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ON BARTH'S CONJECTURE CONCERINING $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$

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To the memory of my Professor Martin Jurchescu

§0. Introduction

A classical, still unsolved problem, is the following: is every connected curve $A \subset \mathbf{P}^3$ a set-theoretic complete intersection? It is clear that if A is a set-theoretic complete intersection then:

a) The algebraic cohomology groups $H^2(\mathbf{P}^3 \setminus A, \mathcal{F})$ vanish for every coherent algebraic sheaf $\mathcal{F}$ on $\mathbf{P}^3$.

b) The analytic cohomology groups $H^2(\mathbf{P}^3 \setminus A, \mathcal{F})$ vanish for every coherent analytic sheaf $\mathcal{F}$ on $\mathbf{P}^3 \setminus A$.

This was one particular motivation for the theory developed by Hartshorne in [8] and which has proved the following result in the algebraic case: If $A \subset \mathbf{P}^n$ is a closed connected analytic subset (possible singular) of positive dimension then for every coherent algebraic sheaf $\mathcal{F}$ on $\mathbf{P}^n$ the algebraic cohomology group $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$ vanishes. In particular a) has a positive answer.

Barth in [2] considered the analytic analogue of Hartshorne's theorem and proved the following partial result: If $A \subset \mathbf{P}^n$ is a closed connected analytic subset of positive dimension and $\mathcal{F}$ is any coherent analytic sheaf on $\mathbf{P}^n \setminus A$ then the topological dual of the analytic cohomology group $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$ vanishes ([2], Satz 2). Unfortunately this result gives no information on the vanishing of $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$ if we do not know that it is separated in the canonical topology. However, if A is smooth, then by the results in [3] and [1] it follows that $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$ has finite dimension, hence for connected smooth A the above result of Barth shows that $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F}) = 0$.

As remarked by Barth in [2] the difficulty for singular $A \subset \mathbf{P}^n$ is that one does not know if $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$ has finite dimension and he conjectured that this would be always true. The aim of this paper is to give a positive answer to Barth's conjecture (Theorem 4.2). In fact we prove more general results

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(Proposition 3.11, Theorem 4.1) giving criteria of cohomological $(n-1)$-convexity for some Zariski open sets in n-dimensional compact complex spaces and which in particular imply that $\dim_{\mathbf{C}} H^{n-1}(\mathbf{P}^n \setminus A, \mathscr{F}) < \infty$ if $\mathscr{F} \in \operatorname{Coh}(\mathbf{P}^n \setminus A)$.

By the finite dimensionality of $H^{n-1}(\mathbf{P}^n \setminus A, \mathscr{F})$ and by the already mentioned result of Barth ([2], Satz 2) it follows:

Main Theorem. *Let $A \subset \mathbf{P}^n$ be a closed analytic subset without isolated points, $k \geqslant 1$ the number of connected components of A and $\mathscr{F}$ a coherent analytic sheaf on $\mathbf{P}^n \setminus A$. Then we have for the analytic cohomology groups:*

$$\dim_{\mathbf{C}} H^{n-1}(\mathbf{P}^n \setminus A, \mathscr{F}) = (k-1) \dim_{\mathbf{C}} H^0(\mathbf{P}^n \setminus A, \mathscr{H}om(\mathscr{F}, \mathscr{K})) < \infty$$

($\mathscr{K}$ is the canonical sheaf of $\mathbf{P}^n$). In particular $H^{n-1}(\mathbf{P}^n \setminus A, \mathscr{F})$ vanishes for connected A.

This result shows that b) has also a positive answer, therefore one cannot construt a counter-example to the complete intersection problem by finding a connected curve $A \subset \mathbf{P}^3$ and $\mathscr{F} \in \operatorname{Coh}(\mathbf{P}^3 \setminus A)$ with non-zero analytic cohomology $H^2(\mathbf{P}^3 \setminus A, \mathscr{F})$.

Let us remark that for coherent algebraic sheaves the finite dimensionality of $H^{n-1}(\mathbf{P}^n \setminus A, \mathscr{F})$ has been obtained also by M. Peternell [15] proving a comparison theorem and using Hartshorne's result. Of course, there are a lot of coherent analytic sheaves on $\mathbf{P}^n \setminus A$ which are not algebraic.

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§1. Preliminaries

We collect in this section some results and definitions which will be needed throughout this paper. All complex spaces are assumed to be reduced and with countable topology. If U is an open subset in $\mathbf{C}^n$, a function $\varphi \in C^\infty(U, \mathbf{R})$ is called q-convex iff its Levi form $L(\varphi)$ has at least $(n-q+1)$ positive (>0) eigenvalues at any point of U. Using local embeddings this notion can be easily extended to complex spaces [1]. A complex space X is called q-convex if there exists a C^∞ function $\varphi: X \to \mathbf{R}$ which is q-convex outside a compact subset $K \subset X$ and such that φ is an exhaustion function on X, i.e. $\{\varphi < c\} \subset\subset X$ for every $c \in \mathbf{R}$. If K may be taken to be the empty set then X is called q-complete.

The most general example of q-completeness is given by the following result

[14]:

THEOREM 1.1. *Let X be a complex space of pure dimension n without compact irreducible components. Then X is n-complete.*

A complex space is said to be:

a) *cohomologically q-convex if* $\dim_{\mathbf{C}} H^i(X, \mathscr{F}) < \infty$ *for any $i \geqslant q$ and any* $\mathscr{F} \in \mathrm{Coh}(X)$

b) *cohomologically q-complete if $H^i(X, \mathscr{F}) = 0$ for any $i \geqslant q$ and any $\mathscr{F} \in$* $\mathrm{Coh}(X)$

The main results of Andreotti and Grauert in [1] *can be stated as follows:*

THEOREM 1.2. *Let X be a complex space. Then:*

a) *X is q-convex $\Rightarrow$ X is cohomologically q-convex*

b) *X is q-complete $\Rightarrow$ X is cohomologically q-complete*

It is also shown in [18] that the cohomological q-convexity and the cohomological q-completeness are invariant under finite surjective morphisms, i.e. one has:

PROPOSITION 1.1. *Let $\Phi : X \to Y$ be a finite surjective morphism of complex spaces. Then:*

a) *X is cohomologically q-convex iff Y is cohomologically q-convex.*

b) *X is cohomologically q-complete iff Y is cohomologically q-complete.*

If $\mathscr{F}$ is an arbitrary coherent sheaf on a complex space X we consider on $H^i(X, \mathscr{F})$ the canonical topology coming from the topology of compact convergence of cocycles. An open subset D of a complex space X is said to be a q-Runge in X if for any compact subset $K \subset D$ there is a q-convex exhaustion function $\varphi : X \to \mathbf{R}$ (which may depend on K) such that $K \subset \{x \in X \mid \varphi(x) < 0\} \subset\subset D$.

In [1] the following approximation result is proved:

THEOREM 1.3. *If $\mathscr{F} \in \mathrm{Coh}(X)$ and $D \subset X$ is q-Runge then the restriction map $H^{q-1}(X, \mathscr{F}) \to H^{q-1}(D, \mathscr{F})$ has dense image.*

One has the following topological property for q-Runge domains [19]:

THEOREM 1.4. *If X is an n-dimensional complex space and $D \subset X$ is q-Runge then the relative homology groups $H_i(X, D; \mathbf{C})$ vanish for $i \geqslant n + q$.*

In general this topological condition is only necessary and it is not sufficient to guarantee the analytic approximation but, for n-Runge domains in non-compact n-dimensional complex spaces, one has the following purely topological characterization [5] (which is a generalization of Theorem 1.1):

THEOREM 1.5. *Let X be a complex normal space of pure dimension n without compact connected components and $D \subset X$ an open set. Then the following conditions are equivalent*:

1) *D is n-Runge in X*

2) *For every $\mathcal{F} \in \mathrm{Coh}(X)$ the restriction map $H^{n-1}(X, \mathcal{F}) \to H^{n-1}(D, \mathcal{F})$ has dense image*

3) *The natural map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(X, \mathbf{C})$ is injective*

4) *$X \setminus D$ has no compact connected components.*

Remark 1.1. a) A similar result holds for non-normal complex spaces but then one has to replace "compact connected components" by "compact irreducible components" according to the following definition: a locally closed subset $F \subset X$ has no compact irreducible components if $\tilde{F} = \nu^{-1}(F)$ has no compact connected components, where $\nu : \tilde{X} \to X$ is the normalization of X. However in this paper we shall need the above theorem only for normal complex spaces.

b) The implication 4) $\Rightarrow$ 2) is also proved in [16] for the sheaves $\mathcal{F} = \Omega^p$, $p \geq 0$ of holomorphic differential p-forms on X.

In the study of the cohomology of $\mathbf{P}^n \setminus A$ we shall need the following result ([15], Theorem 1):

LEMMA 1.1. *Let $A \subset \mathbf{P}^n$ be a closed analytic subset of pure dimension $d \geqq 1$. Then there exist*:

1) *an irreducible projective algebraic space X of dimension n together with a finite surjective holomorphic map $\Phi : X \to \mathbf{P}^n$*

2) *closed analytic subsets $S_0, A_1, \ldots, A_r$ of X and a linear subspace $H \subset \mathbf{P}^n$ of dimension d*

3) *automorphisms $\Psi_1, \ldots, \Psi_r$ of X*

such that

a) $S_0 = \Phi^{-1}(H)$, $A_1 \cup \cdots \cup A_r = \phi^{-1}(A)$

b) $\Psi_i(A_i) = S_0$ *for every* $i = 1, \ldots, r$

Remark 1.2. In Lemma 1.1 the sets $A_1, \ldots, A_r$ are the irreducible components of $\Phi^{-1}(A)$, $r = \deg A$ and $\phi^{-1}(A)$ is connected if A is connected, but we shall not need these facts.

In our proof of the Main theorem some properties of semianalytic sets, or more generally of subanalytic sets, will play an important role. We recall the following:

DEFINITION 1.1. A subset E of a real-analytic manifold M is said to be semianalytic if for any point $x \in M$ there exist an open neighbourhood U of x and finitely many real-analytic functions g_{ij}; f_i $i = 1, \ldots, p$ $j = 1, \ldots, q$ on U such that $E \cap U = \cup_{i=1}^{p}(\cup_{j=1}^{q} \{g_{ij} > 0\} \cap \{f_i = 0\})$.

It is known [11] that given a locally finite family $\{E_\alpha\}$ of semianalytic sets in a real-analytic manifold M there exists on M a locally finite triangulation compatible with $\{E_\alpha\}$.

The class of semianalytic sets is closed to finite unions and intersections, to difference and adherence but it is not closed to proper real-analytic projections. For that reason it was introduced the larger class of subanalytic sets [10].

DEFINITION 1.2. A subset E of a real-analytic manifold M is said to be subanalytic if each point of M admits an open neighbourhood U such that $E \cap U$ is a projection of a relatively compact semianalytic set (i.e. there is a real-analytic manifold N and a relatively compact semianalytic subset A of $M \times N$ such that $E \cap U = \pi(A)$ where $\pi : M \times N \rightarrow M$ is the projection).

