ON A TYPE OF RIEMANNIAN SPACE

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1. Introduction

In connection with Relativity, Kottler [2] introduced the space V_4 whose metric tensor is given by

$$(1.1) \qquad \begin{aligned} &g_{11} = \phi^2, \quad g_{22} = -x_1^2, \quad g_{33} = -x_1^2 \sin^2 x_2 \\ &g_{44} = -\frac{1}{\phi^2}, \quad g_{ij} = 0 \quad (i \neq j), \end{aligned}$$

x_i being space coordinates and ϕ being a function of x_1 only, and showed that if

$$\phi^2 = -\left(1+ax_1^2+\frac{b}{x_1}\right)^{-1}$$

where a and b are arbitrary constants, then the V_4 is an Einstein space. The present paper deals with a type of Riemannian space of n dimensions ($n \geqq 4$) for which the metric tensor is a generalisation of that of Kottler's space V_4 and is given by

$$(1.2) \qquad \begin{aligned} &g_{11} = \phi^2, \quad g_{22} = -x_1^2, \quad g_{hh} = \sin^2 x_{h-1} g_{h-1h-1} \qquad (2 < h < n) \\ &g_{nn} = -\frac{1}{\phi^2}, \quad g_{ij} = 0 \qquad (i \neq j) \end{aligned}$$

where ϕ is a function of x_1 only.

Denoting an n-dimensional space of this kind by T_n the following theorems will be proved in this paper.

THEOREM 1. *If a T_n is conformally flat, then it is of constant Riemannian curvature.*

THEOREM 2. *If a T_n is symmetric in the sense of Cartan, then it is of constant Riemannian curvature.*

THEOREM 3. *If a T_n is Ricci-symmetric, then it is an Einstein space.*

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2. Proofs of Theorems 1 and 2

Let us consider an n-dimensional Riemannian space V_n $(n \geqq 4)$ whose metric tensor is given by (1.2). It can be easily verified that for this space $R_{hijk} = 0$ $(h, i, j, k$ unequal) and $R_{hiik} = 0$ (Eisenhart, [1], p. 44). By actual calculations we get the nonzero components of the Riemann tensor as follows:

$$
(2.1)\quad
\begin{aligned}
&R_{1221} = \frac{x_1}{\phi}\frac{d\phi}{dx_1}, \quad R_{1kk1} = \sin^2 x_{k-1} R_{1k-1k-11} \quad (2 < k < n)\\
&R_{1nn1} = -\frac{3}{\phi^4}\left(\frac{d\phi}{dx_1}\right)^2 + \frac{1}{\phi^3}\frac{d^2\phi}{dx_1^2}, \quad R_{2332} = x_1^2 \sin^2 x_2 \left(1+\frac{1}{\phi^2}\right)\\
&R_{2kk2} = \sin^2 x_{k-1} R_{2k-1k-12} \quad (3 < k < n), \qquad R_{2nn2} = -\frac{1}{\phi^4} R_{1221}\\
&R_{hkkh} = \sin^2 x_{h-1} R_{h-1kkh-1} \quad (2 < h < k \leqq n).
\end{aligned}
$$

Put

$$
(2.2)\qquad L_{hk} = \frac{1}{2-n}\left[R_{hk} - \frac{R}{2(n-1)} g_{hk}\right]
$$

where R_{ij} is the Ricci tensor and R is the scalar curvature. It is known that if an n-dimensional Riemannian space $(n \geqq 4)$ is conformally flat, then

$$
(2.3)\qquad R_{hijk} = g_{ik}L_{hj} - g_{ij}L_{hk} + g_{hj}L_{ik} - g_{hk}L_{ij}
$$

where L_{hk} is given by (2.2).

