambda;r}$ *is the identity on* $\mathcal{T}_\lambda(M)$.

PROOF. We have

$$(DR_{\lambda;r}\eta)(x) = \frac{\partial}{\partial t}[T_\lambda(\Phi_{-t})_{\Phi(t,x)}(R_{\lambda;r}\eta)(\Phi(t,x))]|_{t=0}$$

which is

$$\frac{\partial}{\partial t}\left[T_\lambda(\Phi_{-t})_{\Phi(t,x)}\int_{r(\Phi(t,x))}^{0} T_\lambda(\Phi_{-\tau})_{\Phi(\tau,\Phi(t,x))}\eta(\Phi(\tau,\Phi(t,x)))\,d\tau\right]\Bigg|_{t=0}.$$

Substituting $s = t + \tau$ in the integral and using $r(\Phi(t,x)) = r(x) - t$ we obtain

$$\begin{aligned}(DR_{\lambda;r}\eta)(x) &= \frac{\partial}{\partial t}\left[T_\lambda(\Phi_{-t})_{\Phi(t,x)}\int_{r(x)}^{t} T_\lambda(\Phi_{t-s})_{\Phi(s,x)}\eta(\Phi(s,x))\,ds\right]\Bigg|_{t=0}\\ &= \frac{\partial}{\partial t}\left[\int_{r(x)}^{t} T_\lambda(\Phi_{-s})_{\Phi(s,x)}\eta(\Phi(s,x))\,ds\right]\Bigg|_{t=0}\\ &= \left[T_\lambda(\Phi_{-t})_{\Phi(t,x)}\eta(\Phi(t,x))\right]\Big|_{t=0}\\ &= \eta(x).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(RD\eta)(x) &= \int_{r(x)}^{0} T_\lambda(\Phi_{-\tau})_{\Phi(\tau,x)}(D\eta)(\Phi(\tau,x))\,d\tau\\ &= \int_{r(x)}^{0}\frac{\partial}{\partial t}\left[T_\lambda(\Phi_{-\tau})_{\Phi(\tau,x)}T_\lambda(\Phi_{-t})_{\Phi(t+\tau,x)}\eta(\Phi(t+\tau,x))\right]\Big|_{t=0}\,d\tau\\ &= \int_{r(x)}^{0}\frac{\partial}{\partial s}\left[T_\lambda(\Phi_{-s})_{\Phi(s,x)}\eta(\Phi(s,x))\right]\Big|_{s=\tau}\,d\tau\\ &= T_\lambda(\Phi_{-s})_{\Phi(s,x)}\eta(\Phi(s,x))\Big|_{s=r(x)}^{s=0}\\ &= \eta(x) - T_\lambda(\Phi_{-r(x)})_{\pi_r(x)}\eta(\pi_r(x)).\end{aligned}$$

The initial operator corresponding to this $r \in \mathcal{F}_D(M)$ is $F = \mathrm{id} - RD$, that is, it is given by

(3.3) $$(F\eta)(x) = T_\lambda(\Phi_{-r(x)})_{\pi_r(x)}\eta(\pi_r(x))$$

which is the value of η at the point where the trajectory through x meets the initial submanifold N, 'transported to the fibre at x via the infinitesimal transformation v'.

In particular, if η is a smooth vector field w then $Dw = [v, w]$ and (3.2) gives an expression which can be written as $R_{\lambda;r}w \equiv Rw = \int_0^{r(x)}(\Phi_s)_* w\,ds$, using the notation $(\Phi_s)_* w : x \longmapsto T(\Phi_s)_{\Phi(-s,x)}w(\Phi(-s,x))$, (cf. [1, p. 10]). Note that w is a constant with respect to D if and only if $[v, w] = 0$, that is, the flow of w commutes with Φ.

4. The covariant derivative

The situation is similar in the case of the covariant derivative associated with $v \in \mathscr{T}(M)$. For simplicity, we shall restrict ourselves to finite dimensional smooth manifolds. Thus let $E(M)$ be a vector bundle with a connection. Let $X = \mathscr{E}(M)$ be the $\mathbb{R}$-vector space of smooth sections of $E(M) \to M$. The covariant derivative $D = \nabla_v : \mathscr{E}(M) \to \mathscr{E}(M)$ associated with this connection is given by (cf. [1, p. 114])

$$(4.1)\qquad (D\eta)(x) \equiv (\nabla_v \eta)(x) = \frac{\partial}{\partial t}\left[h_0^t(x)^{-1}\eta(\Phi(t,x))\right]\Big|_{t=0},$$

where $h_0^t(x) : E_x \to E_{\Phi(t,x)}$ denotes the parallel displacement of fibres of $E = E(M) \to M$ along the path $\tau \longmapsto \Phi(\tau, x)$. Observe that each $h_0^t(x)$ is an isomorphism and that $h_0^s(\Phi(t,x)) \circ h_0^t(x) = h_0^{t+s}(x)$.

THEOREM 4.1. *Let $v \in \mathscr{T}(M)$. Then for each $r \in \mathscr{F}_D(M)$ the formula*

$$(4.2)\qquad (R_{\mathscr{E};r}\eta)(x) = \int_{r(x)}^0 h_0^\tau(x)^{-1}\eta(\Phi(\tau,x))\,d\tau$$

defines a right inverse of $D \equiv \nabla_v : \mathscr{E}(M) \to \mathscr{E}(M)$, that is, an $\mathbb{R}$-linear $R_{\mathscr{E};r} : \mathscr{E}(M) \to \mathscr{E}(M)$ such that $D \circ R_{\mathscr{E};r}$ is the identity on $\mathscr{E}_\lambda(M)$.

PROOF. We have

$$(DR_{\mathscr{E};r}\eta)(x) = \frac{\partial}{\partial t}\left[h_0^t(x)^{-1}(R_{\mathscr{E};r}\eta)(\Phi(t,x))\right]\Big|_{t=0}$$

which is

$$\frac{\partial}{\partial t}\left[h_0^t(x)^{-1}\int_{r(\Phi(t,x))}^0 h_0^\tau(\Phi(t,x))^{-1}\eta(\Phi(\tau+t,x))\,d\tau\right]\Bigg|_{t=0}.$$

Substituting $s = t + \tau$ in the integral, using $r(\Phi(t,x)) = r(x) - t$ and the fact that $h_0^t(x)^{-1} \circ h_0^{s-t}(\Phi(t,x))^{-1} = h_0^s(x)^{-1}$ we obtain $(DR_{\mathscr{E};r}\eta)(x) = \eta(x)$ similarly as in the proof of Theorem 3.1.

The same is true about the formula for the initial operator $F_{\mathscr{E};r} = \mathrm{id} - R_{\mathscr{E};r}D$ associated with $r \in \mathscr{F}_D(M)$, namely

$$(4.3)\qquad (F_{\mathscr{E};r}\eta)(x) = h_0^{r(x)}(\pi_r(x))^{-1}\eta(\pi_r(x)),$$

which is the value of η at the point where the trajectory through x meets the initial submanifold N, 'displaced parallelly along the trajectory $t \longmapsto \Phi(t,x)$ to the fibre at x'.

References

[1] S. Kobayashi and K. Nomizu, *Foundations of differential geometry I* (Interscience, New York, 1963).
[2] S. Lang, *Differential manifolds* (Addison-Wesley, Reading, 1972).
[3] D. Przeworska-Rolewicz, *Algebraic analysis* (PWN, Warsaw / Reidel Dordrecht, 1988).
[4] P. L. Renz, 'Equivalent flows on smooth Banach manifolds', *Indiana Univ. Math. J.* **20** (1971), 695–698.

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