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CRITERIA FOR RECURRENCE AND TRANSIENCE OF SEMISTABLE PROCESSES

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1. Introduction

The definition of semistable laws was originally given by Lévy in [11]. Two books, Kagan, Linnik and Rao [6] and Ramachandran and Lau [13], call a probability measure on $\mathbf{R}$ "semistable" when it is nondegenerate and its characteristic function (ch.f.) $f(z)$ does not vanish on $\mathbf{R}$ and satisfies a functional equation of the form

$$f(z) = f(bz)^c, \ \forall \ z \in \mathbf{R} \tag{1.1}$$

for some real numbers b $(0 < |b| < 1)$ and $c > 1$. This is essentially the same as Lévy's definition. Ramachandran and Lau [13] assert in Theorem 3.2.2 that they give a necessary and sufficient condition for "semistable" laws in terms of the Lévy representation of infinitely divisible laws, and in Corollary 3.2.3 that every stable law is "semistable". But the sufficiency part of the first assertion is not true. The second assertion is also incorrect, since, if

$$\log f(z) = i\gamma z - c|z|^{\alpha}, \quad \gamma \neq 0, \quad \alpha \in (0, 1) \cup (1,2], \quad c > 0,$$

then it is stable (in the ordinary sense as well as in their sense), but not "semistable" in their sense. Professor K. Sato indicated that this error could be avoided if the notions of semistable laws and strictly semistable laws are defined appropriately and distinguished from each other. Thus, we make a new definition of semistable laws as follows: a probability measure μ on $\mathbf{R}^d$ or its ch.f. $f(\mathbf{z})$ is semistable if it is not a delta measure, $f(\mathbf{z})$ does not vanish, and there exist $b \in \mathbf{R}$, $c \in \mathbf{R}$, and $\gamma \in \mathbf{R}^d$ such that $0 < |b| < 1$, $c > 1$, and

$$f(\mathbf{z}) = f(b\mathbf{z})^c e^{i\gamma\mathbf{z}}, \quad \forall \ \mathbf{z} \in \mathbf{R}^d. \tag{1.2}$$

Further, we define strictly semistable laws as follows: a probability measure μ on $\mathbf{R}^d$ or its ch.f. $f(\mathbf{z})$ is strictly semistable if it is not a delta measure, $f(\mathbf{z}) \neq 0$ and

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there exist real b and c such that $0 < |b| < 1$, $c > 1$ and

$$f(\mathbf{z}) = f(b\mathbf{z})^c, \quad \forall\ \mathbf{z} \in \mathbf{R}^d. \tag{1.3}$$

The exponent α of a semistable law satisfying (1.2) or a strictly semistable law satisfying (1.3) is defined by

$$c|b|^\alpha = 1. \tag{1.4}$$

Shimizu [16] and Kruglov [10] found characterization of a semistable law on $\mathbf{R}$ as the limit distribution of a normalized subsequence $\{a_n S_{k_n} + b_n\}$ of sums $\{S_n\}$ of independent identically distributed random variables with $\lim_{n\to\infty} k_{n+1}/k_n = r$ for some r. Extension to higher dimensions including Hilbert spaces was done by [8,9]. Jajte [4] made extension to operator semistable measures on $\mathbf{R}^d$, which have been treated and further generalized in [5,7,12].

Since a semistable law is infinitely divisible, it induces a Lévy process, which we call a semistable process. A Lévy process in this paper means a homogeneous Lévy process in the sense of [3], that is, a process with stationary independent increments starting at 0 with sample functions being right-continuous and having left limits. A Lévy process induced by a strictly semistable law is called a strictly semistable process.

The purpose of the present paper is to determine whether a semistable process with exponent α is recurrent or transient. Besides this purpose, we obtain representations for semistable and strictly semistable laws on $\mathbf{R}^d$ in the same manner as is done for stable and strictly stable laws on $\mathbf{R}^d$ (see Sato [14,15]). The characterization is similar to that of [8] but includes the case $b < 0$.

This paper is organized as follows. In Section 2, we begin with showing the infinite divisibility of semistable laws and give necessary and sufficient conditions for a probability measure to be semistable or strictly semistable. We then add some consequences. In Section 3 we obtain three results: the first is that a semistable process with exponent $\alpha \in [1,2]$ on $\mathbf{R}$ is recurrent if and only if it is strictly semistable; the second is that any semistable process with exponent $\alpha \in (0,1)$ on $\mathbf{R}$ is transient; the third is that a genuinely 2-dimensional semistable process with exponent $\alpha \in (0,2]$ on $\mathbf{R}^2$ is recurrent if and only if it is strictly semistable with $\alpha = 2$. These results are analogous to the ones known for stable processes (see [15]) and in accordance with Professor K. Sato's conjecture.

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valuable comments. These comments greatly improved the presentation of this paper. Proposition 2.6 is due to is due to him.

2. Semistable and strictly semistable laws on $\mathbf{R}^d$

For any non-vanishing d-dimensional characteristic function $f(\mathbf{z})$, the function $\log f(\mathbf{z})$ is understood as the distinguished logarithm in the sense of Lemma 2.1.4 of [15] (see [1], Section 7.6, for $d = 1$) and the function $f(\mathbf{z})^c$ is understood to be $e^{c\log f(\mathbf{z})}$.

For $\mathbf{x}, \mathbf{z} \in \mathbf{R}^d$, we denote the inner product of $\mathbf{x}$ and $\mathbf{z}$ by $\mathbf{xz}$ and the Euclidean norm of $\mathbf{x}$ by $|\mathbf{x}|$. We denote $\log f(\mathbf{z})$ by $\psi(\mathbf{z})$.

LEMMA 2.1. *If a ch.f. $f(\mathbf{z})$ is semistable satisfying* (1.2), *then, for any positive integer n,*

$$\psi(z) = c^n \psi(b^n \mathbf{z}) + i\sigma_n \mathbf{z}, \tag{2.1}$$

where $\sigma_n = \gamma(\sum_{j=0}^{n-1} |b|^{-j\alpha} b^j)$.

