Akihiko Morimoto
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PROLONGATIONS OF CONNECTIONS TO TANGENTIAL FIBRE BUNDLES OF HIGHER ORDER

AKIHIKO MORIMOTO

§ Introduction.

In the previous paper [3] we have studied the prolongations of G-structures to tangent bundles of higher order. The purpose of the present paper is to study the prolongations of connections to tangential fibre bundles of higher order, and to generalize the results due to S. Kobayashi [1] for the case of usual tangent bundle —— in fact, the arguments in [1] will be, in a sense, more or less simplified and clarified by using the notion of tangent bundles of higher order. In addition, as a consequence of our results, we shall obtain the prolongations of linear (affine) connections to tangent bundles of higher order.

In § 1, we construct a diffeomorphism $\alpha_M^{r,s}$ of $\overset{s}{T}\overset{r}{T}M$ onto $\overset{r}{T}\overset{s}{T}M$ and prove its naturality. In § 2, we give some properties concerning with the linear parts of zero-preserving differentiable maps of a vector space into another vector space. In § 3, we shall construct the prolongations of connections by making use of the diffeomorphism $\alpha_M^{r,s}$ and the results in § 2. In § 4, we apply the results in § 3 for the case of linear connections and construct the prolongation $\Gamma^{(r)}$ of a linear connection Γ of order r.

We shall investigate the relationships between Γ and $\Gamma^{(r)}$ in a future paper.

In this paper, we keep the same notations as in [3] and all manifolds and mappings are assumed to be differentiable of class C^∞, unless otherwise stated.

§ 1. Diffeomorphism $\alpha_M^{r,s}$.

Let M be a manifold of dimension n. For any function $f \in C^\infty(M)$ we define the ν-extension $f^{(\nu)}$ of f to $\overset{r}{T}M$ as follows:

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$$f^{(\nu)}([\varphi]_r) = \left[\frac{d^\nu(f \circ \varphi)}{dt^\nu}\right]_0 \tag{1.1}$$

for $[\varphi]_r \in \overset{r}{T}M$ and $\nu = 0, 1, \cdots, r$. It is easy to see that $f^{(\nu)} = f_r^{(\nu)}$ is a well-defined differentiable function on $\overset{r}{T}M$.

For any map $\psi: R^2 \longrightarrow M$, we difine ψ_t and ψ^u of R into M by the following equalities:

$$\psi_t(u) = \psi(t, u) = \psi^u(t)$$

for $t, u \in R$.

LEMMA 1.1. Let $\varphi: R \longrightarrow \overset{r}{T}M$ be a differentiable map of R into $\overset{r}{T}M$. Then, there exist a differentiable map $\psi: R^2 \longrightarrow M$ and a positive number δ such that $\varphi(t) = [\psi_t]_r$ holds for $|t| < \delta$.

Proof. Let $\varphi(0) = X_0 \in \overset{r}{T}M$ and $\overset{r}{\pi}X_0 = x^0 \in M$. We take a coordinate neighborhood U of x^0 with coordinate system $\{x_1, \cdots, x_n\}$ such that $x_i(x^0) = 0$ $(i = 1, \cdots, n)$ and denote by $\{\overset{(\nu)}{x}_i | i = 1, \cdots, n; \nu = 0, 1, \cdots, r\}$ the induced coordinate system on $\tilde{U} = (\overset{r}{\pi})^{-1}(U)$. Put $F_i^\nu(t) = \overset{(\nu)}{x}_i(\varphi(t))$ for t such that $\varphi(t) \in \tilde{U}$. Thus $F_i^\nu(t)$ is defined for $|t| < \varepsilon$ with some positive number ε. Let $\Phi: U \to R^n$ be the diffeomorphism of U onto $\Phi(U)$ defined by $\Phi(p) = (x_1(p), \cdots, x_n(p))$ for $p \in U$. Now, define the map $\tilde{\psi}$ as follows:

$$\tilde{\psi}(t, u) = \Phi^{-1}(\cdots, \sum_{\nu=0}^{r} F_i^\nu(t) u^\nu, \cdots)$$

for small $|t|$ and $|u|$. Since $F_i^0(t) = \overset{(0)}{x}_i \varphi(t)) = x_i(\varphi(t))$, there is, for any given $\varepsilon_1 > 0$, a positive number δ_1 such that $|\sum_{\nu=0}^{r} F_i^\nu(t) u^\nu| < \varepsilon_1$ for any t, u satisfying $|t|, |u| < \delta_1$. Therefore, the map $\tilde{\psi}$ is well-defined for any $|t|, |u| < \delta_1$ with some positive number δ_1. Now, it is easy to find a map $\psi: R^2 \to M$ such that $\psi(t, u) = \tilde{\psi}(t, u)$ for any $|t|, |u| < \delta$ with some positive number $\delta < \delta_1$. For this map ψ we have the following equalities:

$$\overset{(\nu)}{x}_i([\psi_t]_r) = \frac{1}{\nu!}\left[\frac{d^\nu(x_i \circ \psi_t)}{du^\nu}\right] = \frac{1}{\nu!}\left[\frac{d^\nu(\sum_\mu F_i^\mu(t) u^\mu)}{du^\nu}\right]_0$$
$$= F_i^\nu(t) = \overset{(\nu)}{x}_i(\varphi(t))$$

for any $|t| < \delta$, which proves $[\psi_t]_r = \varphi(t)$ for $|t| < \delta$. Q.E.D.

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Take and fix two positive integers r and s. We shall prove the following.

