7
Electrodynamics

In this chapter we set up a Lagrangian for a field theory in which electrically charged Dirac particles and antiparticles, for example electrons and positrons, interact with and through the electromagnetic field. To facilitate reference to other texts, and for conciseness, we work with four-component Dirac spinors and the matrices γ^μ introduced in Section 5.5.

7.1 Probability density and probability current

We have seen in previous chapters how conservation laws are associated with symmetries of the Lagrangian. The Lagrangian density (5.31),

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi,$$

is invariant under the transformation

$$\psi(x) \to \psi'(x) = \mathrm{e}^{-\mathrm{i}\alpha}\psi(x), \tag{7.1}$$

where α is a constant phase. These transformations form a group $U(1)$ (see Appendix B) and are said to be *global*: the same at every point in space and time.

If now we allow an arbitrary small space- and time-dependent variation in α, $\alpha \to \alpha'(x) = \alpha + \delta\alpha(x)$, and if the fields satisfy the field equations, the corresponding first-order variation δS in the action must be zero, since S is stationary for the actual fields. The variation comes from the operators ∂_μ acting on $\mathrm{e}^{-\mathrm{i}\delta\alpha(x)}$, so that

$$\begin{aligned}\partial S &= \int \bar{\psi}\gamma^\mu\psi \mathrm{i}\partial_\mu \mathrm{e}^{-\mathrm{i}\delta\alpha}\mathrm{d}^4x \\ &= \int \bar{\psi}\gamma^\mu\psi\partial_\mu(\delta\alpha)\,\mathrm{d}^4x, \text{ to first order.}\end{aligned}$$

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Integrating by parts,

$$\delta S = -\int [\partial_\mu(\bar{\psi}\gamma^\mu\psi)]\delta\alpha \, \mathrm{d}^4x.$$

This is zero for any arbitrary function $\delta\alpha(x)$ only if

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0. \tag{7.2}$$

At each point x of space and time, $\bar{\psi}(x)\gamma^\mu\psi(x)$ transforms like a contravariant four-vector (Section 5.5) and we may define the contravariant field

$$j^\mu(x) = \bar{\psi}\gamma^\mu\psi = (P(x), \mathbf{j}(x)) \tag{7.3}$$

where $P(x) = \bar{\psi}\gamma^0\psi = \psi^\dagger(\gamma^0)^2\psi = \psi_a^*\psi_a = \sum_{a=1}^{4}|\psi_a|^2$. Then (7.2) takes the familiar form

$$\frac{\partial P}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0. \tag{7.4}$$

If $P(x)$ is interpreted as the particle probability density associated with the wave function $\psi(x)$ and $\mathbf{j}(x)$ as the probability current, (7.4) expresses local particle conservation. Integrating over all space, and using the divergence theorem, it follows that for fields that vanish at large distances

$$\frac{\mathrm{d}}{\mathrm{d}t}\int P\mathrm{d}^3\mathbf{x} = 0.$$

Hence

$$\int P(t, \mathbf{x})\,\mathrm{d}^3\mathbf{x} = \int \psi^\dagger\psi\,\mathrm{d}^3\mathbf{x}$$

is a constant independent of time. With $\psi(x)$ taken to be a normalised wave function for a particle, the constant is unity, and we see that a wave function once normalised stays normalised. In Chapter 8 we shall see that in a second quantised field theory, $\int P(t, \boldsymbol{x})\,\mathrm{d}^3\boldsymbol{x}$ is an operator that counts the number of particles minus the number of antiparticles, and thus this number is conserved.

We could have derived (7.2) from the field equation but the device introduced here, whereby the conservation law appears as a consequence of the $U(1)$ symmetry (7.1), is both elegant and economical.

7.2 The Dirac equation with an electromagnetic field

In classical mechanics, the Hamiltonian for a particle carrying charge q moving in an external electromagnetic field specified by the electromagnetic potentials (ϕ,$\mathbf{A}$)

is obtained from the free particle Hamiltonian by the substitution in (3.8)

$$E \to E - q\phi, \quad \mathbf{p} \to \mathbf{p} - q\mathbf{A},$$

or, equivalently

$$p^\mu \to p^\mu - qA^\mu, \tag{7.5}$$

where $p^\mu = (E,\mathbf{p})$ is the energy-momentum four-vector of the particle. (See Problems 4.6 and 4.7.) With the quantisation rule $p_\mu \to \mathrm{i}\partial_\mu$, (7.5) suggests that the Dirac equation in the presence of an electromagnetic field should be

$$[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m]\psi = 0, \tag{7.6}$$

and there should be a corresponding substitution in the Lagrangian density.