It is known [4] that the following conditions are equivalent:

1) E is subanalytic

2) Every point of M has an open neighbourhood U such that $E \cap U = \cup_{i=1}^{p}(f_{i1}(A_{i1}) \setminus f_{i2}(A_{i2}))$ where, for each $i = 1, \ldots, p$ and $j = 1,2$ A_{ij} is a closed analytic subset of a real-analytic manifold N_{ij}, $f_{ij} : N_{ij} \rightarrow U$ is real-analytic and $f_{ij}|_{A_{ij}} : A_{ij} \rightarrow U$ is proper.

THE CURVE SELECTION LEMMA [10]. *Let M be a real-analytic manifold, $A \subset M$ a subanalytic set and $x \in \bar{A} \setminus A$. Then there exists a fundamental system of open neighbourhoods U of x in M such that for any point $y \in U \cap A$ there is a real-analytic map $\varphi : (-1,1) \rightarrow M$ with $\varphi(0) = x$ and $y \in \varphi((0,1)) \subset U \cap A$.*

Let now A be a k-dimensional real-analytic submanifold of the affine space $\mathbf{R}^n$ (not necessarily closed) which is subanalytic. Consider the tangent map $\tau : A \rightarrow G_k(n)$, where $G_k(n)$ is the Grassmannian of k-planes in $\mathbf{R}^n$, given by $x \rightarrow T_xA$, $x \in A$. Then it is known [6] *that τ is a subanalytic map, i.e. its graph is a subanalytic set in $\mathbf{R}^n \times$*

$G_k(n)$.

Let Z be a Whitney stratified subset of a smooth manifold M and $f: M \to N$ a smooth map such that:

a) $f|_Z$ is proper

b) for each stratum A of Z, the restriction $f|_A : A \to N$ is a submersion.

A map with these properties is called a proper stratified submersion. One has:

THEOREM 1.6 (Thom's first isotopy lemma [7]). *Let $f: Z \to \mathbf{R}^n$ be a proper stratified submersion. Then there is a stratum preserving homeomorphism $h: Z \to \mathbf{R}^n \times (f^{-1}(0) \cap Z)$ which is smooth on each stratum and which commutes with the projection to $\mathbf{R}^n$.*

If X is a triangulable space and $A \subset X$ a closed triangulable subspace then it is easy to see [17] that the following conditions are equivalent:

i) the inclusion $A \overset{i}{\hookrightarrow} X$ is a weak homotopy equivalence

ii) the inclusion i is a homotopy equivalence

iii) A is a deformation retract of X

iv) A is a strong deformation retract of X.

By this remark and by the triangulation theorem for semianalytic sets [11] it follows immediately:

LEMMA 1.2. *Let $A \subset W \subset T \subset V$ be closed semianalytic sets in the real-analytic manifold M such that W is a strong deformation retract of V and A is a strong deformation retract of T. Then A is a strong deformation retract of V.*

Since the proof of the Main theorem is very long and technical, we give some general ideas about the steps of the proof. First, by the invariance of cohomological q-convexity under finite surjective holomorphic maps (Proposition 1.1) and by Lemma 1.1 we only have to consider the following situation: a compact complex space X of dimension n is given together with a closed analytic subset $A = A_1 \cup \cdots \cup A_r$ such that on $X \setminus A$ there is an exhaustion function of the type $\varphi = \max(\varphi_1, \ldots, \varphi_r)$ where φ_i are $(n-1)$-convex on $X \setminus A_i$ and $\exp(-\varphi_i)$ are real-analytic on whole X. It is easy to see, by the bumping method of Andreotti and Grauert [1], that on such a space the relatively compact sublevel sets $\{\varphi < c\}$ have finite dimensional cohomology in degree $(n-1)$ for every $c \in \mathbf{R}$ (Lemma 3.4), but the difficult step is to get an approximation result for

$(n-2)$-cohomology classes, and, in fact, this approximation holds only for c sufficiently large. To see how large the constant c must be chosen we use essentially the Curve selection lemma and some properties of semianalytic and subanalytic sets. This is possible because the functions $\exp(-\varphi_i)$ are assumed to be real-analytic on whole X. We show that for large c some sets B_{ij}, considered in the proof, have no compact connected components and, in view of Theorem 1.5, from this topological condition, we get locally an approximation for $(n-1)$-cohomology classes. But, by Mayer-Vietoris sequence, this can be related to an approximation for $(n-2)$-cohomology classes because the functions φ_i are $(n-1)$-convex.

§2. Some real-analytic lemmas

We prove in this section some lemmas about real-analytic functions which will be needed to have a topological control for the bumping used in the next paragraph.

LEMMA 2.1. *Let X be a real-analytic compact manifold and let $A_1, A_2 \subset X$ be closed analytic subsets such that $A_1 = \{\varphi = 0\}$, $A_2 = \{\psi = 0\}$ where φ, ψ are real analytic functions on X and $\varphi \geqslant 0$, $\psi \geqslant 0$. Then there exists a sufficiently small constant $c_0 > 0$ such that at any point $x \in \{\varphi \leqslant c_0, \psi \leqslant c_0\} \setminus (A_1 \cup A_2)$ there is no a relation of the type $d_x\varphi = \alpha d_x\psi$ with $\alpha \leqslant 0$.*

Proof. Let $\mathbf{P}^1(\mathbf{R})$ be the real projective space of dimension 1 and let $[\lambda_0, \lambda_1]$ be the homogeneous coordinates of a point in $\mathbf{P}^1(\mathbf{R})$. We define $M = \{(x, [\lambda_0, \lambda_1]) \in X \times \mathbf{P}^1(\mathbf{R}) \mid x \in X \setminus (A_1 \cup A_2), \lambda_0\lambda_1 \leqslant 0, \lambda_0 d_x\varphi = \lambda_1 d_x\psi\}$. Clearly M is a semianalytic subset of $X \times \mathbf{P}^1(\mathbf{R})$. Assume that the statement of Lemma 2.1 is not true. Then there exist a sequence of points $x_\nu \in X \setminus (A_1 \cup A_2)$ $x_\nu \to x_0 \in A_1 \cap A_2$ and constants $\alpha_\nu \leqslant 0$ such that $d_{x_\nu}\varphi = \alpha_\nu d_{x_\nu}\psi$. Hence $(x_\nu, [1, \alpha_\nu]) \in M$ and we may suppose that $(x_\nu, [1, \alpha_\nu]) \to (x_0, p_0)$. Then $(x_0, p_0) \in \bar{M} \setminus M$ and by the Curve selection lemma for semianalytic sets there is a real-analytic map $\gamma: (-1, 1) \to X \times \mathbf{P}^1(\mathbf{R})$ with $\gamma(0) = (x_0, p_0)$ and $\gamma((0, 1)) \subset M$. Let $\gamma(\tau) = (x(\tau), \lambda(\tau))$ be the components of γ, hence $x(\tau) \in X$, $\lambda(\tau) = [\lambda_0(\tau), \lambda_1(\tau)] \in \mathbf{P}^1(\mathbf{R})$, $x(0) = x_0$, $\lambda(0) = p_0$ and for any $0 < \tau < 1$ $(x(\tau), \lambda(\tau)) \in M$. By the definition of M it follows that for any $0 < \tau < 1$ we have: $x(\tau) \in X \setminus (A_1 \cup A_2)$, $\lambda_0(\tau)\lambda_1(\tau) \leqslant 0$ and

1) $\lambda_0(\tau) d_{x(\tau)}\varphi = \lambda_1(\tau) d_{x(\tau)}\psi$

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Since $\frac{d}{d\tau}(\varphi \circ x(\tau)) = (d_{x(\tau)}\varphi)(\dot{x}(\tau))$ and $\frac{d}{d\tau}(\psi \circ x(\tau)) = (d_{x(\tau)}\psi)(\dot{x}(\tau))$ applying 1) to $\dot{x}(\tau)$ we get

2) $\lambda_0(\tau)\frac{d}{d\tau}(\varphi \circ x(\tau)) = \lambda_1(\tau)\frac{d}{d\tau}(\psi \circ x(\tau))$

But if $\tau > 0$ is sufficiently small then $\frac{d}{d\tau}(\varphi \circ x(\tau)) > 0$, $\frac{d}{d\tau}(\psi \circ x(\tau)) > 0$ (this follows from the following simple remark: if $f : (-1,1) \to \mathbf{R}$ is real-analytic, $f(0) = 0$, $f(\tau) > 0$ for $\tau > 0$ then $f'(\tau) > 0$ for $\tau > 0$ sufficiently small). This gives a contradiction in 2) because $\lambda_0(\tau)\lambda_1(\tau) \leqslant 0$ and $\lambda_0(\tau)$, $\lambda_1(\tau)$ do not vanish simultaneously. Thus Lemma 2.1 is completely proved.

Taking $A_1 = A_2$ and $\varphi = \psi$ in Lemma 2.1 we get:

Corollary 2.1. *Let X be a real-analytic compact manifold and $A \subset X$ a closed analytic set such that $A = \{\varphi = 0\}$ where φ is a real-analytic function on X and $\varphi \geqslant 0$. Then there exists a sufficiently small constant $c_0 > 0$ such that $d_x\varphi \neq 0$ at any point $x \in \{\varphi \leqslant c_0\} \setminus A$.*

Lemma 2.2. *Let X be a real-analytic compact manifold, $A \subset B \subset X$ closed analytic subsets and $\varphi \geqslant 0$ a real-analytic function on X with $A = \{\varphi = 0\}$. Then there exists a sufficiently small constant $c_0 > 0$ such that A is a strong deformation retract of $B \cap \{\varphi \leqslant c\}$ for any $0 \leqslant c \leqslant c_0$.*

Proof. Let Σ be a subanalytic Whitney regular stratification of B with real-analytic strata [10]. From the subanalyticity of the tangent map it follows that if $S \in \Sigma$ then the set $\{x \in S \mid d_x\varphi = 0 \text{ on } T_xS\}$ is subanalytic in X. By the Curve selection lemma there is an open neighbourhood V of A in X such that for any $S \in \Sigma$ and any $x \in S \cap (V \setminus A)$ $d_x\varphi \neq 0$. We choose $c_0 > 0$ sufficiently small such that $\{\varphi \leqslant c_0\} \subset V$. By Thom's first isotopy lemma it follows that for any $0 < b \leqslant c \leqslant c_0$ the set $B \cap \{\varphi \leqslant b\}$ is a strong deformation retract of $B \cap \{\varphi \leqslant c\}$ because $B \cap \{\varphi = b\}$ is a strong deformation retract of $B \cap \{b \leqslant \varphi \leqslant c\}$. Since $(B \cap \{\varphi \leqslant c\}, A)$ is a polyhedral pair [11] and by Lemma 1.2 we get that A is a strong deformation retract of $B \cap \{\varphi \leqslant c\}$ for any $0 \leqslant c \leqslant c_0$, as desired.

Lemma 2.3. *Let X be a real-analytic compact manifold, $A_1, A_2 \subset X$ closed analytic subsets and φ, ψ C^∞ functions on X which are real-analytic in a neighbourhood*

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of $A_1 \cup A_2$ with $A_1 = \{\varphi = 0\}$, $A_2 = \{\psi = 0\}$ *and* $\varphi \geqslant 0$, $\psi \geqslant 0$. *Let* $H = \{\varphi \leqslant 1, \psi \leqslant 1\}$, $L_1 = A_1 \cap \{\psi \leqslant 1\}$, $L_2 = A_2 \cap \{\varphi \leqslant 1\}$ *and assume that*:

1) $d_x\varphi \neq 0$ *for every* $x \in \{\varphi \leqslant 1\} \setminus A_1$ *and* $d_x\psi \neq 0$ *for every* $x \in \{\psi \leqslant 1\} \setminus A_2$
2) *at any point* $x \in H \setminus (L_1 \cup L_2)$ *there is no a relation of the type* $d_x\varphi = \lambda d_x\psi$ *with* $\lambda < 0$

Then each of the sets L_1 *and* L_2 *is a strong deformation retract of* H.