Proof. By mathematical induction, we can get this. If $n = 1$, then (2.1) is the same as (1.2). If (2.1) is true for some n, then it is easy to show (2.1) for $n + 1$. □

LEMMA 2.2. *Every semistable law is infinitely divisible.*

Proof. Denote $\psi(b^n\mathbf{z}) + ic^{-n}\sigma_n\mathbf{z}$ by $g_n(\mathbf{z})$. Then, by Lemma 2.1, $g_n(\mathbf{z}) = c^{-n}\psi(\mathbf{z}) \to 0$ as $n \to \infty$, since $c > 1$. Hence,

$$\begin{aligned}&\exp\{c^n(f(b^n\mathbf{z})\exp(ic^{-n}\sigma_n\mathbf{z}) - 1)\} = \exp\{c^n(\exp(\psi(b^n\mathbf{z}) + ic^{-n}\sigma_n\mathbf{z}) - 1)\}\\ &= \exp\{c^n(g_n(\mathbf{z}) + O(g_n^2(\mathbf{z})))\} = \exp\{\psi(\mathbf{z})(1 + O(g_n(\mathbf{z})))\} \to f(\mathbf{z}),\end{aligned}$$

as $n \to \infty$. By De Finetti's theorem f is infinitely divisible. □

Let $S = \{\mathbf{x} \in \mathbf{R}^d : |\mathbf{x}| = 1\}$, the unit sphere in $\mathbf{R}^d$, and let $\mathbf{R}_+ = (0, \infty)$, the open half line. For $E \subset \mathbf{R}_+$ and $B \subset S$, we denote by EB the set of points $\mathbf{x}$ such that $\mathbf{x} = u\xi$, $u \in E$, $\xi \in B$. The class of Borel subsets of a set T is denoted by $\mathscr{B}(T)$ in general. The set $\{\mathbf{x} \in \mathbf{R}^d : \mathbf{x} = b\mathbf{y}, \mathbf{y} \in B\}$ is denoted by bB, and the set $(-1)B$ is denoted by $-B$. The set $\{\mathbf{x} \in \mathbf{R}^d : \mathbf{x} \neq 0\}$ will be denoted by $\mathbf{R}^d - \{0\}$. For any measure λ on S the measure $\tilde{\lambda}$ is defined by $\tilde{\lambda}(B) = \lambda(-B)$. The set $\{\mathbf{x} : |\mathbf{x}| < 1\}$ is denoted by D and the indicator function of D is denoted by $\mathbf{1}_D(\mathbf{x})$.

Lévy showed that μ is a d-dimensional infinitely divisible distribution if and only if the distinguished logarithm of its characteristic function has the form

$$\psi(\mathbf{z}) = i\gamma_0\mathbf{z} - \frac{1}{2}\mathbf{z}A\mathbf{z} + \int_{\mathbf{R}^d-\{0\}} G(\mathbf{z}, \mathbf{x})\nu(d\mathbf{x}),$$

where

$$G(\mathbf{z}, \mathbf{x}) = e^{i\mathbf{z}\mathbf{x}} - 1 - i\mathbf{z}\mathbf{x}\mathbf{1}_D(\mathbf{x}),$$

γ_0 is a vector in $\mathbf{R}^d$, A is a symmetric nonnegative definite operator, and ν is a measure (called Lévy measure) on $\mathbf{R}^d - \{0\}$ such that

$$\int_{0<|\mathbf{x}|<1} |\mathbf{x}|^2\nu(d\mathbf{x}) < \infty \text{ and } \int_{|\mathbf{x}|\geq 1} \nu(d\mathbf{x}) < \infty.$$

This representation is unique and called the Lévy representation.

PROPOSITION 2.3. *Fix b and c such that $0 < |b| < 1$ and $c > 1$. Define α by* (1.4). *In order that a ch.f. $f(z)$ be semistable satisfying* (1.2) *with some γ, it is necessary and sufficient that it is infinitely divisible and the Lévy representation satisfies one of the following conditions*:

(i) $\alpha = 2$ *and* $\nu = 0$.
(ii) $0 < \alpha < 2$, $A = 0$, *and*

$$\nu(EB) = \int_B \lambda(d\xi) \int_E d\{-H_\xi(u)u^{-\alpha}\},\ {}^\forall B \in \mathcal{B}(S),\ {}^\forall E \in \mathcal{B}(\mathbf{R}_+),$$

where λ is a finite measure on S, $H_\xi(u)$ is nonnegative, right-continuous in u and Borel measurable in ξ, $H_\xi(u)u^{-\alpha}$ is nonincreasing in u, $H_\xi(1) = 1$ and, in addition, the following (2.2) *or* (2.3) *holds*:

(2.2) $b > 0$ *and* $H_\xi(bu) = H_\xi(u)$;
(2.3) $b < 0$, λ *and* $\tilde{\lambda}$ *are mutually absolutely continuous, and* $H_\xi(-bu) = H_{-\xi}(u)c(\xi)$, *where* $c(\xi)$ *is a positive measurable function such that* $\tilde{\lambda}(d\xi) = c(\xi)\lambda(d\xi)$.