Using (4.10) and (5.31), we take the Lagrangian density for the Dirac field together with the electromagnetic field with external charge-current sources J^μ to be

$$\begin{aligned} \mathcal{L} &= \bar{\psi}[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu \\ &= \bar{\psi}[\gamma^\mu\mathrm{i}\partial_\mu - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \left(J^\mu + q\bar{\psi}\gamma^\mu\psi\right)A_\mu. \end{aligned} \tag{7.7}$$

The Lagrangian is still invariant under the transformation $\psi(x) \to \psi'(x) = \mathrm{e}^{-\mathrm{i}\alpha}\psi(x)$ with α constant, and this leads as before to particle conservation:

$$\partial_\mu j^\mu = 0, \ j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{7.8}$$

Variation of the fields A_μ in the action, as in Section 4.2, yields the Maxwell equations, with charge-current density

$$J^\mu + q\bar{\psi}\gamma^\mu\psi = J^\mu + qj^\mu. \tag{7.9}$$

In (7.8) and (7.9), $j^\mu(x)$ is the conserved particle number density current (antiparticles being counted as negative), and $qj^\mu(x)$ is the conserved charge density current. Thus the Lagrangian density (7.7) includes the electromagnetic field produced by the charged particle current as well as the field produced by external sources.

Setting q = the electron charge = $-e$, and m to be the electron mass, the Lagrangian (7.7) is, after quantisation, the Lagrangian of quantum electrodynamics. With the external charge-current distribution $J^\mu(x)$ taken to be that of the atomic nuclei, and including the dynamics of the nuclei as an assembly of point particles, this is the basic Lagrangian that describes and explains most of chemistry and materials science. We shall review some of the astounding successes of quantum electrodynamics in the next chapter.

7.3 Gauge transformations and symmetry

In Chapter 4 we stressed that the four-potential A_μ is not unique: the same physical electric and magnetic fields are obtained after a gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \chi(x)$$

where $\chi(x)$ is an arbitrary function of space and time.

If ψ is a solution of the Dirac equation with the four-potential A_μ, the corresponding solution in the gauge with four-potential A'_μ is given by

$$\psi \to \psi' = \mathrm{e}^{-\mathrm{i}q\chi}\psi.$$

This is easily verified:

$$\begin{aligned}\left(\mathrm{i}\partial_\mu - qA'_\mu\right)\psi' &= \mathrm{e}^{-\mathrm{i}q\chi}\left\{\mathrm{i}\partial_\mu + q\partial_\mu\chi - q(A_\mu + \partial_\mu\chi)\right\}\psi \\ &= \mathrm{e}^{-\mathrm{i}q\chi}(\mathrm{i}\partial_\mu - qA_\mu)\psi.\end{aligned}$$

Hence the Dirac equation (7.6) is equivalent to

$$\left[\gamma^\mu(\mathrm{i}\partial_\mu - qA'_\mu) - m\right]\psi' = 0.$$

The transformations:

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\chi(x) \tag{7.10a}$$

$$\psi(x) \to \mathrm{e}^{-\mathrm{i}q\chi(x)}\psi(x) \tag{7.10b}$$

make up a general local gauge transformation.

The charge-current density $qj^\mu = q\bar{\psi}\gamma^\mu\psi$ is invariant under the transformation and so too is the action provided that (as in Section 4.3) $\partial_\mu J^\mu = 0$. It is also interesting to note that the phase of a charged Dirac field, for example that of an electron, is a gauge artefact without physical significance: this phase cannot be measured.