Proof. We show that L_1 is a strong deformation retract of H (for L_2 the proof is similar). The real-analyticity of φ and ψ near $A_1 \cup A_2$, the triangulation theorem for semianalytic sets [11] and Lemma 1.2 show that it is enough to prove the following: for any $0 < a < b \leqslant 1$ the set $H_a = H \cap \{\varphi \leqslant a\}$ is a strong deformation retract of $H_b = H \cap \{\varphi \leqslant b\}$. We define $H_{a,b} = H \cap \{a \leqslant \varphi \leqslant b\}$ and $P_{a,b} = \{\psi = 1\} \cap H_{a,b}$. By the condition 2) we know that at any point $x \in P_{a,b}$ there is no a relation of the type $d_x\varphi = \lambda d_x\psi$ with $\lambda < 0$. Using a partition of unity it follows that there exists a metric $\{g^{ij}\}$ on the cotangent bundle of X such that $\langle d_x\varphi, d_x\psi\rangle > 0$ if $x \in P_{a,b}$. This metric induces a riemannian metric $\{g_{ij}\}$ on the tangent bundle of X such that $\langle d\varphi, d\psi\rangle = \langle \text{grad}\,\varphi, \text{grad}\,\psi\rangle$ where $\text{grad}\,\varphi$, $\text{grad}\,\psi$ are considered with respect to this riemannian metric (in local coordinates $x_1, \ldots, x_n$ gradf has components $\sum_j g^{ij} \dfrac{\partial f}{\partial x_j}$). Therefore $\langle \text{grad}_x\varphi, \text{grad}_x\psi\rangle > 0$ at any point $x \in P_{a,b}$. Let $\rho : X \to \mathbf{R}$ be a C^∞ function which is equal to $\dfrac{1}{\|\text{grad}\,\varphi\|^2}$ in a neighbourhood of $\{a \leqslant \varphi \leqslant b\}$ and consider on X the vector field Z defined by $Z_x = \rho(x)\,\text{grad}_x\varphi$. Then Z generates a 1-parameter group of diffeomorphisms $\gamma_t : X \to X$, $t \in \mathbf{R}$ [13]. We set $\gamma(t, x) = \gamma_t(x)$. For fixed $x \in X$ consider the function $t \to \varphi(\gamma_t(x))$. If $y_0 = \gamma_{t_0}(x)$ lies in the set $\{a \leqslant \varphi \leqslant b\}$ then

$$\frac{d}{dt}\varphi(\gamma_t(x))\Big|_{t=t_0} = \left\langle \frac{d\gamma_t(x)}{dt}\Big|_{t=t_0}, \text{grad}_{y_0}\varphi \right\rangle = +1$$

Thus the map $t \to \varphi(\gamma_t(x))$ is linear with derivative $+1$ if $a \leqslant \varphi(\gamma_t(x)) \leqslant b$. It follows that $\{\varphi = a\}$ is a strong deformation retract of $\{a \leqslant \varphi \leqslant b\}$ by the homotopy $F : \{a \leqslant \varphi \leqslant b\} \times [0,1] \to \{a \leqslant \varphi \leqslant b\}$ given by $F(x, t) = \gamma(t(a - \varphi(x)), x)$. Setting

$$R(x, t) = \begin{cases} x \text{ if } \varphi(x) \leqslant a \\ F(x, t) \text{ if } a \leqslant \varphi(x) \leqslant b \end{cases}$$

we get that $\{\varphi \leqslant a\}$ is a strong deformation retract of $\{\varphi \leqslant b\}$.

Consider now for a fixed $x \in X$ the function $t \to \psi(\gamma_t(x))$. If $y_0 = \gamma_{t_0}(x)$ lies in the set $P_{a,b}$ then

$$\frac{d}{dt}\psi(\gamma_t(x))\Big|_{t=t_0} = \left\langle \frac{d\gamma_t(x)}{dt}\Big|_{t=t_0}, \operatorname{grad}_{y_0}\psi \right\rangle = \langle Z_{y_0}, \operatorname{grad}_{y_0}\psi \rangle > 0$$

therefore the function $t \to \psi(\gamma_t(x))$ is strictly increasing in a neighbourhood of t_0. It follows that, for a fixed $x \in H_{a,b}$, the trajectory $\gamma(t, x)$ $\alpha \leqslant t \leqslant \beta$ lies in $H_{a,b}$ as soon as this trajectory is contained in $\{a \leqslant \varphi \leqslant b\}$. Hence the homotopy map R induces also a strong deformation retract of H_b onto H_a, and thus the proof of the lemma is complete.

LEMMA 2.4. *Let X be a compact real-analytic manifold, $A_1, \ldots, A_r$ closed analytic subsets of X, $A = A_1 \cup \cdots \cup A_r$, and $f_1, \ldots, f_r$ real-analytic functions on X such that $A_i = \{f_i = 0\}$ and $f_i \geqslant 0$ for every $i = 1, \ldots, r$.*

Then there exists a constant $c_0 > 0$ sufficiently small with the following property:
(P) *for any $\tau_1, \ldots, \tau_r \in C_0^\infty(X \setminus A)$, $\tau_1 \geqslant 0, \ldots, \tau_r \geqslant 0$, there exists a sufficiently small constant $\lambda_0 = \lambda_0(\tau_1, \ldots, \tau_r) > 0$ such that for any constants $0 \leqslant \mu_i \leqslant \lambda_0$ $i = 1, \ldots, r$ the set $A_i \cap A_j$ is a strong deformation retract of the set $C_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c) = \{x \in X \mid f_i(x)e^{\mu_i\tau_i(x)} \leqslant c, f_j(x)e^{\mu_j\tau_j(x)} \leqslant c\}$ for any $i, j = 1, \ldots, r$ and any $0 \leqslant c \leqslant c_0$.*

Proof. Clearly we may assume $r = 2$. We choose a constant $c_0 > 0$ sufficiently small such that, denoting $H' = \{f_1 \leqslant c_0, f_2 \leqslant c_0\}$, the following conditions are satisfied:

1) $d_x f_1 \neq 0$ for every $x \in \{f_1 \leqslant c_0\} \setminus A_1$ and $d_x f_2 \neq 0$ for every $x \in \{f_2 \leqslant c_0\} \setminus A_2$

2) at any point $x \in H' \setminus (A_1 \cup A_2)$ there is no a relation of the type $d_x f_1 = \lambda d_x f_2$ with $\lambda < 0$

3) for every $0 \leqslant c \leqslant c_0$ the set $A_1 \cap A_2$ is a strong deformation retract of $A_1 \cap \{f_2 \leqslant c\}$.

The existence of c_0 follows from Lemma 2.1, Lemma 2.2 and Corollary 2.1. We prove that this c_0 satisfies our property (P). Let $\tau_1, \tau_2 \in C_0^\infty(X \setminus A)$, $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$ and we have to choose the constant $\lambda_0 = \lambda_0(\tau_1, \tau_2) > 0$. The constant λ_0 is chosen sufficiently small such that for any $0 \leqslant \mu_1 \leqslant \lambda_0$, $0 \leqslant \mu_2 \leqslant \lambda_0$ the functions $f_1(x)e^{\mu_1\tau_1(x)}$, $f_2(x)e^{\mu_2\tau_2(x)}$ satisfy the conditions 1) and 2) on the set $H' \cap \{\text{supp}\,\tau_1 \cup \text{supp}\,\tau_2\}$. By the condition 3) and Lemma 2.3 it follows that the set $A_1 \cap A_2$ is a strong deformation retract of $C_{12}(\tau_1, \tau_2, \mu_1, \mu_2, c)$ for any $0 \leqslant c$

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$\leqslant c_0$, $0 \leqslant \mu_1 \leqslant \lambda_0$, $0 \leqslant \mu_2 \leqslant \lambda_0$, which proves Lemma 2.4.

§3. Some other technical results

Beginning with this paragraph by an analytic subset of a complex space we mean a complex analytic subset.

LEMMA 3.1. *Let X be a compact normal complex space of pure dimension n and $A_1, \ldots, A_r$ closed analytic subsets. We set $A = A_1 \cup \cdots \cup A_r$* and *$Y_i = X \backslash A_i$ $i = 1, \ldots, r$. We suppose that on each Y_i an $(n-1)$-convex exhaustion function $\psi_i : Y_i \to \mathbf{R}$ is given. We define $D_i = \{\psi_i < 0\}$, $Y = X \backslash A = Y_1 \cap \cdots \cap Y_r$, $D = D_1 \cap \cdots \cap D_r$, $B_{ij} = Y_i \cup Y_j \backslash (D_i \cup D_j)$ and we assume that for every $1 \leqslant i, j \leqslant r$ the set B_{ij} has no compact connected components.*
Then it follows:

a) *the natural map $H_{2n-1}(D, \mathbf{C}) \to H_{2n-1}(Y, \mathbf{C})$ is bijective*

b) *the natural map $H_{2n-2}(D, \mathbf{C}) \to H_{2n-2}(Y, \mathbf{C})$ is injective.*

Proof. We first remark that by our hypothesis "B_{ij} has no compact connected components" it follows that the map $H_{2n-1}(D_i \cup D_j, \mathbf{C}) \to H_{2n-1}(Y_i \cup Y_j, \mathbf{C})$ is injective and the map $H_{2n}(D_i \cup D_j, \mathbf{C}) \to H_{2n}(Y_i \cup Y_j, \mathbf{C})$ is bijective for any $i, j \in \{1, \ldots, r\}$ (e.g. by Theorem 1.5).

