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ONE DIMENSIONAL FIBERING OVER q-COMPLETE SPACES

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Abstract. We show that if $E \longrightarrow X$ is a locally trivial holomorphic fibrations whose fiber is an open Riemann surface and X is a q-complete space, then E is q-complete.

§1. Introduction

Let X and F be complex manifolds and E a holomorphic fiber bundle over X with typical fiber F. In 1953 Serre [7] posed the following question related to the classical Levi problem for characterizing domains of holomorphy:

($\star$) *Assume that X and F are Stein. Does it follow that E is Stein, too?*

Several particular cases of this were settled (cf. [8] for summaries) until Skoda [9] produced in 1977 a counterexample with fiber $\mathbf{C}^2$, which, however, did not stop the interest around the problem. Mok [5] solved completely the case when F is a complex curve. On the other hand, new questions have appeared, *e.g.*, to study cohomological properties of such an E as in ($\star$). In this direction, Jennane [4] showed that the cohomology of E with coefficients in coherent analytic sheaves is trivial in dimensions ≥ 2. Furthermore, we established a general vanishing theorem for locally q-complete morphisms over p-complete spaces [12]. (The normalization is chosen so that Stein spaces correspond to 1-complete spaces.)

In this paper, by reconsidering the geometrical point of view, we extend Mok's result to the case X is q-complete by proving the following theorem which answers a question raised to the author by Professor Takeo Ohsawa at the Conference on complex analysis in Hayama, Japan in the spring of 1995.

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THEOREM 1. *Let $\pi: E \to X$ be a locally trivial holomorphic fiber space with typical fiber F. Assume that F is an open Riemann surface. If X is q-complete, then E is q-complete, too.*

Note that for fibers of dimensions ≥ 2 there are counterexamples [12]. In fact, for every integer $q > 1$ there is a fiber bundle E with fiber $\mathbf{C}^2$ over a q-complete domain in $\mathbf{C}^{q+1}$ such that $H^q(E, \mathcal{O}_E)$ does not vanish, *a fortiori* E is not q-complete [1].

The method we use in proving Theorem 1 yields also the subsequent interesting criterion of q-completeness which was suggested to me by Professor Mihnea Colţoiu and may be viewed, in a weak sense, as a theorem of Docquier-Grauert type [3] for q-complete manifolds.

THEOREM 2. *Let X be a weakly q-complete complex space such that on every relatively compact open subset there are continuous strongly plurisubharmonic functions. Then X is q-complete.*

§2. Preliminaries

Throughout this paper all complex spaces are assumed to be reduced and with countable topology. By an *open Riemann surface* we mean a non-singular complex curve without compact components.

Let X be a complex space and T_xX denotes the (Zariski) tangent space of X at x. Set $TX = \cup_{x\in X} T_xX$. If $X = \mathbf{C}^n$, T_xX is canonically identified with $\mathbf{C}^n$.

A (local) chart of X at a point $x \in X$ is a holomorphic embedding $\iota: U \to \widehat{U}$, where $U \ni x$ is an open subset of X and $\widehat{U}$ is an open subset of some euclidean space $\mathbf{C}^n$. Holomorphic embedding means that $\iota(U)$ is an analytic subset of $\widehat{U}$ and the induced map $\iota: U \to \iota(U)$ is biholomorphic.

Suppose $\iota: U \to \widehat{U}$ is a local chart at x; then the differential map $\iota_*: T_xX \to \mathbf{C}^n$ of ι at x is an injective homomorphism of complex vector spaces.

Let $D \subset \mathbf{C}^n$ be an open subset. A function $\varphi \in C^\infty(D, \mathbf{R})$ is said to be *q-convex* if the quadratic form

$$L(\varphi, z)(\xi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(z)\xi_i\bar{\xi}_j, \quad \xi \in \mathbf{C}^n,$$

has at least $n - q + 1$ positive eigenvalues for every $z \in D$, or equivalently, there exists a family $\{M_z\}_{z\in D}$ of $(n - q + 1)$-dimensional complex vector subspaces of $\mathbf{C}^n$ such that $L(\varphi, z)_{|M_z}$ is a positive definite form for all $z \in D$.

Let X be a complex space. A function $\varphi \in C^\infty(X, \mathbf{R})$ is said to be *q-convex* if every point of X admits a local chart $\iota: U \to \widehat{U} \subset \mathbf{C}^n$ such that there is an extension $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbf{R})$ of $\varphi_{|U}$ which is q-convex on $\widehat{U}$. (This definition does not depend on the local embeddings.)

We say that X is *q-complete* if there exists a q-convex exhaustion function $\varphi \in C^\infty(X, \mathbf{R})$.

