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ON ABSOLUTE RIESZ SUMMABILITY OF FOURIER SERIES

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Abstract

Let $f(t) \sim \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \sum_1^\infty A_n(t)$, $\phi(t) = \phi(x,t) = \frac{1}{2}\{f(x+t) + f(x-t)\}$, and $l(w) = \exp\{w(\log w)^{-\beta}\}$ $\beta \geqq 0$, $w \geqq A > 1$.

In 1951 Mohanty proved the following theorem:

If $t^{-\delta}\varphi(t) \in BV(0,\pi)$, $\delta > 0$, *then* $\sum A_n(x) \in \left| R, l(w), 1 \right|$, *for* $\beta = 1 + 1/\delta$.

In this paper a general theorem on summability $\left| R, l(w), 1 \right|$ of $\sum A_n(x)$ has been given which improves upon Mohanty's result in different ways (see Corollaries 1, 2 and 3) and it is also shown that some of the results of this note are the best possible.

1. Definitions and notations

Let $\lambda = \lambda(w)$ be continuous, differentiable and monotonic increasing in (C, ∞), where C is some positive number, and $\lambda(w) \to \infty$ as $w \to \infty$. An infinite series $\Sigma\, a_n$ is said to be summable $\left| R, \lambda, r \right|$, $r > 0$, and we write $\Sigma\, a_n \in \left| R, \lambda, r \right|$, if

$$\int_A^\infty \{\lambda'(w)/\lambda^{r+1}(w)\} \left| \sum_{n<w} \{\lambda(w) - \lambda(n)\}^{r-1} \lambda(n) a_n \right| dw < \infty$$

where A is a finite positive number (Obrechkoff (1928) and (1929), Mohanty (1951)).

Let $f(t) \in L(-\pi, \pi)$ and be a 2π-periodic function. Without any loss of generality we may assume that the constant term of the Fourier series of $f(t)$ is zero, so that

$$f(t) \sim \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \sum_1^\infty A_n(t).$$

We use the following notations:

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

$$l(w) = \exp\{w/(\log w)^\beta\}, \quad \beta \geqq 0, \quad w \geqq A > 1,$$

$g(t) \in L(-\pi, \pi)$, an even and 2π-periodic function and

$$g(t) \sim \tfrac{1}{2}d_0 + \sum_{1}^{\infty} d_n \cos nt,$$

$$\gamma(w) = (\log w)^{\alpha},$$

$$\varepsilon = \{\varepsilon_n\} \text{ a sequence of numbers},$$

$$F(w, t, \lambda, g, \varepsilon) = \left\{\frac{\lambda'(w)}{\lambda^{r+1}(w)}\right\} \sum_{n<w} (\lambda(w) - \lambda(n))^{r-1}\lambda(n)\varepsilon_n \int_0^t g(u)\cos nu\, du,$$

K, an absolute constant, not necessarily the same at each occurrence.

2. Introduction

Mohanty (1951) proved the following theorem concerning summability $|R, l(w), 1|$ of a Fourier series at a point.

Theorem A: If $t^{-\delta}\phi(t) \in BV(0, \pi)$, $\delta > 0$, then $\sum A_n(x) \in |R, l(w), 1|$, for $\beta = 1 + 1/\delta$.

Elsewhere (Dikshit (1965a)) the following theorem also has been given in this direction.

Theorem B: There exists a function $f(t)$ such that $\phi(t)\log k/t \in BV(0, \pi)$ but the series $\sum A_n(x) \notin |R, l(w), 1|$, for $\beta = 1$.

The object of this paper is to further investigate the summability $|R, l(w), 1|$ of Fourier series. We first prove a general theorem and then deduce as corollaries some results which improve upon Mohanty's theorem in two ways. Firstly, we relax the condition on the function $\phi(t)$ and get the same conclusion as in Theorem A (Corollary 1). Secondly, we extend the scope of summability in the sense that β may be taken to be any non-negative number, that is, it need not be only greater than 1 (Corollaries 1, 2, and 3), and in the case $\beta > 1$, we have a much stronger result.

It may be remarked that when $\beta \leqq 0$, the summability $|R, l(w), 1|$ becomes ineffective in the sense that it sums only absolutely convergent series (see Mohanty (1951), and Dikshit (1960)).

We have further shown that some of our results are the best possible in a certain sense: $\eta > 0$ in Corollary 2 (and 3) may not be replaced by $\eta = 0$ or, the case $\beta = 1/\delta$, $0 < \delta < 1$, of Corollary 1 cannot be extended to cover the case $\delta = 1$.

For some further existing results on summability $|R, l(w), 1|$ of Fourier series, a reference may also be made to an earlier paper by the author (Dikshit (1965)).

