ON SOME ASYMPTOTIC PROPERTIES CONCERNING BROWNIAN MOTION

TUNEKITI SIRAO and TOSIO NISIDA

1. About the behavior of brownian motion at time point ∞, there are many results by P. Lévy, A. Khintchine etc. P. Lévy cited a theorem by A. Kolmogoroff as the most precise result in his famous book "Processus stochastiques et mouvement brownian" without proof. In this paper we shall prove this theorem, using the similar result about the random sequence by W. Feller,[1] and then, applying the theorem of projective invariance by P. Lévy, we shall find also the behavior of brownian motion at time point 0 from the above theorem.

2. Stating the results. We define the concept of upper class and lower class with respect to Wiener's brownian motion $X(t) \equiv X(t, \omega)$[2] at time point ∞ as follows:

i) If the set of t such that

$$X(t, \omega) > \sqrt{t}\, \phi(t)$$

is bounded (unbounded) for almost all ω, then we say that $\phi(t)$ belongs to upper (lower) class with respect to $\{X(t),\ 0 \leqq t < \infty\}$ at time point ∞ and use the notation $\phi(t) \in \mathfrak{U}_\infty(\mathfrak{L}_\infty)$.

Analogously we define upper class and lower class with respect to $\{X(t),\ 0 \leqq t \leqq 1\}$ at time point 0 as follows:

ii) If the set of t^{-1} such that

$$X(t, \omega) > \sqrt{t}\, \phi(t)$$

is bounded (unbounded) for almost all ω, then we say that $\phi(t)$ belongs to upper (lower) class with respect to $\{X(t),\ 0 \leqq t \leqq 1\}$ at time point 0 and use the notation $\phi(t) \in \mathfrak{U}_0(\mathfrak{L}_0)$.

THEOREM 1 (A. Kolmogoroff). *Let $\{X(t),\ 0 \leqq t < \infty\}$ be a brownian motion of Wiener and $\phi(t)$ be a non-negative, monotone increasing function of t such that*

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[1] W. Feller: "The law of the iterated logarithm for identically distributed random variables." Ann. of Math. Vol. 47 (1946).

[2] ω is the probability parameter.

$$\frac{\phi^2(t)}{t} \to 0 \quad (t \to \infty).$$

[3]

If $\displaystyle\int^{\infty} \frac{1}{t}\phi(t)e^{-\frac{1}{2}\phi^2(t)}\,dt \in \mathfrak{C}(\mathfrak{D})$, *then* $\phi(t) \in \mathfrak{U}_\infty(\mathfrak{L}_\infty)$,

where " $\in \mathfrak{C}(\mathfrak{D})$" *means the convergence (divergence) of the integrals.*

Applying theorem 1 for

$$\phi(t) = \{2\log_2 t + 3\log_3 t + 2\log_4 t + \dots + 2\log_{p-1} t + (2+\delta)\log_p t\}^{\frac{1}{2}},$$

we have

Cor. 1. $\{2\log_2 t + 3\log_3 t + 2\log_4 t + \dots + 2\log_{p-1} t + (2+\delta)\log_p t\}^{\frac{1}{2}}$

$$\in \mathfrak{U}_\infty \quad \text{if} \quad \delta > 0,$$
$$\in \mathfrak{L}_\infty \quad \text{if} \quad \delta \leqq 0.$$

Theorem 2. *Let* $\{X(t),\ 0 \leqq t \leqq 1\}$ *be a brownian motion of Wiener and* $\psi(t)$ *be non-negative monotone decreasing function of* t.

If $\displaystyle\int_0^1 \frac{1}{t}\psi(t)e^{-\frac{1}{2}\psi^2(t)}dt \in \mathfrak{C}(\mathfrak{D})$, *then* $\psi(t) \in \mathfrak{U}_0(\mathfrak{L}_0)$.

For

$$\psi(t) = \{2\log_2 \frac{1}{t} + 3\log_3 \frac{1}{t} + 2\log_4 \frac{1}{t} + \dots + 2\log_{p-1}\frac{1}{t} + (2+\delta)\log_p \frac{1}{t}\}^{\frac{1}{2}},$$

we have

Cor. 2. $\left\{2\log_2 \frac{1}{t} + 3\log_3 \frac{1}{t} + 2\log_4 \frac{1}{t} + \dots + 2\log_{p-1}\frac{1}{t} + (2+\delta)\log_p \frac{1}{t}\right\}^{\frac{1}{2}}$

$$\in \mathfrak{U}_0 \quad \text{if} \quad \delta > 0,$$
$$\in \mathfrak{L}_0 \quad \text{if} \quad \delta \leqq 0.$$

3. Proof.

Proof of Theorem 1.

a) The case of convergence.