A subset $\mathcal{M} \subset TX$ is said to be *a linear set over X (of codimension less than q)* if for every point $x \in X$, $\mathcal{M}_x := \mathcal{M} \cap T_xX \subset T_xX$ is a complex vector subspace (of codimension less than q). If $W \subset X$ is an open set, we have an obvious definition of the restriction $\mathcal{M}_{|W}$. The following is due to Peternell [6].

DEFINITION. Let X be a complex space, $W \subset X$ an open set, $\mathcal{M}$ a linear set over W, and $\varphi \in C^\infty(W, \mathbf{R})$.

(a) Let $x \in W$. Then we say that φ is *weakly 1-convex with respect to $\mathcal{M}_x$* if there are: a local chart $\iota: U \to \widehat{U}$ of X with $x \in U \subset W$, $\widehat{U} \subset \mathbf{C}^n$ open set, and an extension $\widehat{\varphi} \in C^\infty(\widehat{U}, \mathbf{R})$ of $\varphi_{|U}$ such that $L(\widehat{\varphi}, \iota(x))(\iota_*\xi) \geq 0$ for every $\xi \in \mathcal{M}_x$.

We say that φ is *weakly 1-convex with respect to $\mathcal{M}$* if φ is weakly 1-convex with respect to $\mathcal{M}_x$ for every $x \in W$.

(b) The function φ is said to be *1-convex with respect to $\mathcal{M}$* if every point of W admits an open neighborhood $U \subset W$ such that there exists a 1-convex function θ on U with $\varphi - \theta$ weakly 1-convex with respect to $\mathcal{M}_{|U}$.

It is not difficult to see that the extension $\widehat{\varphi}$ of φ is irrelevant for the above definition. In particular, if the functions φ and ψ are (weakly) 1-convex with respect to $\mathcal{M}$, so is their sum $\varphi + \psi$.

On the other hand, a complex space X is q-complete if, and only if, there exists a linear set $\mathcal{M}$ over X of codimension less than q and an exhaustion function $\varphi \in C^\infty(X, \mathbf{R})$ which is 1-convex with respect to $\mathcal{M}$.

Motivated by this observation, we call a complex space X *weakly q-complete* if there exists a linear set $\mathcal{M}$ over X of codimension less than q and an exhaustion function $\varphi \in C^\infty(X, \mathbf{R})$ which is weakly 1-convex with respect to $\mathcal{M}$.

Here, to avoid some technical difficulties in the proofs of the theorems, we introduce the following

DEFINITION. Let X be a complex space and $\mathcal{M}$ a linear set over X. We say that $\varphi \in C^0(X, \mathbf{R})$ is *(weakly) $\mathcal{M}$-convex* if, and only if, every point of X

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admits an open neighborhood D such that there are functions $f_1, \ldots, f_k \in C^\infty(D, \mathbf{R})$, (weakly) 1-convex with respect to $\mathcal{M}$, and $g_1, \ldots, g_k \in \mathrm{Psh}(D) \cap C^0(D, \mathbf{R})$ with

$$(\clubsuit) \qquad \varphi_{|D} = \max\{f_1 + g_1, \ldots, f_k + g_k\}.$$

For instance, if φ is weakly $\mathcal{M}$-convex and ψ is $\mathcal{M}$-convex, then their sum $\varphi + \psi$ is $\mathcal{M}$-convex. Besides, if $h: \mathbf{R} \to \mathbf{R}$ is given locally by $\max\{a_1 t + b_1, \ldots, a_k t + b_k\}$ with $a_i > 0$ for all i, then $h(\varphi)$ is again weakly $\mathcal{M}$-convex.

Remark. In [11] we introduced continuous functions $\varphi: X \to \mathbf{R}$ which we called *pseudoconvex with respect to* $\mathcal{M}$, *i.e.*, one has ($\clubsuit$) without g_i's. Clearly, this notion is stronger than $\mathcal{M}$-convexity.

Therefore, the following approximation result improves [11], Theorem 1.

PROPOSITION 1. *Let $\mathcal{M}$ be a linear set over a complex space X and $\varphi \in C^0(X, \mathbf{R})$ a $\mathcal{M}$-convex function. Then for every $\eta \in C^0(X, \mathbf{R})$, $\eta > 0$, there exists $\widetilde{\varphi} \in C^\infty(X, \mathbf{R})$ which is 1-convex with respect to $\mathcal{M}$ and*

$$\varphi \leq \widetilde{\varphi} < \varphi + \eta.$$

In particular, if $\mathcal{M}$ has codimension less than q, then $\widetilde{\varphi}$ is q-convex.