LEMMA 1.2. *Let* φ (*resp.* φ') *be a map of* R *into* $\overset{r}{T}M$. *We take a map* ϕ (*resp.* ϕ') *of* R^2 *into* M *and a positive number* δ *such that* $\varphi(t) = [\phi_t]_r$ *and* $\varphi'(t) = [\phi'_t]_r$ *for any* $|t| < \delta$. *Consider the map* Ψ: $R \to \overset{s}{T}M$ (*resp.* Ψ') *defined by* $\Psi(u) = [\phi^u]_s$ (*resp.* $\Psi'(u) = [\phi'^u]_s$) *for* $u \in R$. *Suppose* $\varphi \underset{s}{\sim} \varphi'$. *Then* $\Psi \underset{r}{\sim} \Psi'$ *holds.*

Proof. By the assumption, we have $\tilde{f} \circ \varphi \underset{s}{\sim} \tilde{f} \circ \varphi'$ for any differentiable function $\tilde{f}$ on $\overset{r}{T}M$. Take a function f on M, then the ν-extension $f^{(\nu)} = f^{(\nu)}_r$ of f to $\overset{r}{T}M$ is a differentiable function on $\overset{r}{T}M$ for $\nu = 0, 1, \cdots, r$. Hence we get

$$f^{(\nu)} \circ \varphi \sim f^{(\nu)} \circ \varphi' \tag{1.2}$$

for any $f \in C^\infty(M)$ and $\nu = 0, 1, \cdots, r$. In the same way, we see that we have to prove $\tilde{g} \circ \Psi \underset{r}{\sim} \tilde{g} \circ \Psi'$ for any $\tilde{g} \in C^\infty(\overset{s}{T}M)$ and hence to prove $g^{(\mu)} \circ \Psi \underset{r}{\sim} g^{(\mu)} \circ \Psi'$ for every $g \in C^\infty(M)$, where $g^{(\mu)} = g^{(\mu)}_s$ is the μ-extension of g to $\overset{s}{T}M$ with $\mu \leq s$. Now, we have the following equalities:

$$(g^{(\mu)} \circ \Psi)(u) = g^{(\mu)}(\Psi(u)) = \left[\frac{d^\mu(g \circ \phi^u)}{dt^\mu}\right]_0 = \left[\frac{d^\mu g(\phi(t, u))}{dt^\mu}\right]_{t=0}$$

Therefore, we get the following

$$\left[\frac{d^\nu(g^{(\mu)} \circ \Psi)}{du^\nu}\right]_{u=0} = \left[\frac{d^\nu}{du^\nu}\left(\left[\frac{d^\mu g\phi(t, u)}{dt^\mu}\right]_{t=0}\right)\right]_{u=0} \tag{1.3}$$
$$= \left[\frac{\partial^{\nu+\mu} g\phi(t, u)}{\partial u^\nu \partial t^\mu}\right]_{(0,0)},$$

and the similar equalities for Ψ'. On the other hand, since

$$(f^{(\nu)} \circ \varphi)(t) = f^{(\nu)}([\phi_t]_r) = [d^\nu f \circ \phi_t / du^\nu]_{u=0}$$
$= [d^\nu f\phi(t, u)/du^\nu]_{u=0}$, it follows from (1.2) that

$$\left[\frac{d^\mu(f^{(\nu)} \circ \varphi)}{dt^\mu}\right]_{t=0} = \left[\frac{d^\mu(f^{(\nu)} \circ \varphi')}{dt^\mu}\right]_{t=0} \tag{1.4}$$

and the left hand side is equal to

$$\left[\frac{d^\mu}{dt^\mu}\left(\left[\frac{d^\nu f\ \phi(t, u)}{du^\nu}\right]_{u=0}\right)\right]_{t=0},$$

the right hand side being equal to

$$\left[\frac{d^\mu}{dt^\mu}\left(\left[\frac{d^\nu f\phi'(t,u)}{du^\nu}\right]_{u=0}\right)\right]_{t=0}.$$

Therefore, from (1.3) and (1.4) we obtain

$$\left[\frac{d^\nu(g^{(\mu)}\circ\Psi)}{du^\nu}\right]_{u=0}=\left[\frac{d^\nu(g^{(\mu)}\circ\Psi')}{du^\nu}\right]_{u=0}$$

for every $\mu=0,1,\cdots,s$ and $\nu=0,1,\cdots,r$, which shows that $g^{(\mu)}\circ\Psi\underset{r}{\sim}g^{(\mu)}\circ\Psi'$ for any $\mu\leqq s$. Since $g\in C^\infty(M)$ is arbitrary, we see that $\tilde{g}\circ\Psi\underset{r}{\sim}\tilde{g}\circ\Psi'$ for every $\tilde{g}\in C^\infty(\overset{s}{T}M)$, and hence $\Psi\underset{r}{\sim}\Psi'$. Q.E.D.

THEOREM 1.3. *For any manifold M, there exists a cononical diffeomorphism $\alpha^{r,s}=\alpha_M^{r,s}$ of $\overset{s}{T}\overset{r}{T}M$ onto $\overset{r}{T}\overset{s}{T}M$ for any positive integers r, s such that the following diagrams are commutative*

$$(1)\qquad\begin{array}{ccc}\overset{s}{T}\overset{r}{T}M & \xrightarrow{\alpha^{r,s}} & \overset{r}{T}\overset{s}{T}M\\ & \tilde{\pi}\searrow \quad \swarrow \overset{r}{T}\pi & \\ & \overset{r}{T}M & \end{array}$$

$$(2)\qquad\begin{array}{ccc}\overset{s}{T}\overset{r}{T}M & \xrightarrow{\alpha^{r,s}} & \overset{r}{T}\overset{s}{T}M\\ & \overset{s}{T}\pi'\searrow \quad \swarrow \tilde{\pi}' & \\ & \overset{s}{T}M & \end{array}$$

where π, $\tilde{\pi}$, π' and $\tilde{\pi}'$ are natural projections,

$$(3)\qquad\begin{array}{ccc}\overset{s}{T}\overset{r}{T}N & \xrightarrow{\alpha^{r,s}} & \overset{r}{T}\overset{s}{T}M\\ & (f_r^{(\mu)})_s^{(\nu)}\searrow \quad \swarrow (f_s^{(\nu)})_r^{(\mu)} & \\ & R & \end{array}$$

for any $f\in C^\infty(M)$ with any $\mu=0,1,\cdots,r$ and $\nu=0,1,\cdots,s$.