Let us prove a). We show by induction on k that for any $i_1, \ldots, i_k \in \{1, \ldots, r\}$ the map $H_{2n-1}(D_{i_1} \cap \cdots \cap D_{i_k}, \mathbf{C}) \to H_{2n-1}(Y_{i_1} \cap \cdots \cap Y_{i_k}, \mathbf{C})$ is bijective. If $k = 1$ this is clear because $H_{2n-1}(D_i, \mathbf{C}) = H_{2n-1}(Y_i, \mathbf{C}) = 0$ by Theorem 1.4. If $k = 2$ we consider the following commutative diagram with exact lines given by the Mayer-Vietoris sequence:

$$\begin{array}{ccccccc}
 & 0 \quad\quad 0 & & & & 0 \quad\quad 0 & \\
 & \| \quad\quad \| & & & & \| \quad\quad \| & \\
\cdots \to & H_{2n}(D_i, \mathbf{C}) \oplus H_{2n}(D_j, \mathbf{C}) & \to & H_{2n}(D_i \cup D_j, \mathbf{C}) \to H_{2n-1}(D_i \cap D_j, \mathbf{C}) & \to & H_{2n-1}(D_i, \mathbf{C}) \oplus H_{2n-1}(D_j, \mathbf{C}) & \to \cdots \\
 & \downarrow & & \alpha \downarrow \wr \qquad\qquad \downarrow \beta & & \downarrow & \\
\cdots \to & H_{2n}(Y_i, \mathbf{C}) \oplus H_{2n}(Y_j, \mathbf{C}) & \to & H_{2n}(Y_i \cup Y_j, \mathbf{C}) \to H_{2n-1}(Y_i \cap Y_j, \mathbf{C}) & \to & H_{2n-1}(Y_i, \mathbf{C}) \oplus H_{2n-1}(Y_j, \mathbf{C}) & \to \cdots \\
 & \| \quad\quad \| & & & & \| \quad\quad \| & \\
 & 0 \quad\quad 0 & & & & 0 \quad\quad 0 &
\end{array}$$

Since α is an isomorphism it follows that β is an isomorphism too.
Suppose now that $k \geqslant 3$. To simplify the notation we put $i_1 = 1, \ldots, i_k = k$. To apply the induction hypothesis we write:

$$D_1 \cap \cdots \cap D_{k-1} \cap D_k = (D_1 \cap \cdots \cap D_{k-1}) \cap (D_2 \cap \cdots \cap D_k)$$
$$Y_1 \cap \cdots \cap Y_{k-1} \cap Y_k = (Y_1 \cap \cdots \cap Y_{k-1}) \cap (Y_2 \cap \cdots \cap Y_k)$$

and

$$(D_1 \cap \cdots \cap D_{k-1}) \cup (D_2 \cap \cdots \cap D_k) = (D_2 \cap \cdots \cap D_{k-1}) \cap (D_1 \cup D_k)$$
$$(Y_1 \cap \cdots \cap Y_{k-1}) \cup (Y_2 \cap \cdots \cap Y_k) = (Y_2 \cap \cdots \cap Y_{k-1}) \cap (Y_1 \cup Y_k).$$

By the Mayer-Vietoris sequence we get the commutative diagram with exact lines:

$$\begin{array}{ccc}
 & 0 & \\
 & \| & \\
\cdots \to & H_{2n}((D_2 \cap \cdots \cap D_{k-1}) \cap (D_1 \cup D_k), \mathbf{C}) & \to H_{2n-1}(D_1 \cap \cdots \cap D_k, \mathbf{C}) \to \\
 & \downarrow & \downarrow g \\
\cdots \to & H_{2n}((Y_2 \cap \cdots \cap Y_{k-1}) \cap (Y_1 \cup Y_k), \mathbf{C}) & \to H_{2n-1}(Y_1 \cap \cdots \cap Y_k, \mathbf{C}) \to \\
 & \| & \\
 & 0 &
\end{array}$$

$$\begin{array}{ccc}
H_{2n-1}(D_1 \cap \cdots \cap D_{k-1}, \mathbf{C}) \oplus H_{2n-1}(D_2 \cap \cdots \cap D_k, \mathbf{C}) & \to & H_{2n-1}((D_2 \cap \cdots \cap D_{k-1}) \cap (D_1 \cup D_k), \mathbf{C}) \to \cdots \\
\downarrow h & & \downarrow j \\
H_{2n-1}(Y_1 \cap \cdots \cap Y_{k-1}, \mathbf{C}) \oplus H_{2n-1}(Y_2 \cap \cdots \cap Y_k, \mathbf{C}) & \to & H_{2n-1}((Y_2 \cap \cdots \cap Y_{k-1}) \cap (Y_1 \cup Y_k), \mathbf{C}) \to \cdots
\end{array}$$

By induction h is an isomorphism therefore g is injective. To prove that g is surjective it suffices to show that j is injective. To see this we consider the following commutative diagram given by the Mayer-Vietoris sequence:

$$\begin{array}{ccc}
 & 0 & \\
 & \| & \\
\cdots \to & H_{2n}(D_2 \cap \cdots \cap D_{k-1}, \mathbf{C}) \oplus H_{2n}(D_1 \cup D_k, \mathbf{C}) & \to H_{2n}((D_2 \cap \cdots \cap D_{k-1}) \cup (D_1 \cup D_k), \mathbf{C}) \to \\
 & \downarrow m & \downarrow n \\
\cdots \to & H_{2n}(Y_2 \cap \cdots \cap Y_{k-1}, \mathbf{C}) \oplus H_{2n}(Y_1 \cup Y_k, \mathbf{C}) & \to H_{2n}((Y_2 \cap \cdots \cap Y_{k-1}) \cup (Y_1 \cup Y_k), \mathbf{C}) \to \\
 & \| & \\
 & 0 &
\end{array}$$

$$\begin{array}{ccc}
H_{2n-1}((D_2 \cap \cdots \cap D_{k-1}) \cap (D_1 \cup D_k), \mathbf{C}) & \to & H_{2n-1}(D_2 \cap \cdots \cap D_{k-1}, \mathbf{C}) \oplus H_{2n-1}(D_1 \cup D_k, \mathbf{C}) \to \cdots \\
\downarrow j & & \downarrow p \\
H_{2n-1}((Y_2 \cap \cdots \cap Y_{k-1}) \cap (Y_1 \cup Y_k), \mathbf{C}) & \to & H_{2n-1}(Y_2 \cap \cdots \cap Y_{k-1}, \mathbf{C}) \oplus H_{2n-1}(Y_1 \cup Y_k, \mathbf{C}) \to \cdots
\end{array}$$

Clearly m is an isomorphism and n is injective. On the other hand the map $H_{2n-1}(D_2 \cap \cdots \cap D_{k-1}, \mathbf{C}) \to H_{2n-1}(Y_2 \cap \cdots \cap Y_{k-1}, \mathbf{C})$ is bijective by induction and the map $H_{2n-1}(D_1 \cup D_k, \mathbf{C}) \to H_{2n-1}(Y_1 \cup Y_k, \mathbf{C})$ is injective as remarked at the beginning of the proof. It follows that p is injective, and from the

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commutativity of the diagram, j is injective too. Thus a) is completely proved.

We now verify b) proving by induction on k that the map $H_{2n-2}(D_{i_1} \cap \cdots \cap D_{i_k}, \mathbf{C}) \to H_{2n-2}(Y_{i_1} \cap \cdots \cap Y_{i_k}, \mathbf{C})$ is injective for any $i_1, \ldots, i_k \in \{1, \ldots, r\}$. If $k = 1$ the injectivity is clear because D_i is $(n-1)$-Runge in Y_i and we can apply Theorem 1.4.

If $k = 2$ we consider the diagram given by the Mayer-Vietoris sequence:

$$\begin{array}{ccccccccc}
 & 0 \quad\quad 0 & & & & & & & \\
 & \| \quad\quad \| & & & & & & & \\
\cdots \to & H_{2n-1}(D_i, \mathbf{C}) \oplus H_{2n-1}(D_j, \mathbf{C}) & \to & H_{2n}(D_i \cup D_j, \mathbf{C}) & \to & H_{2n-2}(D_i \cap D_j, \mathbf{C}) & \to & H_{2n-2}(D_i, \mathbf{C}) \oplus H_{2n-2}(D_j, \mathbf{C}) & \to \\
 & \downarrow & & \downarrow a & & \downarrow b & & \downarrow c & \\
\cdots \to & H_{2n-1}(Y_i, \mathbf{C}) \oplus H_{2n-1}(Y_j, \mathbf{C}) & \to & H_{2n-1}(Y_i \cup Y_j, \mathbf{C}) & \to & H_{2n-2}(Y_i \cap Y_j, \mathbf{C}) & \to & H_{2n-2}(Y_i, \mathbf{C}) \oplus H_{2n-2}(Y_j, \mathbf{C}) & \to \\
 & \| \quad\quad \| & & & & & & & \\
 & 0 \quad\quad 0 & & & & & & &
\end{array}$$

Note that $H_{2n-1}(D_i, \mathbf{C}) = H_{2n-1}(D_j, \mathbf{C}) = H_{2n-1}(Y_i, \mathbf{C}) = H_{2n-1}(Y_j, \mathbf{C}) = 0$ by Theorem 1.4 because D_i, D_j, Y_i, Y_j are $(n-1)$-complete. Also by Theorem 1.4 the maps $H_{2n-2}(D_i, \mathbf{C}) \to H_{2n-2}(Y_i, \mathbf{C})$ and $H_{2n-2}(D_j, \mathbf{C}) \to H_{2n-2}(Y_j, \mathbf{C})$ are injective, therefore c is injective too. The map a is injective by the remark at the beginning of the proof. If follows that b is injective.

Suppose now that $k \geqslant 3$ and to simplify the notations we put $i_1 = 1, \ldots, i_k = k$. As in a), in order to apply the induction hypothesis, we write:

$$D_1 \cap \cdots \cap D_{k-1} \cap D_k = (D_1 \cap \cdots \cap D_{k-1}) \cap (D_2 \cap \cdots \cap D_k)$$
$$Y_1 \cap \cdots \cap Y_{k-1} \cap Y_k = (Y_1 \cap \cdots \cap Y_{k-1}) \cap (Y_2 \cap \cdots \cap Y_k)$$

and

$$(D_1 \cap \cdots \cap D_{k-1}) \cup (D_2 \cap \cdots \cap D_k) = (D_2 \cap \cdots \cap D_{k-1}) \cap (D_1 \cup D_k)$$
$$(Y_1 \cap \cdots \cap Y_{k-1}) \cup (Y_2 \cap \cdots \cap Y_k) = (Y_2 \cap \cdots \cap Y_{k-1}) \cap (Y_1 \cup Y_k).$$

Consider the following diagram given by the Mayer-Vietoris sequence:

$$\begin{array}{ccccc}
\cdots \to H_{2n-1}(D_1 \cap \cdots \cap D_{k-1}, \mathbf{C}) \oplus H_{2n-1}(D_2 \cap \cdots \cap D_k, \mathbf{C}) & \to & H_{2n-1}((D_2 \cap \cdots \cap D_{k-1}) \cap (D_1 \cup D_k), \mathbf{C}) & \to \\
\downarrow u & & \downarrow j & \\
\cdots \to H_{2n-1}(Y_1 \cap \cdots \cap Y_{k-1}, \mathbf{C}) \oplus H_{2n-1}(Y_2 \cap \cdots \cap Y_k, \mathbf{C}) & \to & H_{2n-1}((Y_2 \cap \cdots \cap Y_{k-1}) \cap (Y_1 \cup Y_k), \mathbf{C}) & \to
\end{array}$$

$$\begin{array}{ccccc}
H_{2n-2}(D_1 \cap \cdots \cap D_k, \mathbf{C}) & \to & H_{2n-2}(D_1 \cap \cdots \cap D_{k-1}, \mathbf{C}) \oplus H_{2n-2}(D_2 \cap \cdots \cap D_k, \mathbf{C}) & \to \cdots \\
\downarrow v & & \downarrow w & \\
H_{2n-2}(Y_1 \cap \cdots \cap Y_k, \mathbf{C}) & \to & H_{2n-2}(Y_1 \cap \cdots \cap Y_{k-1}, \mathbf{C}) \oplus H_{2n-2}(Y_2 \cap \cdots \cap Y_k, \mathbf{C}) & \to \cdots
\end{array}$$

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The map u is bijective by a) and w is injective by induction. On the other hand at a) it was shown that the map j is injective. It follows that v is injective and thus the proof of Lemma 3.1 is complete.