Proof. Choose a locally finite covering $\{U_i\}_{i \in I}$ of X by relatively compact open Stein subsets on which there are functions f_{ij}, g_{ij}, $j = 1, \ldots, n_i$, as in ($\clubsuit$). Then consider open sets $V_i \subset\subset U_i$ and compact sets $K_i \subset V_i$ such that $\{K_i\}_{i \in I}$ is again a covering of X. Let θ_i be smooth functions on X which equal -1 on ∂V_i and 1 on K_i. Choose $\delta_i > 0$ small enough such that $f'_{ij} := \delta_i \theta_i + f_{ij}$, $j = 1, \ldots, n_i$, are 1-convex with respect to $\mathcal{M}$ on V_i, and $2\delta_i < \inf_{V_i} \eta$ for all i. Then choose smooth plurisubharmonic functions g'_{ij} on U_i such that $|g'_{ij} - g_{ij}| < \delta_i$ on V_i. One may define $\varphi': X \to \mathbf{R}$ by: for every $x \in X$ set

$$\varphi'(x) = \sup\{f'_{ij}(x) + g'_{ij}(x);\ i, j \text{ such that } x \in V_i\}.$$

It is straightforward to see that φ' is continuous, 1-convex with respect to $\mathcal{M}$ on X, and $\varphi < \varphi' < \varphi + \eta$. The proposition follows now by [11], Theorem 1, if we approximate φ' in the C^0-topology by $\widetilde{\varphi}$ as required.

The subsequent is proved in [5].

THEOREM 3. *If S is a connected open Riemann surface, then there exists an exhaustion function $\varphi \in \mathrm{Psh}(S) \cap C^0(S, \mathbf{R})$ such that for every automorphism θ of S, $\varphi - \varphi \circ \theta$ is a bounded function.*

§3. The proofs of the theorems

To begin with, we state a proposition, which, by taking into account the definition of a weakly q-complete space, gives immediately a proof of Theorem 2.

PROPOSITION 2. *Let $\mathcal{M}$ be a linear set over a complex space X. Suppose that:*

a) *There exists an exhaustion function $\varphi \in C^0(X, \mathbf{R})$ which is weakly $\mathcal{M}$-convex.*

b) *For every open set $D \subset\subset X$ there exists $\psi \in C^\infty(X, \mathbf{R})$ which is 1-convex with respect to $\mathcal{M}$ on D.*

Then there exists an exhaustion function $\widetilde{\varphi} \in C^\infty(X, \mathbf{R})$ which is 1-convex with respect to $\mathcal{M}$ on X.

Proof. Without any loss of generality, we may assume that $\min_X \varphi = 0$. For $n = 1, 2, \ldots$, we denote $K_n = \{x \in X\,;\, \varphi(x) \le n\}$ and $D_n = \{x \in X\,;\, \varphi(x) < n+2\}$. Let $\psi_n \in C^\infty(X, \mathbf{R})$ be $\mathcal{M}$-convex on D_{n+1}. By taking the exponential, we may suppose $\psi_n > 0$. Set

$$a_n := \max_{K_{n+2}} (\varphi + \psi_n) > 0.$$

Let $h_n \colon \mathbf{R} \to \mathbf{R}$ be defined by $h_n(t) = \max\{t, (1 + a_n)(t - n - 1)\}$, $t \in \mathbf{R}$. Clearly, h_n is strictly increasing, convex, $h_n(t) = t$ for $t \le n+1$, and $h_n(n+2) > a_n$. Therefore $h_n(\varphi) > \varphi + \psi_n$ on the set $\{n+2-\varepsilon < \varphi < n+2\}$ for some $\varepsilon > 0$ sufficiently small; consequently we may define a continuous function $\varphi_n \colon X \to \mathbf{R}$ by:

$$\varphi_n = \begin{cases} \max(\varphi + \psi_n, h_n(\varphi)) & \text{on} \quad D_n, \\ h_n(\varphi) & \text{on} \quad X \setminus D_n. \end{cases}$$

Then φ_n is positive and exhaustive since $\varphi_n \ge h_n(\varphi)$, weakly $\mathcal{M}$-convex on X, and as $\varphi_n = \varphi + \psi_n$ on K_{n+1}, φ_n is a $\mathcal{M}$-convex function on the interior of K_{n+1}.

Now, if the sequence $\{\epsilon_n\}_n$ of positive numbers decreases (fast enough) to 0, we may define an exhaustion $\mathcal{M}$-convex function $\Phi \in C^0(X, \mathbf{R})$ by

$$\Phi = \varphi + \sum \varepsilon_n \varphi_n,$$

and we conclude by Proposition 1.