Proof. Take an element $[\varphi]_s$ of $\overset{s}{T}\overset{r}{T}M$, where φ is a map of R into $\overset{r}{T}M$. By Lemma 1.1., there is a map $\phi:R^2\to M$ and a positive number δ such that $\varphi(t)=[\phi_t]_r$ holds for $|t|<\delta$. Consider the map $\Psi:R\to\overset{s}{T}M$ defined by $\Psi(u)=[\phi^u]_s$ for $u\in R$. Then, by Lemma 1.2, we see that the r-tangent $[\Psi]_r\in\overset{r}{T}\overset{s}{T}M$ is independent of the choice of the map ϕ. Therefore, we can define the map $\alpha_M^{r,s}$ of $\overset{s}{T}\overset{r}{T}M$ into $\overset{r}{T}\overset{s}{T}M$ by

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$$\alpha_M^{r,s}([\varphi]_s) = [\Psi]_r.$$

By this very definition of $\alpha_M^{r,s}$ we see that $\alpha_M^{s,r}([\Psi]_r) = [\varphi]_s$ and hence we have the following relation

$$\alpha_M^{s,r} \circ \alpha_M^{r,s} = 1_{\overset{s}{T}\overset{r}{T}M} \tag{1.5}$$

for any positive integers r and s.

To prove the defferentiability of the map $\alpha_M^{r,s}$, we first prove the commutativity of the diagram (3). Keeping the above notations we calculate as follows:

$$\begin{aligned}
(f^{(\nu)})^{(\mu)} \circ \alpha_M^{r,s}([\varphi]_s) &= (f^{(\nu)})^{(\mu)}([\Psi]_r) = \left[\frac{d^\mu(f^{(\nu)} \circ \Psi)}{du^\mu}\right]_0 \\
&= \left[\frac{d^\mu((f^{(\nu)} \circ \Psi)(u))}{du^\mu}\right]_0 = \left[\frac{d^\mu(f^{(\nu)}([\phi^u]_s))}{du^\mu}\right]_0 \\
&= \left[\frac{d^\mu}{du^\mu}\left(\left[\frac{d^\nu f \cdot \phi^u(t)}{dt^\nu}\right]_{t=0}\right)\right]_{u=0} = \left[\frac{\partial^{\mu+\nu} f(\phi(t,u))}{\partial u^u \partial t^\nu}\right]_{(0,0)} \\
&= \left[\frac{d^\nu}{dt^\nu}\left(\left[\frac{d^\mu f\phi(t,u)}{du^\mu}\right]_{u=0}\right)\right]_{t=0} = \left[\frac{d^\nu}{dt^\nu}\left(\left[\frac{d^\mu(f \circ \phi t)}{du^\mu}\right]_{u=0}\right)\right]_{t=0} \\
&= \left[\frac{d^\nu f^{(\mu)}([\phi_t]_r)}{dt^\nu}\right]_0 = \left[\frac{d^\nu f^{(\mu)}(\varphi(t))}{dt^\nu}\right]_0 \\
&= (f^{(\mu)})^{(\nu)}([\varphi]_s)
\end{aligned}$$

for every $[\varphi]_s \in TTM$, and hence we get $(f^{(\nu)})^{(\mu)} \circ \alpha_M^{r,s}([\varphi]_s) = (f^{(\mu)})^{(\nu)}([\varphi]_s)$, which shows the commutativity of the diagram (3). Since f is an arbitrary function in $C^\infty(M)$, the commutativity of (3) implies that $\tilde{f} \circ \alpha_M^{r,s}$ is always a differentiable function on $\overset{s}{T}\overset{r}{T}M$ for every $\tilde{f} \in C^\infty(\overset{r}{T}\overset{s}{T}M)$, which proves that the map $\alpha_M^{r,s}$ is differentiable. From (1.5) it follows that $\alpha_M^{r,s}$ is a diffeomorphism of $\overset{s}{T}\overset{r}{T}M$ onto $\overset{r}{T}\overset{s}{T}M$.

Next we shall prove the commutativity of the diagram (1). Take an element $[\varphi]_s$ of $\overset{s}{T}\overset{r}{T}M$ with $\varphi : R \to \overset{r}{T}M$. By making use of the above notations we have $\alpha_M^{r,s}([\varphi]_s) = [\Psi]_r$. Since $(\pi \circ \Psi)(u) = \pi([\phi^u]_s) = \phi^u(0) = \phi(0,u) = \phi_0(u)$, we get $\pi \circ \Psi = \phi_0$. Therefore, we have $(\overset{r}{T}\pi)([\Psi]_r) = [\pi \circ \Psi]_r = [\phi_0]_r$. On the other hand, we have $\tilde{\pi}([\varphi]s) = \varphi(0) = [\phi_0]_r$. Hence we obtain $\overset{r}{T}\pi(\alpha_M^{r,s}([\varphi]_s)) = (\overset{r}{T}\pi)([\Psi]_r) = \tilde{\pi}([\varphi]_s)$, which proves the commutativity of (1).

Finally we shall prove the commutativity of the diagram (2). Keeping the notations as above, we have $\tilde{\pi}'([\Psi]_r) = \Psi(0) = [\phi^0]_s$. On the other hand, we calculate as follows: $(\pi' \circ \varphi)(t) = \pi'(\varphi(t)) = \pi'([\phi_t]_r) = \phi_t(0) = \phi(t,0) = \phi^0(t)$

and hence we have $\pi' \circ \varphi = \psi^0$. Therefore, we get $(\overset{s}{T}\pi')([\varphi]_s) = [\pi' \circ \varphi]_s = [\psi^0]_s$ and hence we obtain $\tilde{\pi}'(\alpha_M^{r,s}([\varphi]_s)) = \tilde{\pi}'([\Psi]_r) = (\overset{s}{T}\pi')([\varphi]_s)$, which shows the commutativity of (2). Q.E.D.

Corollary 1.4. *For any positive integers r and s we have the following equality:*

$$\alpha_M^{r,s} \circ \alpha_M^{s,r} = 1_{\overset{r}{T}\overset{s}{T}M},$$

in particular, $\alpha_M^{r,r} \circ \alpha_M^{r,r} = 1_{\overset{r}{T}\overset{r}{T}M}$.