Outline of Proof. Suppose that $f(\mathbf{z})$ is semistable satisfying (1.2). Then we have either $\alpha = 2$ and $\nu = 0$, or $0 < \alpha < 2$, $A = 0$, and $\nu(bEB) = |b|^{-\alpha}\nu(EB)$. Suppose that $0 < \alpha < 2$. Define $\lambda(B) = \nu((1, \infty)B)$ and $N(r, B) = \nu((r, \infty)B)$ for $r \in \mathbf{R}_+$. Then, for any $r \in \mathbf{R}_+$, we can choose n such that $r > (b^2)^n > 0$, so that

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$$0 \leq N(r, B) \leq N((b^2)^n, B) = (b^2)^{-n\alpha}\lambda(B).$$

Hence, $N(r, B)$ is absolutely continuous with respect to λ. Thus, for each $r \in \mathbf{R}_+$, there is a nonnegative measurable function $N_\xi(r)$ of ξ such that

$$N(r, B) = \int_B N_\xi(r)\lambda(d\xi), \; B \in \mathscr{B}(S).$$

By the same method as in [14, 15] for stable probability measures, we can choose $N_\xi(r)$ in such a way that, for λ-almost every ξ, $N_\xi(r)$ is nonincreasing right-continuous in r, and we can show that

$$\nu(EB) = -\int_B \lambda(d\xi) \int_E dN_\xi(u).$$

If $b < 0$, we have that

$$\tilde{\lambda}(B) = \nu((-\infty, -1)B) \leq \nu((-\infty, b)B) = |b|^\alpha \nu((b^2, \infty)B) = |b|^\alpha (b^2)^{-\alpha}\lambda(B),$$

and hence also $\lambda(B) \leq |b|^\alpha (b^2)^{-\alpha}\tilde{\lambda}(B)$. Thus, if $b < 0$, then $\tilde{\lambda}$ and λ are mutually absolutely continuous, so that there is a positive measurable function $c(\xi)$ such that $\tilde{\lambda}(d\xi) = c(\xi)\lambda(d\xi)$ and we have that

$$\nu((r, \infty)B) = |b|^\alpha \int_B N_{-\xi}(-br)c(\xi)\lambda(d\xi).$$

Hence one of the following (2.4) and (2.5) holds:

(2.4) $$b > 0 \text{ and } N_\xi(bu) = |b|^{-\alpha} N_\xi(u)\,;$$

(2.5) $$b < 0 \text{ and } N_{-\xi}(-bu) = |b|^{-\alpha} N_\xi(u)\,\frac{1}{c(\xi)}.$$

Set $N_\xi(u) = H_\xi(u)u^{-\alpha}$. Then we can show (2.2) and (2.3). The converse assertion is easy to check. □

Note that, in (ii) of Proposition 2.3,

$$\psi(\mathbf{z}) = i\gamma_0\mathbf{z} + \int_S \lambda(d\xi) \int_0^\infty G(\mathbf{z}, u\xi)\,d\{-H_\xi(u)u^{-\alpha}\}.$$

Remark 1. Let μ be a semistable law with exponent α $(0 < \alpha < 2)$. Then both $\int |\mathbf{x}|^\beta \mu(d\mathbf{x})$ and $\int_{|\mathbf{x}|>1} |\mathbf{x}|^\beta \nu(d\mathbf{x})$ are finite for $0 < \beta < \alpha$ and infinite for $\beta \geq \alpha$; $\int_{|\mathbf{x}|\leq 1} |\mathbf{x}|^\beta \nu(d\mathbf{x})$ is finite for $\beta > \alpha$ and infinite for $\beta \leq \alpha$.

Proof. Denote by ν_1 and ν_0 the restrictions of ν to the sets $\{|\mathbf{x}| > 1\}$ and $\{|\mathbf{x}| \leq 1\}$, respectively. By Proposition 2.3, we have that

$$\int |\mathbf{x}|^\beta \nu_1(d\mathbf{x}) = \int_S \lambda(d\xi) \int_1^\infty u^\beta d\{-H_\xi(u)u^{-\alpha}\},$$

$$\int |\mathbf{x}|^\beta \nu_0(d\mathbf{x}) = \int_S \lambda(d\xi) \int_0^1 u^\beta d\{-H_\xi(u)u^{-\alpha}\},$$

and see that

$$\int_1^\infty u^\beta d\{-H_\xi(u)u^{-\alpha}\} = \sum_{n=0}^\infty (b^2)^{(\alpha-\beta)n} \int_1^{(b^2)^{-1}} u^\beta d\{-H_\xi(u)u^{-\alpha}\},$$

$$\int_0^1 u^\beta d\{-H_\xi(u)u^{-\alpha}\} = \sum_{n=0}^\infty (b^2)^{(\beta-\alpha)n} \int_{b^2}^1 u^\beta d\{-H_\xi(u)u^{-\alpha}\}.$$

Hence $\int |\mathbf{x}|^\beta \nu_0(d\mathbf{x})$ is finite if $\beta - \alpha > 0$, and infinite if $\beta - \alpha \leq 0$; $\int |\mathbf{x}|^\beta \nu_1(d\mathbf{x})$ is finite if $\alpha - \beta > 0$, and infinite if $\alpha - \beta \leq 0$. Using Theorem 5.2.3 of [15], we conclude that $\int |\mathbf{x}|^\beta \mu(d\mathbf{x})$ is finite for $0 < \beta < \alpha$ and infinite for $\beta \geq \alpha$.

Remark 2. Fix b and c such that $0 < |b| < 1$ and $c > 1$. Define α by (1.4). If $0 < \alpha < 2$ and a ch.f. $f(\mathbf{z})$ is semistable satisfying (1.2) with some γ, then the $\log f(\mathbf{z})$ has the form

$$\psi(\mathbf{z}) = i\gamma_1\mathbf{z} - |\mathbf{z}|^\alpha (R(\mathbf{z}) + iI(\mathbf{z})),$$

where $\gamma_1 \in \mathbf{R}^d$, $R(\mathbf{z})$ is a real-valued continuous bounded function on $\mathbf{R}^d - \{0\}$, satisfying $R(b^n\mathbf{z}) = R(\mathbf{z})$ and $I(\mathbf{z})$ is a real-valued continuous function on $\mathbf{R}^d - \{0\}$ satisfying $I(b^n\mathbf{z}) = I(\mathbf{z})$ for $\alpha \neq 1$ and

$$I(b^n\mathbf{z}) = I(\mathbf{z}) - \mathbf{z}|\mathbf{z}|^{-1}(\gamma_0 - (b|b|^{-1})^n \gamma_0 - \sigma_n),$$

where $\sigma_n = \gamma(\sum_{j=0}^{n-1} |b|^{-j} b^j)$, for $\alpha = 1$. Moreover, if $\mathbf{R}^d$ is spanned by the support of ν, then $\inf\{R(\mathbf{z}) : 0 < |\mathbf{z}| < \infty\} > 0$.