Before stating the next lemma we introduce some notations. Let $\mathcal{U} = (U_i)_{i\in I}$ be a countable Stein open covering of a complex space X such that $\mathcal{U}$ is a base of open sets for the topology of X. If $\mathcal{F} \in \mathrm{Coh}(X)$ we denote by $C^p(\mathcal{U}, \mathcal{F})$ the Fréchet space of Čech cochains, $\delta = \delta^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ the coboundary operator, $Z^p(\mathcal{U}, \mathcal{F}) = \ker \delta^p$ the Fréchet space of cocycles, $B^p(\mathcal{U}, \mathcal{F}) = \mathrm{im}\, \delta^{p-1}$ the space of cobords, $H^p(X, \mathcal{F}) = H^p(\mathcal{U}, \mathcal{F}) = Z^p(\mathcal{U}, \mathcal{F}) / B^p(\mathcal{U}, \mathcal{F})$ the cohomology of X with values in $\mathcal{F}$, with the quotient topology, which clearly does not depend on $\mathcal{U}$. If $D \subset X$ is an open set we define $\mathcal{U}|_D = \{U_i \in \mathcal{U} \mid U_i \subset D\}$.

It is easy to verify that the following conditions are equivalent:

i) The restriction map $H^p(X, \mathcal{F}) \to H^p(D, \mathcal{F})$ has dense image

ii) The restriction map $Z^p(\mathcal{U}, \mathcal{F}) \to Z^p(\mathcal{U}|_D, \mathcal{F}\}$ has dense image for every covering $\mathcal{U}$ as above

iii) The restriction map $Z^p(\mathcal{U}, \mathcal{F}) \to Z^p(\mathcal{U}|_D, \mathcal{F})$ has dense image for one covering $\mathcal{U}$ as above.

LEMMA 3.2. *Let X be a complex space such that $X = D_1 \cup D_2$ where $D_1, D_2 \subset X$ are open sets and let $\mathcal{F} \in \mathrm{Coh}(X)$. Consider the Mayer-Vietoris sequence:*

$$H^i(D_1 \cap D_2, \mathcal{F}) \xrightarrow{u^*} H^{i+1}(D_1 \cup D_2, \mathcal{F}) \to H^{i+1}(D_1, \mathcal{F}) \oplus H^{i+1}(D_2, \mathcal{F})$$

and assume that for some i $H^{i+1}(D_1, \mathcal{F}) = H^{i+1}(D_2, \mathcal{F}) = 0$.

Then the map $H^i(D_1 \cap D_2, \mathcal{F}) \xrightarrow{u^*} H^{i+1}(D_1 \cup D_2, \mathcal{F})$ is open.

Proof. Let $\mathcal{U} = (U_i)_{i\in I}$ be a countable Stein open covering of X such that $\mathcal{U}$ is a base of open sets for the topology of X and such that the following condition is satisfied:

c) $U_{n_0} \cap \cdots \cap U_{n_q} \neq \phi \Rightarrow U_{n_0} \cup \cdots \cup U_{n_q} \subset D_1$ or $U_{n_0} \cup \cdots \cup U_{n_q} \subset D_2$

This condition can be written also as follows:

$$\mathrm{nerf}\, \mathcal{U} = \mathrm{nerf}\, \mathcal{U}|_{D_1} \cup \mathrm{nerf}\, \mathcal{U}|_{D_2}$$

We show that the map $u^* : H^i(\mathcal{U}|_{D_1\cap D_2}, \mathcal{F}) \to H^{i+1}(\mathcal{U}, \mathcal{F})$ is induced by a continuous linear map $u : Z^i(\mathcal{U}|_{D_1\cap D_2}, \mathcal{F}) \to Z^{i+1}(\mathcal{U}, \mathcal{F})$ between cocycles. Let $\gamma \in Z^i(\mathcal{U}|_{D_1\cap D_2}, \mathcal{F})$ and let $\gamma_1 \in C^i(\mathcal{U}|_{D_1}, \mathcal{F})$ be the extension of γ to D_1 by zero, i.e.

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$$\gamma_1(n_0, \ldots, n_i) = \begin{cases} \gamma(n_0, \ldots, n_i) & \text{if } U_{n_0}, \ldots, U_{n_i} \in \mathcal{U}|_{D_1 \cap D_2} \\ 0 & \text{otherwise} \end{cases}$$

Then $\delta\gamma_1 \in Z^{i+1}(\mathcal{U}|_{D_1}, \mathscr{F})$ and we define $u(\gamma)$ to be the extension of $\delta\gamma_1$ to X by zero, i.e.

$$u(\gamma)(n_0, \ldots, n_{i+1}) = \begin{cases} (\delta\gamma_1)(n_0, \ldots, n_{i+1}) & \text{if } U_{n_0}, \ldots, U_{n_{i+1}} \in \mathcal{U}|_{D_1} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $u(\gamma) \in C^{i+1}(\mathcal{U}, \mathscr{F})$ but, by condition C), it follows easily that in fact $u(\gamma) \in Z^{i+1}(\mathcal{U}, \mathscr{F})$. From the definition of u it is obvious that u^* is induced (modulo a sign) by u, hence we have the commutative diagram

$$\begin{array}{ccc} Z^i(\mathcal{U}|_{D_1 \cap D_2}, \mathscr{F}) & \xrightarrow{u} & Z^{i+1}(\mathcal{U}, \mathscr{F}) \\ \downarrow & & \downarrow \\ H^i(\mathcal{U}|_{D_1 \cap D_2}, \mathscr{F}) & \xrightarrow{u^*} & H^{i+1}(\mathcal{U}, \mathscr{F}) \end{array}$$

By our hypothesis u^* is surjective. Therefore the linear continuous map s: $Z^i(\mathcal{U}|_{D_1 \cap D_2}, \mathscr{F}) \oplus C^i(\mathcal{U}, \mathscr{F}) \to Z^{i+1}(\mathcal{U}, \mathscr{F})$ defined by $s(\gamma \oplus \beta) = u(\gamma) - \delta^i(\beta)$ is surjective. By the open mapping theorem s is open, hence u^* is also open. The proof of Lemma 3.2 is complete.

LEMMA 3.3. *Let U be a Stein normal space of pure dimension n, $\varphi_1, \ldots, \varphi_r$ $(n-1)$-convex functions on U and define $D = \{\varphi_1 < 0, \ldots, \varphi_r < 0\}$. Let also $\psi_1 \leqslant \varphi_1$ be an $(n-1)$-convex function on U and put $V = \{\psi_1 < 0, \varphi_2 < 0, \ldots, \varphi_r < 0\}$. Assume that the following topological condition is satisfied:*

the map $H_{2n-2}(D, \mathbf{C}) \to H_{2n-2}(V, \mathbf{C})$ is injective.

Then for every $\mathscr{F} \in \mathrm{Coh}(U)$ the restriction map $H^{n-2}(V, \mathscr{F}) \to H^{n-2}(D, \mathscr{F})$ has dense image.

Proof. We define $A = \{\varphi_1 < 0\}$, $A' = \{\psi_1 < 0\}$, $B = \{\varphi_2 < 0, \ldots, \varphi_r < 0\}$, hence $D = A \cap B$, $V = A' \cap B$. Then A, A' are $(n-1)$-Runge domains in U since U is Stein and φ_1, ψ_1 are $(n-1)$-convex functions on U. In particular $H_{2n-1}(A, \mathbf{C}) = H_{2n-1}(A', \mathbf{C}) = 0$ by Theorem 1.4. Also it follows easily by induction, using the Mayer-Vietoris sequence and Theorem 1.4, that $H_{2n-1}(B, \mathbf{C}) = 0$. We consider now the following commutative diagram given by the Mayer-Vietoris sequence for homology:

$$\begin{array}{ccccccc} & 0 & & 0 & & & \\ & \| & & \| & & & \\ \cdots \to & H_{2n-1}(A, \mathbf{C}) & \oplus & H_{2n-1}(B, \mathbf{C}) & \to H_{2n-1}(A \cup B, \mathbf{C}) & \to H_{2n-2}(D, \mathbf{C}) & \to \cdots \\ & & \downarrow & & \downarrow u & \downarrow v & \\ \cdots \to & H_{2n-1}(A', \mathbf{C}) & \oplus & H_{2n-1}(B, \mathbf{C}) & \to H_{2n-1}(A' \cup B, \mathbf{C}) & \to H_{2n-2}(V, \mathbf{C}) & \to \cdots \\ & \| & & \| & & & \\ & 0 & & 0 & & & \end{array}$$

By hypothesis v is injective, hence u is injective too. From Theorem 1.5 it follows that the restriction map $\rho_3 : H^{n-1}(A' \cup B, \mathscr{F}) \to H^{n-1}(A \cup B, \mathscr{F})$ has dense image. Note also that by Theorem 1.3 the restriction map $\rho_1 : H^{n-2}(A', \mathscr{F}) \to H^{n-2}(A, \mathscr{F})$ has dense image because A is a $(n-1)$-Runge domain in $U \supset A'$. On the other hand $H^{n-1}(A', \mathscr{F}) = H^{n-1}(A, \mathscr{F}) = 0$ since A, A' are $(n-1)$-complete and, by induction, using the Mayer-Vietoris sequence and Theorem 1.1, it follows that $H^{n-1}(B, \mathscr{F}) = 0$.

We write now the Mayer-Vietoris sequence for cohomology in the following commutative diagram:

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \| & & \| \\ \to H^{n-2}(A', \mathscr{F}) \oplus H^{n-2}(B, \mathscr{F}) & \to & H^{n-2}(V, \mathscr{F}) & \to H^{n-1}(A' \cup B, \mathscr{F}) & \to H^{n-1}(A', \mathscr{F}) & \oplus & H^{n-1}(B, \mathscr{F}) \to \cdots \\ \downarrow \rho_1 \otimes \mathrm{id} & & \downarrow \rho_2 & \downarrow \rho_3 & & \downarrow & \\ \to H^{n-2}(A, \mathscr{F}) \oplus H^{n-2}(B, \mathscr{F}) & \to & H^{n-2}(D, \mathscr{F}) & \xrightarrow{u^*} H^{n-1}(A \cup B, \mathscr{F}) & \to H^{n-1}(A, \mathscr{F}) & \oplus & H^{n-1}(B, \mathscr{F}) \to \cdots \\ & & & & \| & & \| \\ & & & & 0 & & 0 \end{array}$$

In this diagram $\rho_1 \oplus \mathrm{id}$ and ρ_3 have dense images and u^* is open by Lemma 3.2. It follows that ρ_2 has also dense image, which proves our Lemma.

Remark 3.1. The condition of normality is not necessary in the above lemma because, as remarked in §1, the first 3 conditions in Theorem 1.5 are equivalent also for non-normal complex spaces.