We prove the following theorem:

THEOREM: *Let* $\delta \geqq 0$, $\beta \geqq 0$ *and* α *be* (i) $(\delta\beta - 1)$ *when* $\beta > 0$, $0 < \delta < 1$, (ii) *a number* < -1, *when* $\delta\beta = 0$ *and* (iii) *any number* $< \beta - 1$, *when* $\delta \geqq 1$.

If $t^{-\delta}\phi(t) \in BV(0, \pi)$, *then* $\sum A_n(x)(\log n)^{\alpha} \in |R, \exp\{w/(\log w)^{\beta}\}, 1|$.

3. Lemmas

We shall make use of the following lemmas towards the proof of our theorem.

LEMMA 1. (Tatchell (1953)). *Necessary and sufficient conditions for*

$$\alpha_s(x) = \int_0^\infty h(x, t)ds(t)$$

be defined and $\in L[0, \infty)$ *whenever* $s(t) \in BV[0, \infty)$ *are that*

(i) $h(x, t)$ *be a function of* t *continuous and bounded on* $[0, \infty)$ *whenever* $x \geqq 0$,

(ii) $h(x, t)$ *be a function of* x *measurable on* $[0, \infty)$ *whenever* $t \geqq 0$, *and*

(iii) *there be real* K, *independent of* t, *such that*

$$\int_0^\infty |h(x, t)|\,dx \leqq K,$$

whenever $t \geqq 0$.

LEMMA 2. *Let* $g(t) \in L(-\pi, \pi)$ *be an even periodic function such that* $\sum d_n \varepsilon_n \in |R, \lambda, r|$, $r > 0$. *If* $\{g(t)\}^{-1}\phi(t) \in BV(0, \pi)$, *then in order that* $\sum A_n(x)\varepsilon_n \in |R, \lambda, r|$, *it is sufficient that*

$$\int_A^\infty |F(w, t, \lambda, g, \varepsilon)|\,dw$$

be uniformly bounded in t *for* $0 \leqq t \leqq \pi$.

PROOF: We have

$$\begin{aligned}
\frac{\pi}{2} A_n(x) &= \int_0^\pi \phi(t)\cos nt\,dt \\
&= \left[s(t) \int_0^t g(u)\cos nu\,du\right]_0^\pi - \int_0^\pi \left[\int_0^t g(u)\cos nu\,du\right] ds(t) \\
&= \frac{\phi(\pi)}{g(\pi)}\, d_n - \int_0^\pi \left[\int_0^t g(u)\cos nu\,du\right] ds(t),
\end{aligned}$$

where $s(t) = \{g(t)\}^{-1}\phi(t)$.
Therefore $\sum A_n(x)\varepsilon_n \in |R, \lambda, r|$ if

$$\int_A^\infty \left\{\frac{\lambda'(w)}{\lambda^{r+1}(w)}\right\} \left| \sum_{n<w} (\lambda(w) - \lambda(n))^{r-1}\lambda(n)\varepsilon_n \int_0^\pi \left[\int_0^t g(u)\cos nu\,du\right] ds(t) \right| dw < K,$$

that is, by Lemma 1,

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$$\int_A^\infty \left|F(w,t,\lambda,g,\varepsilon)\right| dw$$

is uniformly bounded in t for $0 \leqq t \leqq \pi$.

LEMMA 3. *If* $g(t) = |t|^\delta$, $t \in [-\pi,\pi]$, $\delta > 0$, *then* $d_n = O(n^{-\delta'})$ *where* $\delta' = \min(1+\delta, 2)$.

The case $0 < \delta < 1$ is due to Mohanty (1951), and the general case is proved similarly.

4. Proof of the Theorem

It is evident that it is enough to consider for $0 \leqq \delta \leqq 1$. After Lemmas 2 and 3, it is sufficient to show that

$$I \equiv \int_A^\infty \frac{l'(w)}{l^2(w)} \left| \sum_{n<w} l(n)\gamma_n \int_0^t u^\delta \cos nu \, du \right| dw < K,$$

uniformly in $t, 0 \leqq t \leqq \pi$.

$$\text{As } \int_0^t u^\delta \cos nu \, du = \left[\int_0^{1/n} + \int_{1/n}^t\right] u^\delta \cos nu \, du$$

$$= \frac{t^\delta \sin nt}{n} + O\left[\frac{1}{n^{1+\delta}}\right], \text{ if } 0 \leqq \delta \leqq 1,$$

$$I \leqq K \int_A^\infty \frac{l'(w)}{l^2(w)} \sum_{n<w} l(n)\gamma_n n^{-(1+\delta)} dw + t^\delta \int_A^\infty \frac{l'(w)}{l^2(w)} \left| \sum_{n<w} l(n)\gamma_n \frac{\sin nt}{n} \right| dw$$