Proof. We use Proposition 2.3. For $0 < \alpha < 2$ we have that

$$R(\mathbf{z}) = \int_S \lambda(d\xi) \int_0^\infty (1 - \cos(\xi\mathbf{z}|\mathbf{z}|^{-1}v)) d\{-H_\xi(v|\mathbf{z}|^{-1})v^{-\alpha}\},$$

so that $R(\mathbf{z})$ is continuous and $R(b^n\mathbf{z}) = R(\mathbf{z})$. Boundedness of $R(\mathbf{z})$ follows from these properties. For $0 < \alpha < 1$, by Remark 1, we have that

$$I(\mathbf{z}) = -\int_S \lambda(d\xi) \int_0^\infty \sin(\xi\mathbf{z}\,|\,\mathbf{z}\,|^{-1}v)\,d\{-H_\xi(v\,|\,\mathbf{z}\,|^{-1})v^{-\alpha}\}$$

and $\gamma_1 = \gamma_0 - \int_S \xi\lambda(d\xi) \int_{(0,1)} u\,d\{-H_\xi(u)u^{-\alpha}\}$, so that $I(\mathbf{z})$ is continuous and $I(b^n\mathbf{z}) = I(\mathbf{z})$. For $1 < \alpha < 2$, by Remark 1, we have that

$$I(\mathbf{z}) = \int_S \lambda(d\xi) \int_0^\infty (\xi\mathbf{z}\,|\,\mathbf{z}\,|^{-1}v - \sin(\xi\mathbf{z}\,|\,\mathbf{z}\,|^{-1}v))\,d\{-H_\xi(v\,|\,\mathbf{z}\,|^{-1})v^{-\alpha}\}$$

and $\gamma_1 = \gamma_0 + \int_S \xi\lambda(d\xi) \int_{[1,\infty)} u\,d\{-H_\xi(u)u^{-\alpha}\}$, so that $I(\mathbf{z})$ is continuous and $I(b^n\mathbf{z}) = I(\mathbf{z})$. For $\alpha = 1$ we have that

$$I(\mathbf{z}) =$$
$$\int_S \lambda(d\xi) \int_0^\infty (\xi\mathbf{z}\,|\,\mathbf{z}\,|^{-1}v\mathbf{1}_{(0,1)}(v\,|\,\mathbf{z}\,|^{-1}) - \sin(\xi\mathbf{z}\,|\,\mathbf{z}\,|^{-1}v))\,d\{-H_\xi(v\,|\,\mathbf{z}\,|^{-1})v^{-1}\}$$

and $\gamma_1 = \gamma_0$, so that $I(\mathbf{z})$ is continuous and, from Lemma 2.1, we have that

$$I(b^n\mathbf{z}) = I(\mathbf{z}) - \mathbf{z}\,|\,\mathbf{z}\,|^{-1}(\gamma_0 - (b\,|\,b\,|^{-1})^n\gamma_0 - \sigma_n).$$

If $\mathbf{R}^d$ is spanned by the support of ν, then, by Lemma 7.4.8 of [15], there exists $\delta > 0$ such that $\mathrm{Re}\{-\psi(\mathbf{z})\} \geq$ (positive constant) $|\,\mathbf{z}\,|^2 > 0$ for every $0 < |\,\mathbf{z}\,| \leq \delta$. Since $\mathrm{Re}\{-\psi(\mathbf{z})\} = |\,\mathbf{z}\,|R(\mathbf{z})$, $R(\mathbf{z}) > 0$ for every $0 < |\,\mathbf{z}\,| \leq \delta$. There exists n_0 such that $(b^2)^{n_0} < \delta$. From $R(b^n\mathbf{z}) = R(\mathbf{z})$ and continuity of $R(\mathbf{z})$, we can easily check that

$$\inf\{R(\mathbf{z}) : 0 < |\,\mathbf{z}\,| < \infty\} = \inf\{R(\mathbf{z}) : (b^2)^{n_0+1} \leq |\,\mathbf{z}\,| < (b^2)^{n_0}\} > 0. \qquad \square$$

The following Proposition 2.4 is obtained in the same manner as is done for strictly stable laws on $\mathbf{R}^d$ in [15]. The case of $d = 1$ of Proposition 2.4 gives correction of Theorem 3.2.2 of [13]. Another necessary and sufficient condition in the case $d = 1$ and $-1 < b < 0$ is given by Watanabe [17].

PROPOSITION 2.4. *Fix b and c such that $0 < |\,b\,| < 1$ and $c > 1$. Define α by* (1.4). *Suppose that a ch.f. $f(\mathbf{z})$ is semistable satisfying* (1.2) *with some γ. In order that $f(\mathbf{z})$ be strictly semistable satisfying* (1.3). *It is necessary and sufficient that $\psi(\mathbf{z})$ has one of the following forms*:

(i) $\alpha = 2$ *and* $\psi(\mathbf{z}) = -(1/2)\mathbf{z}A\mathbf{z}$.

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(ii) $0 < \alpha < 1$ *and* $\psi(\mathbf{z}) = \int_S \lambda(d\xi) \int_0^\infty \{e^{iu\xi\mathbf{z}} - 1\} d\{-H_\xi(u)u^{-\alpha}\}$.

(iii) $1 < \alpha < 2$ *and* $\psi(\mathbf{z}) = \int_S \lambda(d\xi) \int_0^\infty \{e^{iu\xi\mathbf{z}} - 1 - iu\xi\mathbf{z}\} d\{-H_\xi(u)u^{-\alpha}\}$.