Remark 1.5. The diffeomorphism $\alpha_M^{r,s}$ is, in fact, characterized by the commutativity of the diagram (3). We also note that $\alpha_M^{1,1}$ is the same automorphism as α_M in [1].

Proposition 1.6. *Let $\Phi : M \to N$ be a map of a manifold M into a manifold N. Then, we have the following commutative diagram:*

(1.6)
$$\begin{tikzcd}
\overset{s}{T}\overset{r}{T}M \arrow[r, "\alpha_M^{r,s}"] \arrow[d, "\overset{s}{T}\overset{r}{T}\Phi"] & \overset{r}{T}\overset{s}{T}M \arrow[d, "\overset{r}{T}\overset{s}{T}\Phi"] \\
\overset{s}{T}\overset{r}{T}N \arrow[r, "\alpha_N^{r,s}"] & \overset{r}{T}\overset{s}{T}N.
\end{tikzcd}$$

Proof. Take, as before, an element $[\varphi]_s$ of $\overset{s}{T}\overset{r}{T}M$, where $\varphi : R \to \overset{r}{T}M$ is a map. By Lemma 1.1 there is a map $\psi : R^2 \to M$ and a positive number δ such that $\varphi(t) = [\psi_t]_r$ for $|t| < \delta$. Putting $\Psi(u) = [\psi^u]_s$, we have $\alpha_M^{r,s}([\varphi]_s) = [\Psi]_r$ and hence

(1.7)
$$\overset{r}{T}\overset{s}{T}\Phi(\alpha_M^{r,s}([\varphi]_s)) = [(\overset{s}{T}\Phi) \circ \Psi]_r.$$

Now, we have $\overset{s}{T}\overset{r}{T}\Phi([\varphi]_s) = [(\overset{r}{T}\Phi) \circ \varphi]_s$. We define $\theta : R^2 \to N$ by $\theta = \Phi \circ \psi$. Then, we can calculate as follows: $((\overset{r}{T}\Phi)\circ\varphi)(t) = \overset{r}{T}\Phi(\varphi(t)) = \overset{r}{T}\Phi([\psi_t]_r) = [\Phi\circ\psi_t]_r = [\theta_t]_r$ for $|t| < \delta$. Putting $\Theta(u) = [\theta^u]_s$, we have, by the very definition of the map $\alpha_N^{r,s}$,

(1.8)
$$\alpha_N^{r,s}(\overset{s}{T}\overset{r}{T}\Phi([\varphi]_s)) = [\Theta]_r.$$

We shall show the following

(1.9)
$$(\overset{s}{T}\Phi) \circ \Psi = \Theta.$$

For, we compute as follows: $((\overset{s}{T}\Phi)\circ\Psi)(u)=\overset{s}{T}\Phi(\Psi(u))=\overset{s}{T}\Phi([\psi^u]_s)=[\Phi\circ\psi^u]_s=[\theta^u]_s=\Theta(u)$. Finally form $(1.7)\sim(1.9)$ the commutativity of (1.6) follows. Q.E.D.

Let G be a Lie group. We know that $\overset{r}{T}G$ is again a Lie group with the natural group multiplication (cf. [3] § 2).

Proposition 1.7. *Let G be a Lie group. Then, the diffeomorphism $\alpha_G^{r,s}:\overset{s}{T}\overset{r}{T}G\to\overset{r}{T}\overset{s}{T}G$ is an isomorphism of Lie groups.*

Proof. Take two elements $[\varphi]_s$, $[\varphi']_s$ of $\overset{s}{T}\overset{r}{T}G$, where φ, φ' are maps of R into $\overset{r}{T}G$. By Lemma 1.1, we find maps ψ, ψ' of R_2 into G such that $\varphi(t)=[\psi_t]_r$, $\varphi'(t)=[\psi'_t]_r$ for small t. Put $\Psi(u)=[\psi^u]_s$ and $\Psi'(u)=[\psi'u]_s$. By the definition of $\alpha_G^{r,s}$, we get the following

$$\alpha_G^{r,s}([\varphi]_s)=[\Psi]_r,\ \alpha_G^{r,s}([\varphi']_s)=[\Psi']_r. \tag{1.10}$$

Now, we have $(\varphi\cdot\varphi')(t)=\varphi(t)\cdot\varphi'(t)=[\psi_t]_r\cdot[\psi'_t]_r=[\psi\cdot\psi'_t]_r=[\psi''_t]_r$, where we have put $\psi''(t,u)=\psi(t,u)\cdot\psi'(t,u)$. Putting $\Psi''(u)=[\psi''^u]_s$, we have

$$\alpha_G^{r,s}([\varphi\cdot\varphi']_s)=[\Psi'']_r. \tag{1.11}$$

Since $[\varphi\cdot\varphi']_s=[\varphi]_s\cdot[\varphi']_s$ and since $\Psi''(u)=[\psi''^u]_s=[\psi^u\cdot\psi'^u]_s=[\psi^u]_s\cdot[\psi'^u]_s=\Psi(u)\cdot\Psi'(u)=(\Psi\cdot\Psi')(u)$, it follows, from (1.10), (1.11), that $\alpha_G^{r,s}([\varphi]_s\cdot[\varphi']_s)=\alpha_G^{r,s}([\varphi]_s)\cdot\alpha_G^{r,s}([\varphi']_s)$, which shows that $\alpha_G^{r,s}$ is a homomorphism and hence an isomorphism. Q.E.D.

§ 2. Linear parts of zero preserving mappings.

Definition 2.1. Let V be a finite dimensional real vector space. We define the map θ_V of V into the tangent space $T_0(V)$ to V at zero as follows:

$$\theta_V(V)=[\lambda_v]_1.$$

for $v\in V$, where the map $\lambda_v:R\to V$ is defined by $\lambda_v(t)=t\cdot v$ for $t\in R$. It is trivial to see that θ_V is a linear isomorphism of V onto $T_0(V)$.