LEMMA 3.4. *Let Y be a complex space of dimension n and $\varphi_i : Y \to \mathbf{R}$ $i = 1, \ldots, r$ $(n-1)$-convex functions. Assume that $\varphi = \max(\varphi_1, \ldots, \varphi_r)$ is an exhaustion function on Y and define $Y_c = \{\varphi < c\} c \in \mathbf{R}$. Then for every $\mathscr{F} \in \mathrm{Coh}(Y)$ the restriction map $H^{n-1}(Y, \mathscr{F}) \to H^{n-1}(Y_c, \mathscr{F})$ is surjective. In paricular $\dim_{\mathbf{C}} H^{n-1}(Y_c, \mathscr{F}) < \infty$.*

Proof. We make first the following remark: Let U be a Stein space of dimension n, $\psi_1, \ldots, \psi_r : U \to \mathbf{R}$ $(n-1)$-convex functions, $\psi = \max(\psi_1, \ldots, \psi_r)$ and define $V = \{\psi < 0\}$. Then V is a finite intersection of $(n-1)$-complete domains and it follows by induction, using the Mayer-Vietoris sequence, that $H^{n-1}(V, \mathscr{F}) = 0$ for every $\mathscr{F} \in \mathrm{Coh}(U)$.

We come back now to the proof of Lemma 3.4. It is enough to show, by ([1], Lemme p. 241), that for any $\alpha \in \mathbf{R}$ there exists $\beta > \alpha$ such that the restriction map $H^{n-1}(Y_\beta, \mathscr{F}) \to H^{n-1}(Y_\gamma, \mathscr{F})$ is surjective for every $\alpha \leqslant \gamma \leqslant \beta$. To see this we use the "bumping method" of Andreotti-Grauert in [1]. We consider Stein open subsets U_i $i = 1, \ldots, m$, $U_i \subset\subset Y$, such that $\{\varphi = \alpha\} \subset U_1 \cup \cdots \cup U_m$ and we choose $\theta_t \in C_0^\infty(U_i, \mathbf{R})$, $\theta_i \geqslant 0$ $i = 1, \ldots, m$ with $\sum_{t=1}^m \theta_t(x) > 0$ at any point $x \in \{\varphi = \alpha\}$. Let also $\varepsilon_1 > 0, \ldots, \varepsilon_m > 0$ be sufficiently small constants such that all the functions $\psi_{i,t} = \varphi_i - \varepsilon_1\theta_1 - \ldots - \varepsilon_t\theta_t$ $i = 1, \ldots, r$ $t = 1, \ldots, m$ are $(n-1)$-convex on Y.
We define $D_0(c) = X_c$, $D_1(c) = \{\varphi - \varepsilon_1\theta_1 < c\}, \ldots, D_m(c) = \{\varphi - \varepsilon_1\theta_1 - \ldots - \varepsilon_m\theta_m < c\}$ for $c \in \mathbf{R}$. Since $D_{t+1}(c) \setminus D_i(c) \subset U_{i+1}$, the Mayer-Vietoris sequence and the remark at the beginning of the proof, show that the restriction maps $H^{n-1}(D_{i+1}(c), \mathscr{F}) \to H^{n-1}(D_i(c), \mathscr{F})$ are surjective for $i = 1, \ldots, m-1$. Therefore $H^{n-1}(D_m(c), \mathscr{F}) \to H^{n-1}(Y_c, \mathscr{F})$ is surjective too. Suppose now that $c = \alpha$. In this case $\{\varphi \leqslant \alpha) \subset D_m(\alpha)$, hence there is $\beta > \alpha$ with $Y_\beta \subset D_m(\alpha)$. If $\alpha \leqslant \gamma \leqslant \beta$ then we have the inclusions $Y_\gamma \subset Y_\beta \subset D_m(\alpha) \subset D_m(\gamma)$ and it follows that the restriction map $H^{n-1}(Y_\beta, \mathscr{F}) \to H^{n-1}(Y_\gamma, \mathscr{F})$ is surjective, which proves our lemma.

LEMMA 3.5. *Let X be a compact normal complex space of pure dimension n, $A_1, \ldots, A_r$ closed analytic subsets, $A = A_1 \cup \cdots \cup A_r$. We assume that on each $X \setminus A_i$ an $(n-1)$-convex exhaustion function $\varphi_i : X \setminus A_i \to \mathbf{R}$ is given and define $\varphi : X \setminus A \to \mathbf{R}$ by $= \max(\varphi_1, \ldots, \varphi_r)$. We suppose that the following hypothesis* (H) *is satisfied: There exists a constant $\alpha_0 > 0$ sufficiently large with the following property*: (P′) *for any $\tau_1, \ldots, \tau_r \in C_0^\infty(X \setminus A)$, $\tau_1 \geqslant 0, \ldots, \tau_r \geqslant 0$ there is a sufficiently small constant $\lambda_0(\tau_1, \ldots, \tau_r) > 0$ such that for all constants $0 \leqslant \mu_t \leqslant \lambda_0$ $i = 1, \ldots, r$ the set $B_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c) = (X \setminus A_i) \cup (X \setminus A_j) \setminus (\{x \in X \setminus A_i \mid \varphi_i(x) - \mu_t\tau_t(x) < c\} \cup \{x \in X \setminus A_j \mid \varphi_j(x) - \mu_j\tau_j(x) < c\})$ has no compact connected components for every $c > \alpha_0$. Then, under these assumptions, it follows that for every $\mathscr{F} \in \mathrm{Coh}(X \setminus A)$ and any $c > \alpha_0$ the restriction map $H^{n-1}(X \setminus A, \mathscr{F}) \to H^{n-1}(\{\varphi < c\}, \mathscr{F})$ is bijective. In particular $X \setminus A$ is cohomologically $(n-1)$-convex.*

Remark 3.2. We shall see later (Lemma 3.10) that the hypothesis (H) is always satisfied if the functions $\exp(-\varphi_i)$ are real-analytic on whole X. This will follow from our results in §2.

Proof of Lemma 3.5. In view of Lemma 3.4 and by the approximation result in ([1], p. 250) it is enough to show that for any $\alpha > \alpha_0$ there is $\beta > \alpha$ such that:

1) For any γ with $\alpha \leqslant \gamma \leqslant \beta$ the restriction map $H^{n-2}(X_\beta, \mathscr{F}) \to H^{n-2}(X_\gamma, \mathscr{F})$ has dense image

2) The restriction map $H^{n-1}(X_\beta, \mathscr{F}) \to H^{n-1}(X_\alpha, \mathscr{F})$ is bijective

where $X_c = \{x \in X \backslash A \mid \varphi(x) < c\}$.

So let $\alpha > \alpha_0$ be fixed and we have to choose β.

We choose a Stein open covering $\{U_j\}$ $j = 1, \ldots, m$ of $\{\varphi = \alpha\}$, $U_j \subset\subset X \backslash A$, $\bar{U}_j \cap \{\varphi \leqslant \alpha_0\} = \phi$.

Let also $\theta_j \in C_0^\infty(U_j)$, $\theta_j \geqslant 0$ $j = 1, \ldots, m$ be such that $\sum_{j=1}^m \theta_j(x) > 0$ at any point $x \in \{\varphi = \alpha\}$.

From the $(n-1)$-convexity of the functions φ_i, $i = 1, \ldots, r$ there is an $k_0 > 0$ sufficiently small such that for any ε_j, $0 < \varepsilon_j < k_0$ $j = 1, \ldots, m$ all the functions $\psi_{i,t} = \varphi_i - \varepsilon_1\theta_1 - \ldots - \varepsilon_t\theta_t$ $i = 1, \ldots, r$ $t = 1, \ldots, m$ are $(n-1)$-convex on $X \backslash A_i$ $i = 1, \ldots, r$. Let $\varepsilon' = (\varepsilon'_1, \ldots, \varepsilon'_m)$ be fixed $0 < \varepsilon'_i < k_0$ $i = 1, \ldots, m$.

For every $h \in \mathbf{R}$ we define the open sets $D_{j,k}(\varepsilon', h) \subset X \backslash A$ $j = 1, \ldots, m$ $k = 0, \ldots, r$ as follows:

For $j = 1$ $D_{1,0}(\varepsilon', h) = \{\varphi_1 < h, \ldots, \varphi_r < h\}$, $D_{1,1}(\varepsilon', h) = \{\varphi_1 - \varepsilon'_1\theta_1 < h, \varphi_2 < h, \ldots, \varphi_r < h\} \cdots D_{1,r}(\varepsilon', h) = \{\varphi_1 - \varepsilon'_1\theta_1 < h, \varphi_2 - \varepsilon'_1\theta_1 < h, \ldots, \varphi_r - \varepsilon'_1\theta_1 < h\}$.

In general for arbitrary j and k we define

$$D_{j,k}(\varepsilon', h) = \{\varphi_1 - \varepsilon'_1\theta_1 - \ldots - \varepsilon'_j\theta_j < h, \ldots, \varphi_k - \varepsilon'_1\theta_1 - \ldots - \varepsilon'_j\theta_j < h, \varphi_{k+1} - \varepsilon'_1\theta_1 - \ldots \varepsilon'_{j-1}\theta_{j-1} < h, \ldots, \varphi_r - \varepsilon'_1\theta_1 - \ldots - \varepsilon'_{j-1}\theta_{j-1} < h\}.$$

Hence we have the equalities:

$$\begin{aligned} D_{1,0}(\varepsilon', h) &= \{\varphi < h\} = X_h, \\ D_{1,r}(\varepsilon', h) &= D_{2,0}(\varepsilon', h) = \{\varphi - \varepsilon'_1\theta_1 < h\}, \ldots, D_{j,r}(\varepsilon', h) \\ &= D_{j+1,0}(\varepsilon', h) = \{\varphi - \varepsilon'_1\theta_1 - \ldots - \varepsilon'_j\theta_j < h\}, \ldots, \\ D_{m,r}(\varepsilon', h) &= \{\varphi - \varepsilon'_1\theta_1 - \ldots - \varepsilon'_m\theta_m < h\} \end{aligned}$$

and the sequence of inclusions:

$$\begin{aligned} \{\varphi < h\} = D_{1,0}(\varepsilon', h) \subset \cdots \subset D_{1,r}(\varepsilon', h) = D_{2,0}(\varepsilon', h) \subset \cdots \subset D_{2,r}(\varepsilon', h) \\ = D_{3,0}(\varepsilon', h) \subset \cdots \subset D_{m,r}(\varepsilon', h). \end{aligned}$$

Since all the functions $\varepsilon'_1\theta_1 + \ldots + \varepsilon'_t\theta_t$ belong to $C_0^\infty(X \backslash A)$ it follows by our hypothesis (H) and by Lemma 3.1 that there is some $0 < \tau < 1$ sufficiently small such that:

a') $H_{2n-1}(D_{j,k}(\tau\varepsilon', h), \mathbf{C}) \to H_{2n-1}(X \setminus A, \mathbf{C})$ is bijective

b') $H_{2n-2}(D_{j,k}(\tau\varepsilon', h), \mathbf{C}) \to H_{2n-2}(X \setminus A, \mathbf{C})$ is injective

if $h > \alpha_0$ $j = 1, \ldots, m$ $k = 0, \ldots, r$

We fix this $\tau > 0$ and we put $\varepsilon = \tau\varepsilon'$ and $D_{j,k}(\varepsilon, h) = D_{j,k}(h)$ (from now on ε will remain fixed). Clearly all the functions $\psi_{i,t} = \varphi_i - \varepsilon_1\theta_1 - \ldots - \varepsilon_t\theta_t$ $i = 1, \ldots, r$ $t = 1, \ldots, m$ are $(n-1)$-convex on $X \setminus A_i$ and the conditions a'), b') can be written:

a) $H_{2n-1}(D_{j,k}(h), \mathbf{C}) \to H_{2n-1}(X \setminus A, \mathbf{C})$ is bijective

b) $H_{2n-2}(D_{j,k}(h), \mathbf{C}) \to H_{2n-2}(X \setminus A, \mathbf{C})$ is injective

if $h > \alpha_0$ $j = 1, \ldots, m$ $k = 0, \ldots, r$.