(iv) $\alpha = 1$,

$$\psi(\mathbf{z}) = i\gamma_0\mathbf{z} + \int_S \lambda(d\xi) \int_0^\infty \{e^{iu\xi\mathbf{z}} - 1 - (iu\xi\mathbf{z})\mathbf{1}_{(0,1)}(u)\} d\{-H_\xi(u)u^{-1}\}$$

and, in addition,

$$\int_S \xi\lambda(d\xi) \int_{[b,1)} u\, d\{-H_\xi(u)u^{-1}\} = 0 \quad \text{if} \quad b > 0,$$

$$\int_S \xi\lambda(d\xi) \int_{[-b,1)} u\, d\{-H_\xi(u)u^{-1}\} = 2\gamma_0 \quad \text{if} \quad b < 0.$$

Proof. Use the uniqueness of the Lévy representation and Remark 1 together with Proposition 2.3. □

Remark 3. Let μ be a semistable law with exponent $\alpha \in (1,2)$ on $\mathbf{R}^d$. Then we have that

$$\psi(\mathbf{z}) = i\gamma_2\mathbf{z} + \int_S \lambda(d\xi) \int_0^\infty \{e^{iu\xi\mathbf{z}} - 1 - iu\xi\mathbf{z}\} d\{-H_\xi(u)u^{-\alpha}\}$$

with some $\gamma_2 \in \mathbf{R}^d$. Using Property 1.2.5 (ix) of [15], we can check that γ_2 is the mean of μ. By Proposition 2.4, μ is strictly semistable if and only if μ has mean zero.

PROPOSITION 2.5. *Exponent α of a semistable law is uniquely determined.*

Proof. If μ is a semistable law with exponent 2, then it is Gaussian and we have that $\int |\mathbf{x}|^\beta \mu(d\mathbf{x}) < \infty$, $\forall\, \beta > 0$. Hence, by Remark 1, the exponent α is the supremum of $\beta \in (0,2]$ such that $\int |\mathbf{x}|^\beta \mu(d\mathbf{x})$ is finite. Therefore the exponent is unique. □

The following proposition shows that even if $f(\mathbf{z})$ satisfies (1.2) with $\gamma \neq 0$, it can be strictly semistable. In order to determine whether a given semistable law is strictly semistable or not, it is important to know in what situation such a thing can happen. Propositions 2.6–2.8 answer this question.

PROPOSITION 2.6. *If $f(\mathbf{z})$ is semistable with exponent* 1 *satisfying* (1.2) *with* $-1 < b < 0$, *then* $f(\mathbf{z})$ *is strictly semistable with exponent* 1 *satisfying* $f(\mathbf{z}) = f(b^2\mathbf{z})^{1/(b^2)}$

Proof. We have $f(z) = f(b\mathbf{z})^c e^{i\gamma\mathbf{z}}$ and $f(b\mathbf{z}) = f(b^2\mathbf{z})^c e^{i\gamma b\mathbf{z}}$. Hence,

$$f(z) = \{f(b^2\mathbf{z})^c e^{i\gamma\mathbf{z}}\}^c e^{i\gamma\mathbf{z}} = f(b^2\mathbf{z})^{c^2} e^{i\gamma(bc+1)\mathbf{z}}.$$

Since $bc = -1$, we obtain $f(\mathbf{z}) = f(b^2\mathbf{z})^{c^2}$. $\square$

PROPOSITION 2.7. *Suppose that $f(\mathbf{z})$ is semistable with exponent* 1 *satisfying* (1.2) *with* $\gamma \neq 0$. *If* $0 < b < 1$, *then $f(\mathbf{z})$ cannot be strictly semistable. If* $-1 < b < 0$, *then there is no* b_1 *satisfying* $f(\mathbf{z}) = f(b_1\mathbf{z})^{c_1}$, $-1 < b_1 < 0$, $c_1 |b_1| = 1$.

Proof. Suppose that there exists $b_1 \in \mathbf{R}$ such that $0 < |b_1| < 1$ and $|b_1| \psi(\mathbf{z}) = \psi(b_1\mathbf{z})$. Then we divide the proof into three cases: (i) $b > 0$ and $b_1 > 0$, (ii) $b > 0$ and $b_1 < 0$, and (iii) $b < 0$ and $b_1 < 0$.

(i) $b > 0$ and $b_1 > 0$. Using the uniqueness of the Lévy representation, we see that

$$\int_S \xi\lambda(d\xi) \int_{[b,1)} u\, d\{-H_\xi(u)u^{-1}\} = -\gamma$$

and

$$\int_S \xi\lambda(d\xi) \int_{[b_1,1)} u\, d\{-H_\xi(u)u^{-1}\} = 0.$$

Hence we have that

$$\int_S \xi\lambda(d\xi) \int_{[b^n,1)} u\, d\{-H_\xi(u)u^{-1}\} = -n\gamma \tag{2.6}$$

and

$$\int_S \xi\lambda(d\xi) \int_{[b_1^n,1)} u\, d\{-H_\xi(u)u^{-1}\} = 0. \tag{2.7}$$

Since $\gamma \neq 0$, we have $\limsup_{a\downarrow 0} \left| \int_S \xi\lambda(d\xi) \int_{[a,1)} u\, d\{-H_\xi(u)u^{-1}\} \right| = \infty$ by (2.6). On the other hand, since

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$$\int_S \xi\lambda(d\xi) \int_{[a,b_1^n)} u\, d\{-H_\xi(u)u^{-1}\} = \int_S \xi\lambda(d\xi) \int_{[b_1^{-n}a,1)} u\, d\{-H_\xi(u)u^{-1}\},$$

for $b_1^{n+1} \leq a < b_1^n$, we have $\limsup_{a\downarrow 0} \left| \int_S \xi\lambda(d\xi) \int_{[a,1)} u\, d\{-H_\xi(u)u^{-1}\} \right| < \infty$ by (2.7). This leads to contradiction.

(ii) $b > 0$ and $b_1 < 0$. In this case, we have that $b_1{}^2\psi(\mathbf{z}) = \psi(b_1{}^2\mathbf{z})$. Hence we have a contradiction by (i).