Here we start the proof of Theorem 1. First assume that the fiber F is connected (for steps 1 and 2); second, the general case (step 3), will follow by a canonical reduction procedure if we note some facts on q-complete spaces.

Step 1. By Theorem 3 and standard arguments, there exists a locally finite covering $\{U_i\}_{i\in I}$ of X by relatively compact open Stein subsets which trivialize E and such that the following property holds:

(•) There are continuous plurisubharmonic exhaustion functions $\varphi_i: E_i \to \mathbf{R}$, $E_i = \pi^{-1}(U_i)$, such that for every compact set $K \subset U_i \cap U_j$, $\varphi_i - \varphi_j$ are bounded functions on $\pi^{-1}(K)$ for every indices i and j. Clearly, we may suppose $\varphi_i > 0$.

Now consider open sets $W_i \subset\subset V_i \subset\subset U_i$ such that $\cup W_i = X$; then choose non-negative smooth functions p_i' with compact support contained in V_i which equal 1 on W_i. Since on $\pi^{-1}(W_i \cap \partial V_j)$ one has: $\varphi_i - \varphi_j$ is a bounded function, $p_i' \circ \pi = 1$, and $p_j' \circ \pi = 0$, there are large enough constants $C_i > 0$ such that on $\pi^{-1}(W_i \cap \partial V_j)$ one has:

$$C_i p_i' + \varphi_i > \varphi_j = C_j p_j' + \varphi_j$$

for all indices i and j. Put $p_i := C_i p_i'$, $i \in I$, and define $u: E \to (0, \infty)$ by

$$u(\zeta) = \max\{p_i(\pi(\zeta)) + \varphi_i(\zeta);\, i \in I(\zeta)\},$$

where for $\zeta \in E$, $I(\zeta) := \{i \in I;\, \pi(\zeta) \in V_i\}$. One checks readily that u is continuous and for an arbitrary compact set $L \subset X$ the restriction of u to $\pi^{-1}(L)$ is exhaustive.

Step 2. Let $\mathcal{M} = \{\mathcal{M}_x\}_{x\in X}$ be a linear set over X of codimension less than q and $\varphi' \in C^\infty(X, \mathbf{R})$ an exhaustion function which is 1-convex with respect to $\mathcal{M}$. Select a smooth non-negative function λ from $\mathbf{R}$ into itself which is rapidly increasing and convex such that $\lambda(\varphi') + p_i$, $i \in I$, are 1-convex with respect to $\mathcal{M}$. Set $\varphi := \lambda(\varphi')$.

Now define a linear set $\mathcal{N}$ over E by $\mathcal{N}_\zeta = \pi_{*,\zeta}^{-1}(\mathcal{M}_{\pi(\zeta)})$ for $\zeta \in E$. Here $\pi_{*,\zeta}$ means the differential map of π at ζ from $T_\zeta E$ into $T_{\pi(\zeta)}X$. Obviously, $\mathcal{N}$ has codimension less than q.

Put $\sigma = \varphi \circ \pi + u$. Then σ is weakly $\mathcal{N}$-convex on E, and by what we said in Step 1, σ is exhaustive.

We claim that for every open set $\Omega \subset\subset E$ there exists a smooth function ψ on E which is 1-convex with respect to $\mathcal{N}$ on Ω.

Indeed, we cover $\overline{\pi(\Omega)}$ with finitely many W_i's, say W_i, $i = 1, \dots, m$; then choose positive 1-convex functions ψ_i on $\pi^{-1}(U_i)$. Straightforward computations show that there exists a constant $A_0 > 0$ large enough such that the function $\psi_A \in C^\infty(E, \mathbf{R})$ given by

$$\psi_A = A \cdot (\varphi \circ \pi) + \sum_{i+1}^{m} (p_i \circ \pi) \cdot \psi_i$$

is 1-convex with respect to $\mathcal{N}$ on Ω for every $A \geq A_0$. Now we conclude the proof of the theorem by applying Proposition 2.

Step 3. Here we consider the general case. In order to do this, decompose $F = \cup F_j$ so that in F_j appear only connected components isomorphic to each other and non-isomorphic to connected components of F_s, for $s \neq j$. Each F_j is invariant under transition automorphisms of E, so that E splits into a disjoint union of fiber bundles E_j with base X and fiber F_j. Thus it suffices to assume that the fiber F consists of isomorphic connected components. Then the transition automorphisms can permute the connected components of F and we have a two-step fibration $E \to \widetilde{X} \to X$ where $\widetilde{X}$ is a topological covering of X and the first fibration has a connected fiber. Since $\widetilde{X}$ is q-complete by [2], the theorem follows now from the preceding case.

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