We have $\{\varphi \leqslant \alpha\} \subset D_{m,r}(\alpha)$ because $\sum_{j=1}^m \theta_j(x) > 0$ at any point $x \in \{\varphi = \alpha\}$, therefore there is some $\beta > \alpha$ with $X_\beta = \{\varphi < \beta\} \subset D_{m,r}(\alpha)$. We shall prove that this β satisfies our conditions 1) and 2) stated at the beginning of the proof.

Step 1. We show the density of the image of the restriction map

(*) $H^{n-2}(D_{m,r}(h), \mathscr{F}) \to H^{n-2}(X_h, \mathscr{F})$ for every $h > \alpha_0$.

By the conditions a) and b) we get:

a_1) $H_{2n-1}(D_{j,k}(h), \mathbf{C}) \to H_{2n-1}(D_{j,k+1}(h), \mathbf{C})$ is bijective

b_1) $H_{2n-2}(D_{j,k}(h), \mathbf{C}) \to H_{2n-2}(D_{j,k+1}(h), \mathbf{C})$ is injective

if $h > \alpha_0$.

Since $\operatorname{supp} \theta_j \subset U_j$ $j = 1, \ldots, m$ it follows that $D_{j,k+1}(h) \setminus D_{j,k}(h) \subset U_j$ therefore $D_{j,k+1}(h) = D_{j,k}(h) \cup (D_{j,k+1}(h) \cap U_j)$.

We consider the Mayer-Vietoris sequence for homology:

$$\begin{array}{c} 0 \\ \| \\ \cdots \to H_{2n-1}(D_{j,k}(h), \mathbf{C}) \oplus H_{2n-1}(D_{j,k+1}(h) \cap U_j, \mathbf{C}) \to H_{2n-1}(D_{j,k+1}(h), \mathbf{C}) \to \\ H_{2n-2}(D_{j,k}(h) \cap U_j, \mathbf{C}) \to H_{2n-2}(D_{j,k}(h), \mathbf{C}) \oplus H_{2n-2}(D_{j,k+1}(h) \cap U_j, \mathbf{C}) \to \\ H_{2n-2}(D_{j,k+1}(h), \mathbf{C}) \to \cdots \end{array}$$

It is easy to see (by induction and by Mayer-Vietoris sequence) that $H_{2n-1}(D_{j,k+1}(h) \cap U_j, \mathbf{C}) = 0$ because $D_{j,k+1}(h) \cap U_j$ is a finite intersection of $(n-1)$-Runge domains in the Stein space U_j.

From the exactness of the above sequence and by the conditions a_1) and b_1) it follows that the map $H_{2n-2}(D_{j,k}(h) \cap U_j, \mathbf{C}) \to H_{2n-2}(D_{j,k+1}(h) \cap U_j, \mathbf{C})$ is injective.

In view of Lemma 3.3 the restriction map $H^{n-2}(D_{j,k+1}(h) \cap U_j, \mathscr{F}) \to$

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$H^{n-2}(D_{j,k}(h) \cap U_j, \mathscr{F})$ has dense image.
We consider now the Mayer-Vietoris sequence for cohomology:

$$\cdots \to H^{n-2}(D_{j,k+1}(h), \mathscr{F}) \to H^{n-2}(D_{j,k}(h), \mathscr{F}) \oplus H^{n-2}(D_{j,k+1}(h) \cap U_j, \mathscr{F}) \to H^{n-2}(D_{j,k}(h) \cap U_j, \mathscr{F}) \to H^{n-1}(D_{j,k+1}(h), \mathscr{F}) \to \cdots$$

Since the map $H^{n-2}(D_{j,k}(h), \mathscr{F}) \oplus H^{n-2}(D_{j,k+1}(h) \cap U_j, \mathscr{F}) \to H^{n-2}(D_{j,k}(h) \cap U_j, \mathscr{F})$ comes from a map of cocycles and has finite dimensional cokernel (by Lemma 3.4 $\dim_{\mathbf{C}} H^{n-1}(D_{j,k+1}(h), \mathscr{F}) < \infty$) it follows that this map is quasi-open (i.e. open onto its image). Since we have shown that the restriction $H^{n-2}(D_{j,k+1}(h) \cap U_j, \mathscr{F}) \to H^{n-2}(D_{j,k}(h) \cap U_j, \mathscr{F})$ has dense image it follows that $H^{n-2}(D_{j,k+1}(h), \mathscr{F}) \to H^{n-2}(D_{j,k}(h), \mathscr{F})$ has also dense image which proves Step 1.

Step 2. We verify the condition 1) stated at the beginning of the proof. Let γ be such that $\alpha \leqslant \gamma \leqslant \beta$. By the definition of β we have $X_\beta \subset D_{m,r}(\alpha)$, hence the inclusions $X_\gamma \subset X_\beta \subset D_{m,r}(\alpha) \subset D_{m,r}(\gamma)$ and by Step 1 $H^{n-2}(D_{m,r}(\gamma), \mathscr{F}) \to H^{n-2}(X_\gamma, \mathscr{F})$ has dense image because $\gamma > \alpha_0$. Therefore the restriction map $H^{n-2}(X_\beta, \mathscr{F}) \to H^{n-2}(X_\gamma, \mathscr{F})$ has dense image too.

Step 3. We verify the condition 2) stated at the beginning of the proof. By Lemma 3.4 (the surjectivity assertion) it is enough to show that the restriction map $H^{n-1}(D_{m,r}(\alpha), \mathscr{F}) \to H^{n-1}(X_\alpha, \mathscr{F})$ is injective. Exactly as in Step 1 we have only to verify the injectivity of the maps $H^{n-1}(D_{j,k+1}(\alpha), \mathscr{F}) \to H^{n-1}(D_{j,k}(\alpha), \mathscr{F})$. To see this we consider the Mayer-Vietoris sequence:

$$\cdots \to H^{n-2}(D_{j,k}(\alpha), \mathscr{F}) \oplus H^{n-2}(D_{j,k+1}(\alpha) \cap U_j, \mathscr{F}) \xrightarrow{\mu} H^{n-2}(D_{j,k}(\alpha) \cap U_j, \mathscr{F}) \to H^{n-1}(D_{j,k+1}(\alpha), \mathscr{F}) \to H^{n-1}(D_{j,k}(\alpha), \mathscr{F}) \oplus \underset{\underset{0}{\parallel}}{H^{n-1}(D_{j,k+1}(\alpha) \cap U_j, \mathscr{F})} \to \cdots$$

One has $H^{n-1}(D_{j,k+1}(\alpha) \cap U_j, \mathscr{F}) = 0$ because $D_{j,k+1}(\alpha) \cap U_j$ is a finite intersection of $(n-1)$-Runge domains in the Stein space U_j. Also by Step 1 we know that u has dense image. On the other hand $H^{n-1}(D_{j,k+1}(\alpha), \mathscr{F})$ is finite dimensional by Lemma 3.4, therefore u must be surjective. It follows that the restriction map $H^{n-1}(D_{j,k+1}(\alpha), \mathscr{F}) \to H^{n-1}(D_{j,k}(\alpha), \mathscr{F})$ is injective, which proves the assertion of Step 3. Thus the proof of Lemma 3.5 is complete.

LEMMA 3.6. *Let $A \subset \mathbf{P}^n$ be a closed analytic subset without isolated points. Then there exist:*

1) *a projective algebraic space X of pure dimension n together with a finite surjective holomorphic map $\phi : X \to \mathbf{P}^n$*

2) *closed analytic subsets $A_1, \ldots, A_r$ of X*

3) *ample line bundles* $L_1, \ldots, L_r$ *on* X *and for each bundle* L_ι *sections* $s_{i,1}, \ldots, s_{\iota,k_i} \in \Gamma(X, L_i)$ $k_i \leqslant n-1$
such that:

a) $\phi^{-1}(A) = A_1 \cup \cdots \cup A_r$

b) $A_i = \{s_{i,1} = \ldots = s_{i,k_i} = 0\}$ $i = 1, \ldots, r$

Remark 3.3. By taking the normalization of X we always may assume in the above lemma that X is normal.

Proof of Lemma 3.6.

Step 1. We assume that A has pure dimension $d \geqslant 1$. We can apply Lemma 1.1 and we take $L_\iota = (\phi \circ \psi_\iota)^*(\mathcal{O}_{\mathbf{P}^n}(1))$.

Step 2. We prove the following statement:
Let $A = A_1 \cup A_2$ with A_1, A_2closed analytic subsets of $\mathbf{P}^n$ without isolated points. Assume that for each A_j, $j = 1,2$, there exists $(X_j, \phi_j, A_\iota^j, L_\iota^j, s_{i,\iota}^j)$ with the properties stated in Lemma 3.6. Then there exists $(X, \phi, A_i, L_i, s_{\iota,\iota})$ associated to A and which satisfies the conditions of Lemma 3.6.

To see this we consider the commutative diagram:

$$\begin{array}{ccc} & X & \\ {\scriptstyle pr_1}\swarrow & & \searrow{\scriptstyle pr_2} \\ X_1 & & X_2 \\ {\scriptstyle \phi_1}\searrow & & \swarrow{\scriptstyle \phi_2} \\ & \mathbf{P}^n & \end{array}$$

where $X = X_1 \times_{\mathbf{P}^n} X_2$ is the reduced fiber product, i.e. $X = \{x = (x_1, x_2) \in X_1 \times X_2 \mid \phi_1(x_1) = \phi_2(x_2)\}$ and $\phi = \phi_1 \circ \mathrm{pr}_1 = \phi_2 \circ \mathrm{pr}_2$. Clearly X has pure dimension n and ϕ, pr_1, pr_2 are finite surjective morphisms. If we write $\phi^{-1}(A) = \phi^{-1}(A_1) \cup \phi^{-1}(A_2) = \mathrm{pr}_1^{-1}(\phi_1^{-1}(A_1)) \cup \mathrm{pr}_2^{-1}(\phi_2^{-1}(A_2)) = \mathrm{pr}_1^{-1}(\cup_\iota A_i^1) \cup \mathrm{pr}_2^{-1}(\cup_i A_i^2)$ then everything is clear because pr_1, pr_2 are finite surjective morphisms, hence $\mathrm{pr}_1^*(L_i^1)$, $\mathrm{pr}_2^*(L_i^2)$ are ample line bundles on X. This ends the proof of Step 2. Obviously Lemma 3.6 follows from Step 1 and Step 2 by an induction argument.

LEMMA 3.7. *Let* $D \subset \mathbf{C}^n$ *be defined by* $D = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_1|^2 + \cdots + |z_q|^2 \neq 0\}$ *where* $1 \leqslant q \leqslant n$ *is a fixed positive integer, and let* $\varphi : D \to \mathbf{R}$ *be given by* $\varphi(z) = \log(|z_1|^2 + \ldots + |z_q|^2)$. *Then for every* $z \in D$ *there is a complex linear subspace* $M_z \subset T_z D = \mathbf{C}^n$ *with* $\dim M_z = n - (q-1)$ *such that the Levi form* $L(\varphi)_z(w) = 0$ *for any* $w \in M_z$.