(iii) $b < 0$ and $b_1 < 0$. In this case, we have that $b_1 b(\psi(\mathbf{z}) - i\gamma\mathbf{z}) = \psi(b_1 b\mathbf{z})$. On the other hand, $b_1^2\psi(\mathbf{z}) = \psi(b_1^2\mathbf{z})$. Hence we have a contradiction by (i). □

PROPOSITION 2.8. *Suppose that $f(\mathbf{z})$ is semistable with exponent α satisfying (1.2) with $\gamma \neq 0$. If $\alpha \neq 1$, then $f(\mathbf{z})$ cannot be strictly semistable.*

Proof. Suppose that there exists $b_1 \in \mathbf{R}$ and $c_1 \in \mathbf{R}$ such that $0 < |b_1| < 1$, $c_1 > 1$, and

$$f(\mathbf{z}) = f(b_1\mathbf{z})^{c_1}, \ \forall\, \mathbf{z} \in \mathbf{R}^d.$$

Define α_1 by $c_1 |b_1|^{\alpha_1} = 1$. Then $\alpha = \alpha_1$ by Proposition 2.5. If $\alpha = 2$, then we get obvious contradiction by Proposition 2.4. Consider the case of $0 < \alpha < 1$. Then we have that

$$\psi(\mathbf{z}) = i\gamma_3\mathbf{z} + \int_S \lambda(d\xi) \int_0^\infty \{e^{iu\xi\mathbf{z}} - 1\} d\{-H_\xi(u)u^{-\alpha}\}$$

with some γ_3. Since $|b|^\alpha(\psi(\mathbf{z}) - i\gamma\mathbf{z}) = \psi(b\mathbf{z})$, we have that $\gamma_3 = \dfrac{|b|^\alpha\gamma}{|b|^\alpha - b}$.

Similarly it follows from $|b_1|^\alpha\psi(\mathbf{z}) = \psi(b_1\mathbf{z})$ that $\gamma_3 = 0$. Hence $\gamma = 0$. This is contradiction. In the same way we get contradiction in the case of $1 < \alpha < 2$. □

EXAMPLE 1. It is well-known that a probability measure with ch.f. $f(\mathbf{z})$ is stable if and only if, for any $c > 1$, there exist $0 < b < 1$ and $\gamma \in \mathbf{R}^d$ satisfying (1.2). Similarly, a probability measure with ch.f. $f(\mathbf{z})$ is strictly stable if and only if, for any $c > 1$, there exists $0 < b < 1$ such that (1.3) holds. Hence every stable law is semistable; every strictly stable law is strictly semistable.

EXAMPLE 2. For $d = 1$, every strictly stable law with exponent 1 has a symmetric Lévy measure. But there are strictly semistable laws with nonsymmetric Lévy measures even in case $\alpha = 1$ and $d = 1$. We consider two examples for

$d = 1$.

(i) This is similar to examples in [10, 17]. For $-1 < b < 0$ and $0 < \alpha < 2$, let

$$H_1(u)u^{-\alpha} = \int_0^\infty h(\log v)v^{-(\alpha+1)}dv$$

and

$$H_{-1}(u)u^{-\alpha} = \int_0^\infty h(\log v + \log(-b))v^{-(\alpha+1)}dv,$$

where $h(v)$ is a nonnegative measurable bounded periodic function on $(0, \infty)$ with period $-2\log(-b)$. Then $H_1(-bu) = H_{-1}(u)$, $H_1(b^2u) = H_1(u)$ and $H_{-1}(b^2u) = H_{-1}(u)$. Let $\lambda(\{1\}) = \lambda(\{-1\}) = 1$ and consider $\psi(z)$ described in Proposition 2.4 (ii), (iii), (iv). We obtain a strictly semistable law satisfying (1.3) with $-1 < b < 0$ and exponent α. In fact, in case $\alpha = 1$, we choose a constant γ_0 such that

$$\int_0^\infty u\mathbf{1}_{[-b,1)}(u)d\{-H_1(u)u^{-1}\} + \int_0^\infty u\mathbf{1}_{[-b,1)}(u)d\{H_1(u)u^{-1}\} = 2\gamma_0$$

and consider

$$\psi(z) = i\gamma_0\mathbf{z} + \int_0^\infty \{e^{iuz} - 1 - iuz\mathbf{1}_{(0,1)}(u)\}d\{-H_1(u)u^{-1}\}$$
$$+ \int_{-\infty}^0 \{e^{iuz} - 1 - iuz\mathbf{1}_{(-1,0)}(u)\}d\{H_{-1}(|u|)|u|^{-1}\}.$$

Note that this has nonsymmetric Lévy measure in general.

(ii) In [11], there is a semistable example with discrete Lévy measure. Modifying it, we consider the case $-1 < b < 0$. Let $d = 1$, $0 < \alpha < 2$, and

$$H_1(u)u^{-\alpha} = \sum_{(-b)^{-2n} > u} (-b)^{2n\alpha}$$

and

$$H_{-1}(u)u^{-\alpha} = \sum_{(-b)^{-(2n+1)} > u} (-b)^{(2n+1)\alpha}.$$

Then $H_1(-bu) = H_{-1}(u)$, $H_1(b^2u) = H_1(u)$ and $H_{-1}(b^2u) = H_{-1}(u)$. Hence we obtain a strictly semistable law with nonsymmetric Lévy measure similarly to (i). In this example, in case $\alpha = 1$, we choose γ_0 equal to $-1/2$ and consider

$$\psi(z) = i\gamma_0 z + \sum_{n=1}^{\infty} (e^{ib^n z} - 1 - ib^n z)|b|^{-n} + \sum_{n=-\infty}^{0} (e^{ib^n z} - 1)|b|^{-n}.$$

3. Criteria for recurrence and transience of the semistable processes

Let $\{X(t)\}$ be a Lévy processes on $\mathbf{R}^d$. If $X(1)$ has a semistable law with exponent α, then we call $\{X(t)\}$ a semistable process with exponent α. If $X(1)$ has a strictly semistable law with exponent α, then we call $\{X(t)\}$ a strictly semistable process with exponent α. Let $f(\mathbf{z})$ be the ch.f. of the distribution $X(1)$ and $\psi(\mathbf{z}) = \log f(\mathbf{z})$. Denote $(\mathrm{Re}\{-\psi(\mathbf{z})\})^{-1} = H(\mathbf{z})$, $\mathrm{Re}\{1/(-\psi(\mathbf{z}))\} = G(\mathbf{z})$, $\mathrm{Re}\{-\psi(\mathbf{z})\} = K(\mathbf{z})$, and $\mathrm{Im}\{-\psi(\mathbf{z})\} = J(\mathbf{z})$.