Let us define the sequence $\{t_k\}$ as follows:

$$t_{k+1} = t_k(1 + 1/\phi^2(t_k)),$$

where t_1 is chosen sufficiently large, enough to satisfy the condition $t_k \to \infty$ as $k \to \infty$.

Now we put

[3] This condition is not any essential restriction, since A. Khintchine has already proved that $\varepsilon > 0$ (< 0) implies $\{(2+\varepsilon)\log_2 t\}^{1/2} \in \mathfrak{U}_\infty(\mathfrak{L}_\infty)$.

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$$p_k = \frac{1}{\phi(t_k)} e^{-\frac{1}{2}\phi^2(t_k)} .$$

Then, since $\phi(t)e^{-\frac{1}{2}\phi^2(t)}$ is monotone decreasing in t, we have

$$\begin{aligned}\int_{t_1}^{\infty} \frac{1}{t}\phi(t)e^{-\frac{1}{2}\phi^2(t)}\,dt &\geqq \sum_k \frac{t_{k+1}-t_k}{t_{k+1}}\phi(t_{k+1})e^{-\frac{1}{2}\phi^2(t_{k+1})} \\ &= \sum_k \frac{t_k}{t_{k+1}}\frac{1}{\phi^2(t_k)}\phi(t_{k+1})e^{-\frac{1}{2}\phi^2(t_{k+1})} \\ &\geqq \frac{1}{1+1/\phi^2(t_1)}\sum_k \frac{1}{\phi^2(t_{k+1})}\phi(t_{k+1})e^{-\frac{1}{2}\phi^2(t_{k+1})} \\ &= \frac{1}{1+1/\phi^2(t_1)}\sum_k p_{k+1} .\end{aligned}$$

Hence, the convergence of this integral implies the convergence of $\sum_k p_k$. By the monotony of $\phi(t)$, we have

$$\max_{t_{k-1}\leqq t\leqq t_k} \frac{X(t)}{\sqrt{t}\,\phi(t)} \leqq \frac{\max_{t_{k-1}\leqq t\leqq t_k} X(t)}{\sqrt{t_{k-1}}\,\phi(t_{k-1})} \leqq \frac{\max_{0\leqq t\leqq t_k} X(t)}{\sqrt{t_{k-1}}\,\phi(t_{k-1})} .$$

Using the theorem by P. Lévy with respect to the maximum function of brownian motion

$$\begin{aligned}\Pr\left\{\frac{\max_{0\leqq t\leqq t_k} X(t)}{\sqrt{t_{k-1}}\,\phi(t_{k-1})} \geqq 1\right\} &= \Pr\{\max_{0\leqq t\leqq t_k} X(t) \geqq \sqrt{t_{k-1}}\,\phi(t_{k-1})\} \\ &= \sqrt{\frac{2}{\pi t_k}}\int_{\sqrt{t_{k-1}}\,\phi(t_{k-1})}^{\infty} e^{-\frac{x^2}{2t_k}}\,dx \leqq \sqrt{\frac{2}{\pi}}\,\frac{\sqrt{t_k}}{\sqrt{t_{k-1}}\,\phi(t_{k-1})}e^{-\frac{t_{k-1}}{2t_k}\phi^2(t_{k-1})} .\end{aligned}$$

By the definition of t_k, (if k is sufficiently large) we have

$$\frac{t_{k-1}}{t_k} = \frac{1}{1+1/\phi^2(t_{k-1})} > 1 - \frac{1}{\phi^2(t_{k-1})}$$

$$\frac{t_k}{t_{k-1}} = 1 + \frac{1}{\phi^2(t_{k-1})} < 2 .$$

Therefore we obtain

$$\begin{aligned}\Pr\left\{\frac{\max_{0\leqq t\leqq t_k} X(t)}{\sqrt{t_{k-1}}\,\phi(t_{k-1})} \geqq 1\right\} &\leqq \sqrt{\frac{2}{\pi}}\sqrt{2}\,e^{\frac{1}{2}}\frac{1}{\phi(t_{k-1})}e^{-\frac{1}{2}\phi^2(t_{k-1})} \\ &= Cp_{k-1}, \quad C = \frac{2}{\sqrt{\pi}}e^{\frac{1}{2}} ,\end{aligned}$$

and so

$$\sum_k \Pr\left\{\max_{t_{k-1}\leqq t\leqq t_k}\frac{X(t)}{\sqrt{t}\,\phi(t)} \geqq 1\right\} \leqq \sum_k \Pr\left\{\frac{\max_{0\leqq t\leqq t_k} X(t)}{\sqrt{t_{k-1}}\,\phi(t_{k-1})} \geqq 1\right\} \leqq C\sum_k p_{k-1} .$$

Since the last series converges by the assumption, the events

$$\max_{t_{k-1}\leqq t\leqq t_k} \frac{X(t)}{\sqrt{t}\,\phi(t)} \geqq 1$$

arise only for finitely many k with probability 1, by virtue of the lemma of Borel-Cantelli. In other words, there exists a number $k(\omega)$ which is finite with probability 1 such that

$$\max_{t_{k-1}\leqq t\leqq t_k} \frac{X(t)}{\sqrt{t}\,\phi(t)}<1 \quad \text{for} \quad k>k(\omega),$$

which implies

$$X(t)<\sqrt{t}\,\phi(t) \quad \text{for} \quad t>t(\omega)=t_{k(\omega)}.$$

Thus we have $\phi(t)\in\mathfrak{U}_\infty$.

b) The case of divergence.