Proof. We put $s = |z_1|^2 + \ldots + |z_q|^2$. By a direct computation it follows that:

$$L(\varphi)_z(w) = \frac{1}{s^2}\left(\sum_{i,j} |z_i w_j - z_j w_i|^2\right)$$

Let $z = (z_1, \ldots, z_n) \in D$ be fixed and let $1 \leqslant i \leqslant q$ be such that $z_i \neq 0$. We define M_z by the following $(q-1)$ equations in $T_z D = \mathbf{C}^n$:

$$z_i w_1 - z_1 w_i = 0, \ldots, z_i w_{i-1} - z_{i-1} w_i = 0, z_i w_{i+1} - z_{i+1} w_i = 0, \ldots, z_i w_q - z_q w_i = 0.$$

Clearly M_z satisfies the required conditions.

LEMMA 3.8. *Let X be a complex space, $U \subset X$ an open subset, $f_1, \ldots, f_q \in \mathcal{O}(U)$ with $\{x \in U \mid f_1(x) = \ldots = f_q(x) = 0\} = \emptyset$ (empty set) and ψ a 1-convex function on U. Then $\psi - \log(|f_1|^2 + \ldots + |f_q|^2)$ is q-convex on U.*

Proof. Obviously we may assume that U is an open subset of $\mathbf{C}^m$. Let $t_1, \ldots, t_m$ be the coordinates on $\mathbf{C}^m$. If we consider the embedding $U \hookrightarrow \mathbf{C}^q \times U \subset \mathbf{C}^n$ $n = q + m$ given by $t = (t_1, \ldots, t_m) \to (f_1(t), \ldots, f_q(t), t_1, \ldots, t_m)$ then Lemma 3.8 follows immediately from Lemma 3.7;

LEMMA 3.9. *Let X, Y be locally compact Hausdorff spaces and $\pi : X \to Y$ a continuous map which is proper and surjective. Let $A \subset Y$ be a locally closed subset and $\tilde{A} = \pi^{-1}(A)$.*

Then the following statements hold:

1) *If $\tilde{A}$ has no compact connected components then A has no compact connected components*
2) *Assume additionally that π has connected fibers. Then the condition "A has no compact connected components" implies "$\tilde{A}$ has no compact connected components".*

The proof of this lemma is a simple exercise of topology and so it is omitted (see e.g. [5]).

LEMMA 3.10. *Let X be a compact complex space of pure dimension n, $A_1, \ldots, A_r$ closed analytic subsets, $A = A_1 \cup \cdots \cup A_r$ and for each $i = 1, \ldots, r$ let $\varphi_i : X \setminus A_i \to \mathbf{R}$ be an exhaustion function such that $\exp(-\varphi_i)$ is real-analytic on X. Then there exists a sufficiently large constant $\alpha_0 > 0$ with the following property:*

(P′) *for any $\tau_1, \ldots, \tau_r \in C_0^\infty(X \setminus A)$, $\tau_1 \geqslant 0, \ldots, \tau_r \geqslant 0$ there is a sufficiently small*

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constant $\lambda_0 = \lambda_0(\tau_1, \ldots, \tau_r) > 0$ *such that for all constants* $0 \leqslant \mu_1 \leqslant \lambda_0$ $i = 1, \ldots, r$ *the set*

$$B_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c) = (X \setminus A_i) \cup (X \setminus A_j) \setminus (\{x \in X \setminus A_i \mid \varphi_i(x) - \mu_i \tau_i(x) < c\} \cup \{x \in X \setminus A_j \mid \varphi_j(x) - \mu_j \tau_j(x) < c\})$$

has no compact connected components if $c > \alpha_0$.

Proof. If we set $f_i = e^{-\varphi_i}$ then f_i are real-analytic on X, $f_i \geqslant 0$, $A_i = \{f_i = 0\}$ and $B_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c) = \{x \in X \mid f_i(x)\, e^{\mu_i \tau_i(x)} \leqslant e^{-c}, f_j(x) e^{\mu_j \tau_j(x)} \leqslant e^{-c}\} \setminus (A_i \cap A_j)$. With the notation in Lemma 2.4 $B_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c) = C_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, e^{-c}) \setminus (A_i \cap A_j)$. Let $\pi : \tilde{X} \to X$ be a resolution of singularities [9] and define $\tilde{A}_i = \pi^{-1}(A_i)$, $\tilde{A} = \cup \tilde{A}_i$, $\tilde{f}_i = f_i \circ \pi$, $\tilde{\tau}_i = \tau_i \circ \pi$ and the corresponding sets $\tilde{B}_{ij}$, $\tilde{C}_{ij}$. By Lemma 2.4 there is a constant $c_0 > 0$ sufficiently small with the property (P). Since $\tilde{A}_i \cap \tilde{A}_j$ is a strong deformation retract of $\tilde{C}_{ij}(\tilde{\tau}_i, \tilde{\tau}_j, \mu_i, \mu_j, e^{-c})$, $c > -\log c_0 = \alpha_0$, it follows in particular the surjectivity of the map $H_0(\tilde{A}_i \cap \tilde{A}_j, \mathbf{C}) \to H_0(\tilde{C}_{ij}(\tilde{\tau}_i, \tilde{\tau}_j, \mu_i, \mu_j, e^{-c}), \mathbf{C})$ which implies that the set $\tilde{B}_{ij}(\tilde{\tau}_i, \tilde{\tau}_j, \mu_i, \mu_j, c) = \tilde{C}_{ij}(\tilde{\tau}_i, \tilde{\tau}_j, \mu_i, \mu_j, e^{-c}) \setminus (\tilde{A}_i \cap \tilde{A}_j)$ has no compact connected components. By Lemma 3.9 the set $B_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c)$ also has no compact connected components. Therefore the constant $\alpha_0 = -\log c_0$ satisfies the conditions required in our lemma.

Remark 3.4. It follows by our proof that the sets $B_{ij}(\tau_i, \tau_j, \mu_i, \mu_j, c)$ $c > \alpha_0$ have no compact irreducible components.

PROPOSITION 3.1 *Let* X *be a compact normal complex space of pure dimension* n, $A_1, \ldots, A_r$ *closed analytic subsets,* $A = A_1 \cup \cdots \cup A_r$, *and assume that for each* $i = 1, \ldots, r$ *there exists an* $(n-1)$*-convex exhaustion function* $\varphi_i : X \setminus A_i \to \mathbf{R}$ *such that* $\exp(-\varphi_i)$ *is real-analytic on* X. *Then* $X \setminus A$ *is cohomologically* $(n-1)$*-convex.*

Proof. It is a direct consequence of Lemma 3.5 and Lemma 3.10.

Remark 3.5. In fact the condition of normality is not necessary in the above proposition as one can easily see by replacing in the previous lemmas "compact connected components" by "compact irreducible components" and dropping the normality assumption on X.

§4. Proof of the main results

THEOREM 4.1. *Let* X *be a compact projective algebraic space of pure dimension* n,

$A_1, \ldots, A_r$ *closed analytic subsets,* $A = A_1 \cup \cdots \cup A_r$. *Assume that there exist ample line bundles* $L_1, \ldots, L_r$ *on* X *and for each bundle* L_i *sections* $s_{i,1}, \ldots, s_{i,k_i} \in \Gamma(X, L_i)$, $k_i \leqslant n-1$, such that $A_i = \{s_{i,1} = \ldots = s_{i,k_i} = 0\}$ $i = 1, \ldots, r$.
Then $X \setminus A$ *is cohomologically* $(n-1)$*-convex.*

Proof. By Proposition 1.1 and the invariance of ampleness under finite surjective morphisms we may assume that X is normal. Also, replacing L_i by a sufficiently large power of it, we may assume that L_i is very ample. Hence there is an embedding $X \overset{\mu_i}{\hookrightarrow} \mathbf{P}^N$ such that $L_i \cong \mu_i^*(\mathcal{O}_{\mathbf{P}}N(1))$ and thus we have on L_i a real-analytic metric induced by the standard metric of positive curvature on $\mathcal{O}_{\mathbf{P}}N(1)$. Let h_i be this metric on L_i. Let $(U_\alpha)_\alpha$ be an open covering of X such that all the line bundles L_i are trivial when restricted to U_α and let $h_i = (h_{i,\alpha})_\alpha$ be the local representation of the metric h_i, hence $-\log h_{i,\alpha}$ are 1-convex functions on U_α. If $(g_{\iota,\alpha\beta})$ are the transition functions for L_ι then $h_{i,\alpha} = |g_{i,\beta\alpha}|^2 h_{i,\beta}$. Let also $(s_{i,j,\alpha})$ be the local representations for the sections s_{ij} $j = 1, \ldots, k_i$ corresponding to (U_α). Hence $s_{i,j,\alpha} = g_{i,\alpha\beta} s_{i,j,\beta}$.
If we set $f_i = h_{i,\alpha} \sum_{j=1}^{k_i} |s_{i,j,\alpha}|^2$ then f_i is well-defined, is real-analytic on X, $f_i \geqslant 0$, $A_i = \{f_\iota = 0\}$. On the other hand by Lemma 3.8 the function $\varphi_i = -\log f_i$ is $(n-1)$-convex on $X \setminus A_i$. Now Theorem 4.1 is a direct consequence of Proposition 3.1.

THEOREM 4.2. *Let* $A \subset \mathbf{P}^n$ *be a closed analytic subset without isolated points. Then* $\mathbf{P}^n \setminus A$ *is cohomologically* $(n-1)$*-convex.*

Proof. This follows directly by Theorem 4.1, Lemma 3.6 and Proposition 1.1.

We recall the main result in [2]:

THEOREM 4.3 ([2], Satz 2). *Let* $A \subset \mathbf{P}^n$ *be a closed analytic subset without isolated points,* $\mathcal{F} \in \mathrm{Coh}(\mathbf{P}^n \setminus A)$, $\mathcal{K}$ *the canonical sheaf of* $\mathbf{P}^n$, $k \geqslant 1$ *the number of connected components of* A. *Then one has:*

$$\begin{aligned}\dim \mathrm{Ext}_c^1(\mathbf{P}^n \setminus A; \mathcal{F}; \mathcal{K}) &= \dim H_c^1(\mathbf{P}^n \setminus A, \mathcal{H}om(\mathcal{F}, \mathcal{K})) \\ &= (k-1)\dim H^0(\mathbf{P}^n \setminus A, \mathcal{H}om(\mathcal{F}, \mathcal{K})) < \infty.\end{aligned}$$

By Theorem 4.2, Theorem 4.3 and Serre duality it follws:

MAIN THEOREM. *Let* $A \subset \mathbf{P}^n$ *be a closed analytic. subset without isolated points,* $k \geqslant 1$ *the number of connected components of* A *and* $\mathcal{F} \in \mathrm{Coh}(\mathbf{P}^n \setminus A)$.

Then we have for the analytic cohomology groups:

$$\dim_{\mathbf{C}} H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F}) = (k-1) \dim_{\mathbf{C}} H^0(\mathbf{P}^n \setminus A, \mathcal{H}om(\mathcal{F}, \mathcal{K})) < \infty$$

($\mathcal{K}$ *is the canonical sheaf of* $\mathbf{P}^n$). *In particular* $H^{n-1}(\mathbf{P}^n \setminus A, \mathcal{F})$ *vanishes for connected* A.

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