THEOREM 3.1. *Assume that $\{X(t)\}$ is a semistable process with exponent $\alpha \in [1,2]$ on $\mathbf{R}$. Then $\{X(t)\}$ is recurrent if and only if $\{X(t)\}$ is strictly semistable.*

Proof. Corollary 7.4.5 in [15] says that if $\int_{-\delta}^{\delta} G(z)\,dz = \infty$ for some $\delta > 0$, then the process is recurrent. Since $G(z)$ is an even function, we need only to consider $\int_0^{\delta} G(z)\,dz$. Suppose that $\{X(t)\}$ is strictly semistable with exponent $\alpha \in [1,2]$ and satisfies (1.3). Since it is nondegenerate, there exists $\delta > 0$ such that $K(z) > 0$ for every $z \in (0, \delta]$ by Theorem 2, §14 in [2]. Since $G(z)$ is continuous, there exists $C > 0$ such that $G(z) \geq C$ on $[b^2\delta, \delta]$, Hence

$$\int_{b^2\delta}^{\delta} G(z)\,dz \geq C\int_{b^2\delta}^{\delta} dz > 0.$$

We claim that $\int_0^{\delta} G(z)\,dz = \sum_{n=0}^{\infty}\int_{(b^2)^{n+1}\delta}^{(b^2)^n\delta} G(z)\,dz = \sum_{n=0}^{\infty}(b^2)^{n(1-\alpha)}\int_{b^2\delta}^{\delta} G(z)\,dz$. Indeed, noticing that Lemma 2.1 holds with $\sigma_n = 0$, we have that $K(b^{2n}z) = (b^2)^{n\alpha}K(z)$ and $J(b^{2n}z) = (b^2)^{n\alpha}J(z)$. Hence

$$\begin{aligned}\int_{b^{2n+2}\delta}^{b^{2n}\delta} G(z)\,dz &= \int_{b^{2n+2}\delta}^{b^{2n}\delta} K(z)/\{K(z)^2 + J(z)^2\}\,dz \\ &= b^{2n}\int_{b^2\delta}^{\delta} K(b^{2n}z)/\{K(b^{2n}z)^2 + J(b^{2n}z)^2\}\,dz = (b^2)^{n(1-\alpha)}\int_{b^2\delta}^{\delta} G(z)\,dz.\end{aligned}$$

It follows that $\int_0^{\delta} G(z)\,dz = \infty$, because $(b^2)^{1-\alpha} \geq 1$. Hence $\{X(t)\}$ is recurrent.

Suppose that $\{X(t)\}$ is not strictly semistable. Then consider three cases: (i) $\alpha = 1$, (ii) $\alpha = 2$, and (iii) $1 < \alpha < 2$.

(i) $\alpha = 1$. We will show $\limsup_{p\downarrow 0}\int_0^{\delta}\mathrm{Re}\{(p - \psi(z))^{-1}\}\,dz < \infty$, which implies transience of $\{X(t)\}$ by Theorem 7.4.4 in [15]. Since $\{X(t)\}$ is not strictly semi-stable, b has to be positive by Proposition 2.6. We have that

$$\int_0^\delta \mathrm{Re}\{(p-\psi(z))^{-1}\}dz \leq \int_0^\delta \frac{p}{p^2+K(z)^2}dz + \int_0^\delta \frac{K(z)}{K(z)^2+J(z)^2}dz.$$

By Remark 2, we have that $K(z) = |z| R(z)$ and $\inf\{R(z) : z \in (0,\infty)\} > 0$. Hence

$$\int_0^\delta \frac{p}{p^2+K(z)^2}dz = \int_0^\delta \frac{p}{p^2+z^2R(z)}dz \leq \int_0^\delta \frac{1}{1+z^2R(pz)^2}dz \tag{3.1}$$

$$\leq 1 + \int_1^\infty (zR(pz))^{-2}dz \leq 1 + (\inf\{R(z) : z \in (0,\infty)\})^{-2}\int_1^\infty z^{-2}\,dz < \infty.$$

By Lemma 2.1, we have $-\psi(b^n z) = -b^n\psi(z) + inb^n\gamma z$, where $\gamma \neq 0$. Hence $K(b^n z) = b^n K(z)$ and $J(b^n z) = b^n J(z) + nb^n\gamma z$. Thus we have

$$\int_0^\delta \frac{K(z)}{K(z)^2+J(z)^2}dz = \int_0^\delta G(z)dz,$$

$$\int_{b^{n+1}\delta}^{b^n\delta} G(z)dz = b^n\int_{b\delta}^\delta G(b^n z)dz$$

$$= b^n\int_{b\delta}^\delta \{b^n K(z)\}/\{(b^n K(z))^2 + (b^n J(z) + nb^n\gamma z)^2\}dz$$

$$= \int_{b\delta}^\delta K(z)/\{K(z)^2 + (J(z)+n\gamma z)^2\}dz,$$

and

$$n^2\int_{b^{n+1}\delta}^{b^n\delta} G(z)dz = \int_{b\delta}^\delta K(z)/\{(K(z)^2/n^2) + ((J(z)/n)+\gamma z)^2\}dz$$

$$\to \int_{b\delta}^\delta K(z)/(\gamma^2 z^2)dz \quad \text{as } n\to\infty.$$

Therefore $\int_{b^{n+1}\delta}^{b^n\delta} G(z)dz \leq$ (positive constant)$/(n^2)$. Thus,

$$\int_0^\delta G(z)dz = \textstyle\sum_{n=0}^\infty \displaystyle\int_{b^{2n+2}\delta}^{b^{2n}\delta} G(z)dz < \infty. \tag{3.2}$$

By (3.1) and (3.2), $\limsup_{p\downarrow 0}\int_0^\delta \mathrm{Re}\{(p-\psi(z))^{-1}\}dz < \infty$. Hence the process is transient.