Definition 2.2. Let V and W be vector spaces and let $\varphi:V\to W$ be a differentiable map such that $\varphi(0)=0$. Define the linear map $\varphi^0:V\to W$ by $\varphi^0=\theta_W^{-1}\circ(T_0\varphi)\circ\theta_V$, where $T_0\varphi$ is the tangent to φ (or the differential of φ) at zero, i.e. the following diagram is commutative:

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$$\begin{array}{ccc} V & \xrightarrow{\varphi^0} & W \\ {\scriptstyle\theta_V}\downarrow & & \downarrow{\scriptstyle\theta_W} \\ T_0(V) & \xrightarrow{T_0\varphi} & T_0(W). \end{array}$$

The linear map φ^0 will be called the linear part of the map φ.

LEMMA 2.3. *Let $\varphi : V \to W$ be a linear map of V into W. Then, $\varphi = \varphi^0$.*

Proof. Since φ is linear, we see that $\varphi \circ \lambda_v = \lambda_{\varphi(v)}$ for any $v = V$. Therefore, we calculate as follows: $\varphi^0(v) = \theta_W^{-1} \circ (T_0\varphi) \circ \theta_V(v) = \theta_W^{-1}((T_0\varphi)([\lambda_v]_1)) = \theta_W^{-1}([\varphi \circ \lambda_v]_1) = \theta_W^{-1}([\lambda_{\varphi(v)}]_1) = \theta_W^{-1}(\theta_W(\varphi(v))) = \varphi(v)$ for $v \in V$, and hence we get $\varphi = \varphi^0$. Q.E.D.

LEMMA 1.4. *Let U, V, W be vector spaces, and let $\varphi : U \to V$ and $\psi : V \to W$ be differentiable maps such that $\varphi(0) = 0$ and $\psi(0) = 0$. Then we have the following*

$$(\psi \circ \varphi)^0 = \psi^0 \circ \varphi^0. \tag{2.2}$$

Proof. Since $T_0(\psi \circ \varphi) = (T_0\psi) \circ (T_0\varphi)$, the equality (2.2) follows from the following commutative diagram:

$$\begin{array}{ccccc} U & \xrightarrow{\varphi^0} & V & \xrightarrow{\psi^0} & W \\ {\scriptstyle\theta_U}\downarrow & & {\scriptstyle\theta_V}\downarrow & & {\scriptstyle\theta_W}\downarrow \\ T_0(U) & \xrightarrow[T_0\varphi]{} & T_0(V) & \xrightarrow[T_0\psi]{} & T_0(W). \end{array}$$

Q.E.D.

LEMMA 2.5. *Let $\varphi : V \to W$ be a map of V into W. Suppose there is a vector subspace V_1 of V such that the restriction φ_1 of φ to V_1 is a linear map of V_1 into W, in particular $\varphi(0) = 0$. Then, we have the following*

$$\varphi^0 | V_1 = \varphi_1. \tag{2.3}$$

Proof. Let $\eta : V_1 \to V$ be the inclusion map. Since $\varphi_1 = \varphi \circ \eta$, it follows from Lemma 2.3 and 2.4 that $\varphi_1 = \varphi_1^0 = (\varphi \circ \eta)^0 = \varphi^0 \circ \eta^0 = \varphi^0 \circ \eta = \varphi^0 | V_1$.

Q.E.D.

§ 3. Prolongations of connections.

Let $P(M, \pi, G)$ be a principal fibre bundle with bundle space P, base M, projection π and structure group G. Consider a connection on P whose connection form will be denoted by ω. For each point $u \in P$, ω is a linear map of the tangent space $T_u(P)$ to P at u with values in the tangent space $T_e(G)$ to G at unit element e. This form ω can be considered as a differentiable map of TP into TG satisfying the following conditions (cf. [1]),

$$\omega(u\cdot\bar{s}) = s^{-1}\cdot\bar{s}, \tag{3.1}$$

$$\omega(\bar{u}\cdot s) = s^{-1}\cdot\omega(\bar{u})\cdot s \tag{3.2}$$

for every $u\in P$, $s\in G$, $\bar{u}\in T_u(P)$ and $\bar{s}\in T_s(G)$, where $\bar{u}\cdot s$ is, by definition, $(TR_s)(\bar{u})$ with right translation $R_s: P\to P$ and where $u\cdot\bar{s}$ is defined as follows: Consider the map $L_u: G\to P$ defined by $L_u(a) = u\cdot a = R_a(u)$ for $a\in G$, then $u\cdot\bar{s}$ is defined to be $(TL_u)(\bar{s})$.

DEFINITION 3.1. Let $\omega: TP\to TG$ be a connection form on a principal fibre bundle $P(M,\pi,G)$ as above. Define the map $\omega_r: T\overset{r}{T}P\to T\overset{r}{T}G$ as follows:

$$\omega_r = \alpha_G^{1,r}\circ(\overset{r}{T}\omega)\circ\alpha_P^{r,1}. \tag{3.3}$$

Let $\overset{r}{T}P(\overset{r}{T}M, \overset{r}{T}\pi, \overset{r}{T}G)$ be the tangential principal fibre bundle of order r to the bundle $P(M,\pi,G)$ (cf. [3] §6). We shall prove, for ω_r, equalities similar to (3.1) and (3.2).

LEMMA 3.2. *Let $\tilde{u}\in\overset{r}{T}P$, $\tilde{s}\in\overset{r}{T}G$ and $\bar{\bar{s}}\in T_{\tilde{s}}(\overset{r}{T}G)$. Then we have the following equality:*

$$\omega_r(\tilde{u}\cdot\bar{\bar{s}}) = \tilde{s}^{-1}\cdot\bar{\bar{s}}; \tag{3.4}$$

in particular, $\omega_r(0_{\tilde{u}}) = 0$ for any zero tangent vector $0_{\tilde{u}}$ at $\tilde{u}$.