We shall make use of the following theorem of W. Feller:[4)]

Let $\{X_n\}$ be a sequence of mutually independent random variables having the same distribution function $F(t)$ which have the properties:

$$(*)\begin{cases}\displaystyle\int_{-\infty}^{+\infty} t\,dF(t)=0\\ \displaystyle\int_{-\infty}^{+\infty} t^2 dF(t)=1\\ \displaystyle\int_{|t|>x} t^2 dF(t)=0\,((\log\log x)^{-1}) \quad (x\to\infty).\end{cases}$$

Then the monotone increasing sequence $\{\phi_n,\ \phi_n>0\}$ belongs to lower (upper) class[5)] if and only if the series

$$\sum_n \frac{\phi_n}{n} e^{-\frac{1}{2}\phi_n^2}$$

diverges (converges).

If we put

$$X_n=X(n)-X(n-1) \quad (n=1,2,\ldots),$$

then X_n are mutually independent and subject to the standard Gaussian distribution $\Phi_G(0.1)$. Moreover

$$\int_{|t|>x} t^2 d\Phi_G=\sqrt{\frac{2}{\pi}}\int_x^\infty t^2 e^{-\frac{t^2}{2}}dt=\sqrt{\frac{2}{\pi}}\left[xe^{-\frac{x^2}{2}}+\int_x^\infty e^{-\frac{t^2}{2}}dt\right]=0(x^{-1}) \quad (x\to\infty).$$

Thus $\{X_n\}$ satisfies the above-cited conditions $(*)$.

Since $\phi(t)e^{-\frac{1}{2}\phi^2(t)}$ is monotone decreasing,

4) loc. cit. 1).

5) A numerical monotonic sequence $\{\phi_n\}$ will be said to belong to the lower class $\mathfrak{L}$ if with probability one the inequality

(A) $$S_n>\sqrt{n}\,\phi_n, \quad S_n=X_1+\ldots+X_n,$$

be satisfied for infinitely many n; on the contrary, if with probability one (A) be satisfied only for finitely many n, then $\{\phi_n\}$ will be said to belong to the upper class $\mathfrak{U}$ (the terminology due to P. Lévy).

$$\int_1^\infty \frac{1}{t}\phi(t)e^{-\frac{1}{2}\phi^2(t)}dt \leqq \sum_{n=1}^\infty \frac{1}{n}\phi(n)e^{-\frac{1}{2}\phi^2(n)}.$$

Thus we see, by the assumption, that the right hand side diverges and by the above Feller's theorem, this $\phi(n)$ belongs to lower class, namely that

$$X(n) = \sum_{k=1}^n X_k > \sqrt{n}\,\phi(n),$$

for infinitely many n-values with P-measure 1, which implies

$$\phi(t) \in \mathfrak{L}_\infty .$$

Proof of Theorem 2. According to the theorem of projective invariance by P. Lévy for the transformation $t \to t^{-1}$, '$X(t)/\sqrt{t}$, $0 < t \leqq 1$,' and '$\sqrt{t}\,X\left(\frac{1}{t}\right)$, $0 < t \leqq 1$,' yield the same probability distribution on the space of continuous functions defined on [0,1]. Therefore the following two sets are bounded with the same probability;

$$\left\{\frac{1}{t};\ \frac{X(t)}{\sqrt{t}} > \psi(t)\right\}, \quad \left\{\frac{1}{t};\ \sqrt{t}\,X\left(\frac{1}{t}\right) > \psi(t)\right\}.$$

But the latter set coincides with the set

$$\left\{t;\ X(t) > \sqrt{t}\,\psi\left(\frac{1}{t}\right)\right\},$$

which is bounded with P-measure 1 or 0 by Theorem 1 according as

$$\int_1^\infty \frac{1}{t}\psi\left(\frac{1}{t}\right)e^{-\frac{1}{2}\psi^2\left(\frac{1}{t}\right)}dt \quad \text{i.e.} \quad \int_0^1 \frac{1}{t}\psi(t)e^{-\frac{1}{2}\psi^2(t)}dt \in \mathfrak{C} \quad \text{or} \quad \in \mathfrak{D}.$$

Thus our theorem is proved.

Mathematical Institute,
Nagoya University
and
Institute for Mathematical Statistics,
Kôbe University

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