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(ii) $\alpha = 2$. The distribution of $X(1)$ is Gaussian with $EX(1) \neq 0$. Hence we know through Theorem 7.4.13 in [15] that $\{X(t)\}$ is transient.
(iii) $1 < \alpha < 2$. By Remark 3 we have that $EX(1) \neq 0$. Hence $\{X(t)\}$ is transient. $\square$

THEOREM 3.2. *Assume that $\{X(t)\}$ is a semistable process with exponent $\alpha \in (0, 1)$ on $\mathbf{R}$. Then $\{X(t)\}$ is transient.*

Proof. If $\int_{-\delta}^{\delta} H(z)\,dz < \infty$, then the process is transient (Corollary 7.4.5 in [15]). We have $\int_{-\delta}^{\delta} H(z)\,dz = 2\int_{0}^{\delta} H(z)\,dz$. By a similar discussion to the proof of Theorem 3.1, there exist $\delta > 0$ and $C > 0$ such that $H(z) \leq C$ on $[b^2\delta, \delta]$. Hence $\int_{b^2\delta}^{\delta} H(z)\,dz$ is finite. We have that

$$\int_0^{\delta} H(z)\,dz = \sum_{n=0}^{\infty} \int_{b^{2n+2}\delta}^{b^{2n}\delta} H(z)\,dz.$$

Using Lemma 2.1, we get that

$$\{H(b^{2n}z)\}^{-1} = K(b^{2n}z) = (b^2)^{n\alpha}K(z) = (b^2)^{n\alpha}\{H(z)\}^{-1}.$$

Hence,

$$\int_{b^{2n+2}\delta}^{b^{2n}\delta} H(z)\,dz = b^{2n}\int_{b^2\delta}^{\delta} H(b^{2n}z)\,dz = (b^2)^{n(1-\alpha)}\int_{b^2\delta}^{\delta} H(z)\,dz.$$

Thus, we have that

$$\int_0^{\delta} H(z)\,dz = \sum_{n=0}^{\infty} (b^2)^{n(1-\alpha)}\int_{b^2\delta}^{\delta} H(z)\,dz < \infty,$$

because $(b^2)^{1-\alpha} < 1$. Hence $\{X(t)\}$ is transient. $\square$

THEOREM 3.3. *Assume that $\{X(t)\}$ is a semistable process with exponent $\alpha \in (0,2]$ on $\mathbf{R}^2$, where $\mathbf{R}^2$ is spanned by the support Σ of $\{X(t)\}$. The support Σ of $\{X(t)\}$ is defined to be the set of points $\mathbf{x}$ such that, for each $\varepsilon > 0$, there exists $t \geq 0$ satisfying $P(|X(t) - \mathbf{x}| < \varepsilon) > 0$. Then $\{X(t)\}$ is recurrent if and only if $\{X(t)\}$ is strictly semistable with $\alpha = 2$.*

Proof. (i) We first consider the case of $\alpha = 2$. We know that $E|X(1)|^2 < \infty$. By Theorem 7.4.11 of [15], $\{X(t)\}$ is recurrent if $EX(1) = 0$; $\{X(t)\}$ is transient if $EX(1) \neq 0$.

(ii) $0 < \alpha < 2$. By Lemma 7.4.8 of [15], there exists $\delta > 0$ such that $K(\mathbf{z}) \geq$ (positive constant) $|\mathbf{z}|^2$ for $|\mathbf{z}| < \delta$. Hence $H(\mathbf{z})$ is continuous on the set $\{\mathbf{z} : 0 < |\mathbf{z}| \leq \delta\}$ and the integral $C = \int_{b^2\delta < |\mathbf{z}| \leq \delta} H(\mathbf{z})\,d\mathbf{z}$ is finite. Let $U = \{\mathbf{z} : |\mathbf{z}| < \delta\}$ and $U_n = \{\mathbf{z} : b^{2(n+1)}\delta \leq |\mathbf{z}| < b^{2n}\delta\}$. We have that

$$\int_U H(\mathbf{z})\,d\mathbf{z} = \sum_{n=0}^{\infty} \int_{U_n} H(\mathbf{z})\,d\mathbf{z}.$$

Since $K(b^{2n}\mathbf{z}) = (b^2)^{n\alpha} K(\mathbf{z})$, we have that

$$\begin{aligned}\int_{U_n} H(\mathbf{z})\,d\mathbf{z} &= \int_0^{2\pi} \int_{b^{2(n+1)}\delta}^{b^{2n}\delta} H(u\cos\theta,\, u\sin\theta)\,u\,du\,d\theta \\ &= (b^2)^{n(2-\alpha)} \int_0^{2\pi} \int_{b^2\delta}^{\delta} H(u\cos\theta,\, u\sin\theta)\,u\,du\,d\theta.\end{aligned}$$

Hence, $\int_U H(\mathbf{z})\,d\mathbf{z} = \sum_{n=0}^{\infty} (b^2)^{n(2-\alpha)}\, C < \infty$. Therefore the process is transient. □

Remark 4. Let $d \geq 3$. If $\{X(t)\}$ is a Lévy process on $\mathbf{R}^d$ such that $\mathbf{R}^d$ is spanned by the support Σ of $\{X(t)\}$, then $\{X(t)\}$ is transient, by Theorem 7.4.7 of [15].

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