Proof. Since $\bar{\bar{s}}$ is an element of $\overset{1}{T}\overset{r}{T}G$, there exist, by Lemma 1.1, maps $\varphi: R\to\overset{r}{T}G$ and $\psi: R^2\to G$ such that $\bar{\bar{s}} = [\varphi]_1$, $\varphi(0) = \tilde{s} = [\psi_0]_r$ and $\varphi(t) = [\psi_t]_r$ for small t. Define $\psi': R^2\to G$ by $\psi'(t,u) = \psi(0,u)^{-1}\cdot\psi(t,u)$ for $t,u\in R$. Consider the left translation $L_{\tilde{s}}$ of $\overset{r}{T}G$ by the element $\tilde{s}\in\overset{r}{T}G$, i.e. $L_{\tilde{s}}(\tilde{x}) = \tilde{s}\cdot\tilde{x}$ for $\tilde{x}\in\overset{r}{T}G$. Then we have the following equalities: $(L_{\tilde{s}^{-1}}\circ\varphi)(t) = L_{\tilde{s}^{-1}}(\varphi(t)) = \tilde{s}^{-1}\cdot\varphi(t) = [\psi_0^{-1}]_1\cdot[\psi_t]_1 = [\psi_0^{-1}\psi_t]_1 = [\psi'_t]_1$. Put $\Psi'(u) = [\psi'^u]_1$ for $u\in R$. Since we have $\tilde{s}^{-1}\cdot\bar{\bar{s}} = \overset{1}{T}L_{\tilde{s}^{-1}}(\bar{\bar{s}}) = \overset{1}{T}L_{\tilde{s}^{-1}}(([\varphi]_1) = [L_{\tilde{s}^{-1}}\circ\varphi]_1$, by the definition of $\alpha_G^{r,1}$, we get the following relation

$$\alpha_G^{r,1}(s^{-1}\cdot\bar{\bar{s}}) = [\Psi']_r. \tag{3.5}$$

On the other hand, since $\tilde{u}$ is an element of $\overset{1}{T}P$, there is a map $\eta : R \to P$ such that $\tilde{u} = [\eta]_1$. Define $\eta' : R^2 \to P$ by $\eta'(t, u) = \eta(u) \cdot \phi(t, u)$ for $t, u \in R$. Then we have the following equalities $(L_{\tilde{u}} \circ \varphi)(t) = \tilde{u} \cdot \varphi(t) = [\eta]_r \cdot [\phi_t]_r = [\eta \cdot \phi_t]_r = [\eta'_t]_r$ for small t. Put $Y'(u) = [\eta'^u]_1$ for $u \in R$. Then, since we have $\tilde{u} \cdot \bar{s} = \overset{1}{T}L_{\tilde{u}}(\bar{s}) = \overset{1}{T}L_{\tilde{u}}([\varphi]_1) = [L_{\tilde{u}} \circ \varphi]_1$, we get, by the definition of $\alpha_P^{r,1}$, $\alpha_P^{r,1}(\tilde{u} \cdot \bar{s}) = [Y']_r$. Therefore, we obtain the following relation

$$(3.6) \qquad (\overset{r}{T}\omega \circ \alpha_P^{r,1})(\tilde{u} \cdot \bar{s}) = [\omega \circ Y']_r.$$

Now, since we have $\eta'^u(t) = \eta'(t, u) = \eta(u) \cdot \phi(t, u) = \eta(u) \cdot \phi^u(t)$, we get $[\eta'^u]_1 = [L_{\eta(u)} \circ \phi^u]_1 = \overset{1}{T}L_{\eta(u)}([\phi^u]_1) = \eta(u) \cdot [\phi^u]_1$, and hence we obtain $(\omega \circ Y')(u) = \omega([\eta'^u]_1) = \omega(\eta(u) \cdot [\phi^u]_1) = \phi^u(0)^{-1} \cdot [\phi^u]_1$, where we have used the condition (3.1) in the last equality. On the other hand, we have $\phi'^u(t) = \phi'(t, u) = \phi(0, u)^{-1} \cdot \phi(t, u) = \phi(0, u)^{-1} \cdot \phi^u(t) = \phi^u(0)^{-1} \cdot \phi^u(t)$. Therefore, we get $\Psi'(u) = [\phi'^u]_1 = \phi^u(0)^{-1} \cdot [\phi^u]_1 = (\omega \circ Y')(u)$ and hence $\Psi' = \omega \circ Y'$. Finally, by (3.5) and (3.6) we obtain $(\overset{r}{T}\omega \circ \alpha_P^{r,1})(\tilde{u} \cdot \bar{s}) = \alpha_G^{r,1}(\bar{s}^{-1} \cdot \bar{s})$ and hence, by Corollary 1.4, the equality (3.4) is proved. Q.E.D.

LEMMA 3.3. *Let $\tilde{\omega}$ be a map of R into $T(G)$ such that $\tilde{\omega}(R) \subset T_e(G)$, and let η be a map of R into G. Then the element $\alpha_G^{1,r}([\tilde{\omega}]_r)$ is am elenent of $T_{e_r}(\overset{r}{T}G)$, where e_r is the unit element of $\overset{r}{T}G$, and we have the following relation*:

$$(3.7) \qquad \alpha_G^{1,r}([\eta^{-1} \cdot \tilde{\omega} \cdot \eta]_r) = [\eta]_r^{-1} \cdot \alpha_G^{1,r}([\tilde{\omega}]_r) \cdot [\eta]_r.$$

Proof. First, we shall show that $\alpha_G^{1,r}([\tilde{\omega}]_r) \in T_{e_r}(\overset{r}{T}G)$. Applying Lemma 1.1. for $\tilde{\omega} = \varphi$, we get a map $\sigma : R^2 \to G$ such that $\tilde{\omega}(t) = [\sigma_t]_1$ and that $\sigma_t(0) = e$ for small t. Since $\sigma^0(t) = \sigma(t, 0) = \sigma_t(0) = e = \gamma_e(t)$, we get $\sigma^0(t) = \gamma_e(t)$ for small t, where we have used the notation of constant map γ_e, i.e. $\gamma_e(t) = e$ for every $t \in R$. Put $\Sigma(u) = [\sigma^u]_r$. Then, by the definition of $\alpha_G^{1,r}$, we see that

$$(3.8) \qquad \alpha_G^{1,r}([\tilde{\omega}]_r) = [\Sigma]_1.$$

Now, since $\Sigma(0) = [\sigma^0]_r = [\gamma_e]_r = e_r$ is the unit element of $\overset{r}{T}G$, we see that $[\Sigma]_1 \in T_{e_r}(\overset{r}{T}G)$, which proves our assertion.