We can look at this transformation from a different point of view. The Lagrangian (7.7) is invariant under the global $U(1)$ transformation $\psi \to \psi' = \mathrm{e}^{-\mathrm{i}\alpha}\psi$ where α is constant. If we now ask for the Lagrangian to be invariant under a similar but *local* transformation, $\psi \to \psi'(x) = \mathrm{e}^{-\mathrm{i}q\chi}\psi(x)$, where $\chi(x)$ is an arbitrary function of space and time, we are forced into introducing the *gauge field* A_μ, with the transformation property $A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi$, in order to cancel out the additional terms which arise.

From this point of view, the electromagnetic field appears as a consequence of the invariance of the Lagrangian under a local symmetry transformation. This idea will be generalised in later chapters.

7.4 Charge conjugation

Charge conjugation is the operation of replacing matter by antimatter so that, for example, an electron is interpreted as the antiparticle of the positron, which is then the particle. This would be the natural point of view if the Universe contained antimatter rather than matter. An interchange is achieved if we replace the Dirac field by its complex conjugate. Consider a positive energy solution of the field equation that has a phase factor e^{-iEt}. After complex conjugation it has a phase factor e^{iEt}, and with the standard phase convention is a negative energy solution. In the 'hole' interpretation, negative energy solutions are associated with antiparticles. However, the operation of complex conjugation does not leave $\mathcal{L}$ invariant: additional manipulations are needed to display the symmetry.

Taking the complex conjugate of the Dirac equation (7.6) gives

$$[(\gamma^\mu)^*(-i\partial_\mu - qA_\mu) - m]\psi^* = 0.$$

Now in the chiral representation γ^0, γ^1 and γ^3 are real and $(\gamma^2)^* = -\gamma^2$. Multiplying the equation above by γ^2 and using the anticommuting properties of the γ matrices gives

$$[\gamma^\mu(i\partial_\mu + qA_\mu) - m](\gamma^2\psi^*) = 0,$$

or

$$\left[\gamma^\mu\left(i\partial_\mu - qA^c_\mu\right) - m\right](\gamma^2\psi^*) = 0.$$

Hence if ψ is a positive energy solution of the Dirac equation for a particle carrying charge q, $(\gamma^2\psi^*)$ is a negative energy solution in the charge conjugate field $A^c_\mu = -A_\mu$, which we introduced in Section 4.6.

There is some freedom of choice in the details of the transformation. We shall define the charge conjugate field ψ^c by

$$\psi^c = -\,i\gamma^2\psi^* \tag{7.11a}$$

or, in terms of two-component spinors

$$\psi^c_L = -i\sigma^2\psi^*_R, \qquad \psi^c_R = i\sigma^2\psi^*_L. \tag{7.11b}$$

Using $\left(\gamma^2\right)^2 = -\mathbf{I}$, $\left(\gamma^2\right)^* = -\gamma^2$, we can invert the transformation (7.11a), obtaining

$$\psi = -i\gamma^2\left(\psi^c\right)^* \tag{7.12a}$$

or

$$\psi_L = -i\sigma^2\left(\psi^c_R\right)^*, \qquad \psi_R = i\sigma^2\left(\psi^c_L\right)^*. \tag{7.12b}$$

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Then (noting $(\gamma^2)^\dagger = -\gamma^2$) we have

$$\psi^\dagger = -\mathrm{i}(\psi^c)^\mathrm{T}\gamma^2 \tag{7.13a}$$

or

$$\psi_\mathrm{L}^\dagger = \mathrm{i}\big(\psi_\mathrm{R}^c\big)^\mathrm{T}\sigma^2, \qquad \psi_\mathrm{R}^\dagger = -\mathrm{i}\big(\psi_\mathrm{L}^c\big)^\mathrm{T}\sigma^2. \tag{7.13b}$$

Let us see how the various terms in the Lagrangian density (7.7) transform. Consider

$$\bar\psi\psi = \psi^\dagger\gamma^0\psi = -(\psi^c)^\mathrm{T}\gamma^2\gamma^0\gamma^2(\psi^c)^* = -(\psi^c)^\mathrm{T}\gamma^0(\psi^c)^*,$$

(using the properties of the γ-matrices).

To display the invariance of $\mathcal{L}$ we must anticipate Chapter 8. As operators, spinor fields anticommute: if a product of two fields is interchanged, a minus sign is introduced. For example, $\psi_a{}^*\psi_b = -\psi_b\psi_a{}^*$. Thus in transposing the last expression above we introduce a minus sign, and hence recover the form of the original term:

$$\bar\psi\psi = (\bar\psi^c)\psi^c$$

(since $(\gamma^0)^\mathrm{T} = \gamma^0$).