Let now T_n be a conformally flat space. Then from (2.3) we get

$$
(2.4)\qquad \frac{-x_1^2}{\phi}\frac{d^2\phi}{dx_1^2} + \frac{3x_1^2}{\phi^2}\left(\frac{d\phi}{dx_1}\right)^2 + \frac{2x_1}{\phi}\frac{d\phi}{dx_1} + (1+\phi^2) = 0.
$$

The solution of (2.4) is given by

$$
(2.5)\qquad \phi^2 = -(1+ax_1^2+bx_1)^{-1}
$$

where a and b are arbitrary constants.

As proved by Hlavatý ([3], p. 477), in a conformally flat V_n $(n \geqq 4)$ a consequence of (2.3) is

$$
(2.6)\qquad L_{ij,k} = L_{ik,j}
$$

where comma denotes covariant differentiation with respect to g_{ij}. From (2.6) it follows that $b = 0$. Hence from (2.5) we get $\phi^2 = -(1+ax_1^2)^{-1}$ and therefore $R_{ij} = R/n\, g_{ij}$. Thus the space T_n is a conformally flat Einstein space and therefore it is a space of constant Riemannian curvature (Eisenhart, [1], p. 93).

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Let us next suppose that T_n is symmetric in the sense of Cartan, i.e. let

$$R_{hijk,l} = 0. \tag{2.7}$$

In consequence of (2.7) we get

$$\left(1+\frac{1}{\phi^2}\right)+\frac{x_1}{\phi^3}\frac{d\phi}{dx_1} = 0. \tag{2.8}$$

Solving (2.8) we get

$$\phi^2 = -(1+ax_1^2)^{-1} \tag{2.9}$$

where a is an arbitrary constant. From (2.9) it follows that

$$R_{hijk} = a(g_{hj}g_{ik}-g_{ij}g_{hk}).$$

Hence the T_n is of constant Riemannian curvature.

3. Proof of Theorem 3

Let us now suppose that T_n is Ricci-symmetric, i.e., let $R_{ij,k} = 0$. Taking covariant derivatives of R_{ii} we obtain

$$R_{11,1} = -\frac{1}{\phi}\frac{d^3\phi}{dx_1^3}+\frac{d^2\phi}{dx_1^2}\left[\frac{9}{\phi^2}\frac{d\phi}{dx_1}-\frac{(n-2)}{x_1\phi}\right] -\frac{12}{\phi^3}\left(\frac{d\phi}{dx_1}\right)^3+\frac{3(n-2)}{x_1\phi^2}\left(\frac{d\phi}{dx_1}\right)^2+\frac{n-2}{x_1^2\phi}\frac{d\phi}{dx_1}$$

$$R_{11,p} = 0, \quad p \neq 1$$

$$R_{22,1} = \frac{2}{x_1\phi^4}\left[\phi x_1^2\frac{d^2\phi}{dx_1^2}-3x_1^2\left(\frac{d\phi}{dx_1}\right)^2 +(n-4)\phi x_1\frac{d\phi}{dx_1}+(n-3)\phi^2(1+\phi^2)\right]$$

$$R_{22,p} = 0 \quad p\neq 1, \qquad R_{hh,1} = \sin^2 x_{h-1} R_{h-1\,h-1,1}$$

$$R_{hh,p} = 0 \quad p \neq 1 \quad (2 < h < n)$$

$$R_{nn,1} = -\frac{1}{\phi^4}R_{11,1} \qquad R_{nn,p} = 0 \quad p \neq 1.$$

Since T_n is Ricci-symmetric

$$\phi x_1^2\frac{d^2\phi}{dx_1^2}-3x_1^2\left(\frac{d\phi}{dx_1}\right)^2+(n-4)\phi x_1\frac{d\phi}{dx_1}+(n-3)\phi^2(1+\phi^2) = 0. \tag{3.1}$$

Solving (3.1) we get

$$\phi^2 = -(1+ax_1^2+bx_1^{3-n})^{-1} \tag{3.2}$$

where a and b are arbitrary constants.

But a T_n for which the value of ϕ is given by (3.2) is an Einstein space (H. Sen, [4]). Hence the T_n is, in this case, an Einstein space.

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References

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