Next, consider the map $\eta^{-1} \cdot \tilde{\omega} \cdot \eta : R \to T(G)$. We have the following equalities: $(\eta^{-1} \cdot \tilde{\omega} \cdot \eta)(t) = \eta^{-1}(t) \cdot \tilde{\omega}(t) \cdot \eta(t) = \eta^{-1}(t) \cdot [\sigma_t]_1 \cdot \eta(t) = [\eta^{-1}(t) \cdot \sigma_t \cdot \eta(t)]_1 = [\sigma'_t]_1$ for small t, where we have put $\sigma'(t, u) = \eta^{-1}(t) \cdot \sigma(t, u) \cdot \eta(t)$ for $t, u \in R$. Put $\Sigma'(u) = [\sigma'^u]_r$, then we get the following

$$(3.9) \qquad \alpha_G^{1,r}([\eta^{-1}\cdot\tilde{\omega}\cdot\eta]_r) = [\Sigma']_1.$$

Now, since we have $\Sigma'(u)=[\sigma'^u]_r=[\eta^{-1}\cdot\sigma^u\cdot\eta]_r=[\eta]_r^{-1}\cdot[\sigma^u]_r\cdot[\eta]_r=[\eta]_r^{-1}\cdot\Sigma(u)\cdot[\eta]_r$, we get $[\Sigma']_1=[\eta]_r^{-1}\cdot[\Sigma]_1\cdot[\eta]_r$, and hence, by (3.8) and (3.9) we obtain the relation (3.7). Q.E.D.

Lemma 3.4. *Let* $\tilde{s}\in\overset{r}{T}G$, $\tilde{u}\in T_{\tilde{u}}\overset{r}{T}P$. *Then,* $\omega_r(\tilde{u})$ *is an element of* $T_{e_r}(\overset{r}{T}G)$ *and we have the following relation*

$$(3.10) \qquad \omega_r(\tilde{u}\cdot\tilde{s}) = \tilde{s}^{-1}\cdot\omega_r(\tilde{u})\cdot\tilde{s}.$$

Proof. Since $\tilde{s}\in\overset{r}{T}G$, there is a map $\eta: R\to G$ such that $\tilde{s}=[\eta]_r$. Since $\tilde{u}\in\overset{1}{T}\overset{r}{T}P$, there are maps $\varphi: R\to\overset{r}{T}P$ and $\phi: R^2\to P$ such that $\tilde{u}=[\varphi]_1$ and $\varphi(t)=[\phi_t]_1$ for small t. Consider the right translation $R_{\tilde{s}}$ of $\overset{r}{T}P$ by the element $\tilde{s}\in\overset{r}{T}G$. Then, we compute as follows: $(R_{\tilde{s}}\circ\varphi)(t)=[\phi_t]_r\cdot[\eta]_r=[\phi_t\cdot\eta]_r=[\phi_t']_r$, where we have put $\phi'(t,u)=\phi(t,u)\cdot\eta(u)$ for $t,u\in R$. Define $\Psi': R\to\overset{1}{T}P$ by $\Psi'(u)=[\phi'^u]_1$ for $u\in R$. Since we have $\tilde{u}\cdot\tilde{s}=TR_{\tilde{s}}(\tilde{u})=TR_{\tilde{s}}([\varphi]_1)=[R_{\tilde{s}}\circ\varphi]_1$, and since $(R_{\tilde{s}}\circ\varphi)(t)=[\phi_t']_r$, we get the following

$$(3.11) \qquad \alpha_P^{r,1}(\tilde{u}\cdot\tilde{s}) = [\Psi']_r.$$

In particular, we get $\alpha_P^{r,1}(\tilde{u})=[\Psi]_r$, where $\Psi(u)=[\phi^u]_1$ for $u\in R$.

Now, we calculate as follows: $(\omega\circ\Psi')(u)=\omega([\phi'^u]_1)=\omega([\phi^u]_1\cdot\eta(u))=\eta(u)^{-1}\cdot\omega([\phi^u]_1)\cdot\eta(u)=\eta(u)^{-1}\cdot\omega(\Psi(u))\cdot\eta(u)=\eta(u)^{-1}\cdot(\omega\circ\Psi)(u)\cdot\eta(u)=(\eta^{-1}\cdot(\omega\circ\Psi)\cdot\eta)(u)$, where we have used the condition (3.2) in the third equality. Therefore, by (3.11) we get $(\overset{r}{T}\omega\circ\alpha_P^{r,1})(\tilde{u}\cdot\tilde{s})=[\omega\circ\Psi']_r=[\eta^{-1}\cdot(\omega\circ\Psi)\cdot\eta]_r=[\eta]_r^{-1}\cdot[\omega\circ\Psi]_r\cdot[\eta]_r=\tilde{s}^{-1}\cdot[\omega\circ\Psi]_r\cdot\tilde{s}$. In particular, we get $\overset{r}{T}\omega\circ\alpha_P^{r,1}(\tilde{u})=[\omega\circ\Psi]_r$. Put $\tilde{\omega}=\omega\circ\Psi$ and apply Lemma 3.3 for this $\tilde{\omega}$ and η. Then we can conclude that $\omega_r(\tilde{u})=\alpha_G^{1,r}([\tilde{\omega}]_r)$ is an element of $T_{e_r}(\overset{r}{T}G)$ and that $\omega_r(\tilde{u}\cdot\tilde{s})=\alpha_G^{1,r}([\eta^{-1}\cdot\tilde{\omega}\cdot\eta]_r)=[\eta]_r^{-1}\cdot\alpha_G^{1,r}([\tilde{\omega}]_r)\cdot[\eta]_r=\tilde{s}^{-1}\cdot\omega_r(\tilde{u})\cdot\tilde{s}$. Q.E.D.