Other terms likewise acquire a minus sign:

$$\begin{aligned}\bar\psi\gamma^\mu\psi &= -(\psi^c)^\mathrm{T}\gamma^2\gamma^0\gamma^\mu\gamma^2(\psi^c)^*\\ &= (\psi^c)^\dagger(\gamma^2\gamma^0\gamma^\mu\gamma^2)^T(\psi^c).\end{aligned}$$

But, as the reader may verify,

$$(\gamma^2\gamma^0\gamma^\mu\gamma^2)^\mathrm{T} = -\gamma^0\gamma^\mu.$$

Hence

$$\bar\psi\gamma^\mu\psi = -(\bar\psi^c)\gamma^\mu(\psi^c).$$

Finally,

$$\begin{aligned}\bar\psi\gamma^\mu\mathrm{i}\partial_\mu\psi &= -(\psi^c)^\mathrm{T}\gamma^2\gamma^0\gamma^\mu\gamma^2\mathrm{i}\partial_\mu(\psi^c)^*\\ &= \mathrm{i}\partial_\mu(\psi^c)^\dagger(\gamma^2\gamma^0\gamma^\mu\gamma^2)^\mathrm{T}(\psi^c)\\ &= -\mathrm{i}\partial_\mu(\psi^c)^\dagger\gamma^0\gamma^\mu(\psi^c).\end{aligned}$$

Integration by parts in the action allows us to replace this last term by $(\bar\psi^c)\gamma^\mu\mathrm{i}\partial_\mu(\psi^c)$ in the Lagrangian density.

The Lagrangian can be seen to be of exactly the same form after charge conjugation, provided that the charge conjugate potentials A_μ^c are defined to be $A_\mu^c = -A_\mu$ (as in Section 4.6) and any external charge-current density J_μ also changes sign. In ordinary matter, where the Dirac particles are electrons, the external J_μ arise from the atomic nuclei, and these currents also change sign under charge conjugation.

7.5 The electrodynamics of a charged scalar field

In Section 3.5 we introduced the Klein–Gordon equation,

$$-\partial_\mu \partial^\mu \phi - m^2 \phi = 0,$$

which describes the motion of an uncharged scalar particle. The corresponding equation for a charged scalar particle is obtained from the Klein–Gordon equation by making the substitution (7.5), $\mathrm{i}\partial_\mu \to \mathrm{i}\partial_\mu - qA_\mu$, which gives

$$[(\mathrm{i}\partial_\mu - qA_\mu)(\mathrm{i}\partial^\mu - qA^\mu) - m^2]\Phi = 0. \tag{7.14}$$

A solution of (7.14) is necessarily complex. Thus a charged particle of zero spin in an electromagnetic field must be described by a complex, or two-component, wave function $\Phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$. We introduced complex scalar fields in Section 3.7. A real Lagrangian density that yields (7.14) and is Lorentz invariant is

$$\mathcal{L} = -\left[(\mathrm{i}\partial_\mu + qA_\mu)\Phi^*\right]\left[(\mathrm{i}\partial^\mu - qA^\mu)\,\Phi\right] - m^2\Phi^*\Phi. \tag{7.15}$$

$\mathcal{L}$ is invariant under a local gauge transformation, $\Phi \to \mathrm{e}^{-\mathrm{i}q\chi}\Phi$. Note that, since zero spin particles are bosons, the fields Φ and Φ^* commute.

Taking the complex conjugate of equation (7.14), we see that if $\Phi(x)$ is a solution for a particle carrying charge q in a given external field, then $\Phi^*(x)$ is a solution for a particle carrying a charge $-q$. We define the field $\Phi^c(x) = \Phi^*(x)$ to be the charge conjugate of Φ. The Lagrangian density (7.15) is invariant under charge conjugation, $\Phi \to \Phi^c$, if the charge conjugate potentials are again defined to be $A^c_\mu = -A_\mu$.

The charged π^+ and π^- mesons are composite, spin zero, particles whose overall motion is described by the generalised Klein–Gordon equation (7.14). We shall meet these particles and the fields Φ and Φ^* in the phenomenological discussions of Chapter 9.

7.6 Particles at low energies and the Dirac magnetic moment

In an electromagnetic field, the coupled Dirac equations (5.10) become

$$\begin{aligned}(\mathrm{i}\partial_0 - qA_0)\,\psi_\mathrm{L} - \sigma^i\,(\mathrm{i}\partial_i - qA_i)\,\psi_\mathrm{L} - m\psi_\mathrm{R} = 0\\ (\mathrm{i}\partial_0 - qA_0)\,\psi_\mathrm{R} + \sigma^i\,(\mathrm{i}\partial_i - qA_i)\,\psi_\mathrm{R} - m\psi_\mathrm{L} = 0\end{aligned} \tag{7.16}$$

where the σ^i are the Pauli spin matrices.

From Section 6.1, solutions of the Dirac equation that correspond to particles at low energies have $\psi_{\mathrm{L}} \approx \psi_{\mathrm{R}}$. We shall now show that at low energies the two-component wave function

$$\phi = \mathrm{e}^{\mathrm{i}mt}\,(\psi_{\mathrm{L}} + \psi_{\mathrm{R}}) \tag{7.17a}$$

corresponds closely to the Schrödinger wave function for the particle. The factor e^{imt} has been inserted so that, as in the Schrödinger equation, the rest mass energy of the particle is omitted. If we define the orthogonal combination

$$\chi = e^{\mathrm{i}mt}\,(\psi_{\mathrm{L}} - \psi_{\mathrm{R}})\,, \tag{7.17b}$$

then by adding and subtracting the equations (7.16) we obtain an equivalent pair of equations:

$$\begin{aligned} &(\mathrm{i}\partial_0 - qA_0)\,\phi - \sigma^i\,(\mathrm{i}\partial_i - qA_i)\,\chi = 0, \\ &(\mathrm{i}\partial_0 - qA_0 + 2m)\,\chi - \sigma^i\,(\mathrm{i}\partial_i - qA_i)\,\phi = 0. \end{aligned} \tag{7.18}$$

The Schrödinger equation results if the term $(\mathrm{i}\partial_0 - qA_0 + 2m)\,\chi$ is replaced by $2m\chi$. This approximation is reasonable if the Coulomb potential energy qA_0 and the kinetic energy are small compared with the rest mass of the particle. Then

$$\chi = (1/2m)\,\sigma^i\,(\mathrm{i}\partial_i - qA_i)\,\phi,$$

and by substitution

$$\mathrm{i}\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}\sigma^i\,(\mathrm{i}\partial_i - qA_i)\,\sigma^j(\mathrm{i}\partial_j - qA_j) + qA_0\right]\phi. \tag{7.19}$$

The Pauli spin matrices have the property

$$\sigma^i\sigma^j = \mathrm{i}\varepsilon_{ijk}\sigma^k + \delta_{ij}\sigma^0,$$

and from the antisymmetry of ε_{ijk},

$$\varepsilon_{ijk}\partial_i\partial_j\phi = 0, \qquad \varepsilon_{ijk}A_iA_j = 0.$$

Also $\varepsilon_{ijk}[\partial_i(A_j\phi) + A_i\partial_j\phi] = \varepsilon_{ijk}[\partial_i(A_j\phi) - A_j\partial_i\phi] = \varepsilon_{ijk}(\partial_iA_j)\phi$, and recalling $A_\mu = (\phi, -\mathbf{A})$, $\varepsilon_{ijk}(\partial_iA_j) = B_k = -B^k$ gives the magnetic field $\mathbf{B}$. Using these results, we write (7.19) as

$$\mathrm{i}\frac{\partial\phi}{\partial t} = \left[\frac{1}{2m}\,(-\mathrm{i}\nabla - q\mathbf{A})^2 + qA_0 - \left(\frac{q\boldsymbol{\sigma}}{2m}\right)\cdot\mathbf{B}\right]\phi. \tag{7.20}$$

Without the term $-(q\boldsymbol{\sigma}/2m)$. $\mathbf{B}$, this would be the Schrödinger equation for a charged particle in an electromagnetic field. The additional term we interpret as the energy in a magnetic field of an intrinsic magnetic moment associated with a Dirac particle. This is another remarkable consequence of the Dirac equation. For an

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electron, with $q = -e$, the magnetic moment is the Bohr magneton $\mu_B = e\hbar/2m$, anti-aligned with the electron spin. The observed magnetic moment agrees to better than 1% (cf. Section 8.5).