Summarizing Lemma 3.3 and 3.4, we obtain the following

Proposition 3.5. *The map* $\omega_r: T\overset{r}{T}P\to T\overset{r}{T}G$ *defined by* (3.3) *have the following properties:*

$$(0) \qquad I_m\omega_r\subset T_{e_r}(\overset{r}{T}G),$$

$$(1) \qquad \omega_r(\tilde{u}\cdot\bar{s}) = \tilde{s}^{-1}\cdot\bar{s},$$

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(2) $$\omega_r(\tilde{u}\cdot\tilde{s}) = \tilde{s}^{-1}\cdot\omega_r(\tilde{u})\cdot\tilde{s}$$

for every $\bar{s}\in \overset{r}{T}G$, $\bar{u}\in \overset{r}{T}P$, $\tilde{s}\in T_{\bar{s}}(\overset{r}{T}G)$ *and* $\tilde{u}\in T_{\bar{u}}\overset{r}{T}P$.

Now, consider a principal fibre bundle $P(M,\pi,G)$ and a map $\eta: TP\to TG$ such that $\eta(TP)\subset T_eG$ and that $\eta(0_z)=0$ for every tangent zero vector 0_z to P at $z\in P$. We note that the restriction $\eta_z=\eta\,|\,T_z(P)$ of η to the tangent space T_zP is not necessarily a linear map of T_zP into T_eG. Take the linear part η^0_z of η_z for each $z\in P$ (cf. § 2)). It is easy to see that $\{\eta^0_z\}_{z\in P}$ defines a differentiable map η^0 of TP into TG.

DEFINITION 3.6. We call η^0 *the linear part* of η.

PROPOSITION 3.7. *Let η be a map of TP into TG such that $\eta(TP)\subset T_eG$ and that $\eta(0_z)=0$ for any $z\in P$. Suppose that η satisfies the following conditions:*

(3.12) $$\eta(z\cdot\bar{s}) = s^{-1}\cdot\bar{s},$$

(3.13) $$\eta(\bar{z}\cdot s) = s^{-1}\cdot\eta(\bar{z})\cdot s$$

for every $s\in G$, $z\in P$, $\bar{s}\in T_sG$ and $\bar{z}\in T_zP$. Then, the linear part η^0 of η satisfies the following conditions:

(3.14) $$\eta^0(z\cdot\bar{s}) = s^{-1}\cdot\bar{s}$$

(3.15) $$\eta^0(\bar{z}\cdot s) = s^{-1}\cdot\eta^0(\bar{z})\cdot s,$$

namely η^0 is a connection form on P.

Proof. Fix $z\in P$ and $s\in G$ and consider the vector subspace $V_1=\{z\cdot\bar{s}\,|\,\bar{s}\in T_sG\}$ of the vector space $T_{z\cdot s}P$. Since $\eta(z\cdot\bar{s})=s^{-1}\cdot\bar{s}$ for $\bar{s}\in T_sG$, the restriction η_1 of η to V_1 is a linear map of V_1 into T_eG. Therefore, by Lemma 2.5, we have $\eta^0|V_1=\eta_1$, which shows that (3.14) holds.

Next, from (3.12) it follows that $\eta_{z\cdot s}\circ TR_s = ad(s^{-1})\circ\eta_z$ on T_zP, where we have defined $ad(s^{-1})$ by $ad(s^{-1})X=s^{-1}\cdot X\cdot s$ for $X\in T_e(G)$. Since the maps $TR_s: T_zP\to T_{z\cdot s}P$ and $ad(s^{-1}): T_eG\to T_eG$ are both linear, we have by Lemma 2.3 and 2.4 the following equalities: $\eta^0_{z\cdot s}\circ TR_s=\eta^0_{zs}\circ(TR_s)^0=(\eta_{z\cdot s}\circ TR_s)^0$ $=(ad(s^{-1})\circ\eta_z)^0=ad\,(s^{-1})^0\circ\eta^0_z=ad\,(s^{-1})\circ\eta^0_z$, and hence $\eta^0_{z\cdot s}\circ TR_s=ad\,(s^{-1})\circ\eta^0_z$, which shows the condition (3.15). Q.E.D.

DEFINITION 3.8. Let ω be a connection form on a principal fibre bundle $P(M,\pi,G)$. We denote by $\omega^{(r)}$ the linear part of the map ω_r, i.e. $\omega^{(r)}=(\omega_r)^0$ (cf. Def. 3.1 and 3.6).

By virtue of Proposition 3.5, we can apply Proposition 3.7 for ω_r and $\overset{r}{T}P$ and we conclude that $\omega^{(r)}$ is a connection form on $\overset{r}{T}P$. Thus we have proved the following

THEOREM 3.9. *Let ω be a connection form on a principal fibre bundle P. Then there exists canonically a connection form $\omega^{(r)}$ on the tangential fibre bundle $\overset{r}{T}P$ to P of order r for every positive integer r. We shall call $\omega^{(r)}$ the prolongation of the connection ω of order r.*

Remark 3.10. For the case $r=1$ we see that $\omega^{(1)}$ coincides with the connection tangential to ω due to Kobayashi (cf. [1] p. 152). We note that ω_r itself is not in general a connection form for $r \geqq 2$, although $\omega_1 = (\omega_1)^0 = \omega^{(1)}$.

§ 4. Prolongations of linear (affine) connections.

In this section we apply the result in the previous section to the linear connections on a manifold. As in [2] we denote by $FM(M, \pi, GL(n))$ the frame bundle of M.

THEOREM 4.1. *Let Γ be a linear connection on a manifold M. Then, there exists canonically a linear connection $\Gamma^{(r)}$ on the tangent bundle $\overset{r}{T}M$ of order r to M.*

Proof. Let ω be the connection form on $F(M)$ defining the connection Γ. The prolongation $\omega^{(r)}$ of ω is a connection form on $\overset{r}{T}FM$. By making use of the bundle homomorphism $j_M^{(r)}$ of $\overset{r}{T}FM$ into $F\overset{r}{T}M$ (cf. [3] § 7), we obtain canonically a connection $\Gamma^{(r)}$ on the principal fibre bundle $F\overset{r}{T}M$.

Q.E.D.

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Nagoya University.