At the level of approximation of (7.20), the magnetic moment would play no role in a purely electrostatic field A_0. In better approximations, or indeed solving the Dirac equation directly, 'spin–orbit coupling' terms appear, which are of some importance in atomic physics and materials science.

Problems

7.1 Using the plane wave expansion (6.24), show that the conserved particle number can be written

$$\int P\left(x^0, \mathbf{x}\right) \mathrm{d}^3\mathbf{x} = \int \psi^\dagger \psi \, \mathrm{d}^3\mathbf{x} = \sum_{\mathbf{p},\varepsilon} (b^*_{\mathbf{p}\varepsilon} b_{\mathbf{p}\varepsilon} + d_{\mathbf{p}\varepsilon} d^*_{\mathbf{p}\varepsilon}).$$

7.2 Show that the charge conjugation operation acting on the positive energy solutions (7.12) and (7.13) yields the negative energy solutions (7.17).

7.3 Show that, taking the fields to be anticommuting and neglecting the neutrino mass, the neutrino Lagrangian density

$$\mathcal{L} = \mathrm{i}\psi_\mathrm{L}^\dagger \tilde{\sigma}^\mu \partial_\mu \psi_\mathrm{L}$$

is invariant under the combined operations of parity and charge conjugation. (Note equations (5.26) and (5.27).)

7.4 Show that $\mathrm{i}\,\sigma^2 \psi_\mathrm{R}^*$ transforms like a left-handed spinor under a Lorentz transformation.

7.5 Obtain the Klein–Gordon equation (7.14) from the Lagrangian density (7.15).

7.6 Using the method of Section 7.1, show that the global $U(1)$ symmetry $\Phi \to \mathrm{e}^{\mathrm{i}\alpha}\Phi$ of the Lagrangian density (7.15) leads to a conserved charge density current

$$qj^\mu = \mathrm{i}q[\Phi^*(\partial^\mu \Phi) - (\partial^\mu \Phi^*)\Phi] - 2q^2 A^\mu \Phi^* \Phi.$$

(Note that, in contrast to the result (7.9) for the Dirac Lagrangian, the current of a complex scalar field contains a term proportional to A^μ.)

7.7 Show that for the positive energy solutions (6.12) and (6.13) of the Dirac equation,

$$qj^\mu = -e\bar{\psi}\gamma^\mu \psi = -e\,(\cosh\theta, 0, 0, \sinh\theta) = -\,(eE/m)\,(1, 0, 0, \upsilon)$$

and also for the 'negative energy' solutions (6.17),

$$qj^\mu = -\,(eE/m)\,(1, 0, 0, \upsilon)\,.$$

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With Dirac's interpretation, the hole that remains when this state is removed from the sea corresponds to a particle carrying charge e moving with velocity v along the z-axis.

7.8 Show that after the operation of charge conjugation a proton has negative charge and an electron has positive charge.

7.9 How do the electromagnetic potentials transform under the operation of time reversal, $t \to t' = -t$? Show that $\gamma^1\gamma^3\psi^*(t)$ is a solution of the time reversed Dirac equation, if $\psi(t)$ is a solution of the Dirac equation.

7.10 Show that, for a Dirac particle in a magnetic field **B** given by the vector potential **A**, both ψ_L and ψ_R satisfy the equation

$$\left[-\frac{\partial^2}{\partial t^2} - (-\mathrm{i}\nabla - q\mathbf{A})^2 - m^2 + q\sigma \cdot \mathbf{B}\right]\psi = 0.$$

Note that this differs from the Klein–Gordon equation for a charged scalar particle in a magnetic field, by the additional term $q\boldsymbol{\sigma} \cdot \mathbf{B}$.

7.11 Using the parity transformations (4.18) and (5.27), show that the Lagrangian density (7.7) is invariant under space inversion.