$$= KI_1 + I_2, \text{ say.}$$

Since $l(w)\gamma(w)/w^{1+\delta}$ is a logarithmico-exponential function which is ultimately increasing, we have

$$I_1 \leqq \int_A^\infty \frac{l'(w)}{l^2(w)} \int_A^w \frac{l(x)\gamma(x)}{x^{1+\delta}} dx \, dw + \int_A^\infty \frac{l'(w)}{l^2(w)} \frac{l(w)\gamma(w)}{w^{1+\delta}} dw$$

$$= \int_A^\infty \frac{l(x)\gamma(x)}{x^{1+\delta}} \int_x^\infty \frac{l'(w)}{l^2(w)} dw \, dx + K_1 = \int_A^\infty \frac{\gamma(x)}{x^{1+\delta}} dx + K_1 \leqq K_2.$$

This also terminates the proof of the theorem for the case $\delta = 0$, and hence also for the case $\delta\beta = 0$.

For $\delta\beta > 0$

$$I_2 = t^\delta \left\{\int_A^\tau + \int_\tau^\infty\right\} \frac{l'(w)}{l^2(w)} \left| \sum_{n<w} l(n)\gamma_n \frac{\sin nt}{n} \right| dw$$

$$= I_3 + I_4, \text{ say,}$$

where $\tau = \exp(t^{-1/\beta})$.

Now

$$
\begin{aligned}
I_3 &\leqq t^{\delta}\left\{\int_A^{\tau}\frac{l'(w)}{l^2(w)}\int_A^{w}\frac{l(x)\gamma(x)}{x}\,dx+\int_A^{\tau}\frac{l'(w)\gamma(w)}{wl(w)}\,dw\right\}\\
&\leqq t^{\delta}\int_A^{\tau}\frac{l(x)\gamma(x)}{x}\int_x^{\tau}\frac{l'(w)}{l^2(w)}\,dw+Kt^{\delta}\int_A^{\tau}\frac{(\log w)^{\alpha-\beta}}{w}\,dw\\
&\leqq t^{\delta}\int_A^{\tau}\frac{(\log x)^{\alpha}}{x}\,dx+K_1\leqq K_2.
\end{aligned}
$$

Hence to complete the proof of the theorem it is sufficient to show that I_4 is uniformly bounded in $t, 0 < t \leqq \pi$.

As $l(n)\gamma(n)/n$ is increasing for $n \geqq n_0, n_0$ a finite number, by Abel's Lemma

$$
\left|\sum_{n<w} l(n)\gamma_n\frac{\sin nt}{n}\right| \leqq Kl(w)\gamma(w)/(wt).
$$

Therefore,

$$
\begin{aligned}
I_4 &\leqq Kt^{\delta-1}\int_{\tau}^{\infty}\frac{l'(w)\gamma(w)}{wl(w)}\,dw\\
&\leqq K_1t^{\delta-1}\int_{\tau}^{\infty}\frac{(\log w)^{\alpha-\beta}}{w}\,dw\\
&\leqq K_2,
\end{aligned}
$$

and the proof is completed.

4.1. The following results are obtained as special cases of our theorem.

COROLLARY 1: *Let* $0 < \delta \leqq 1$ *and* $t^{-\delta}\phi(t) \in BV(0,\pi)$. *Then*

$$
\sum A_n(x) \in \left|R, \exp\{w/(\log w)^{\beta}\}, 1\right|,
$$

where $\beta \geqq 1/\delta$, $\delta \neq 1$, *and* $\beta > 1$ *when* $\delta = 1$.

The inequality in '$\beta \geqq 1/\delta$' is an outcome of the 'second theorem of consistency' for absolute Riesz summability (see Dikshit (1958)).

COROLLARY 2: *If* $t^{-1}\phi(t) \in BV(0,\pi)$, *then*

$$
\sum_{n=2}^{\infty} A_n(x)/(\log n)^{\eta} \in \left|R, \exp\{w/\log w\}, 1\right|, \text{ for } \eta > 0.
$$

COROLLARY 3: *If* $\phi(t) \in BV(0,\pi)$, *then* $\sum_{n=2}^{\infty} A_n(x)/(\log n)^{1+\eta}$, $\eta > 0$, *is absolutely convergent.*

The result in Corollary 3 was first given by Cheng (1948) (see also Dikshit (1961)).

4.2 $\beta = 1/\delta$ in Corollary 1 may not be extended to cover the case $\delta = 1$, that is, $\eta > 0$ of Corollary 2 may not be replaced by $\eta = 0$. For, consider

$$\phi(t) = f(t) = \begin{cases} 0, & \text{when } t \in [-\pi/2, \pi/2] \\ \dfrac{\pi}{2}, & \text{when } |t| \in (\pi/2, \pi], \end{cases}$$

so that

$$f(t) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nt}{(2n-1)}.$$

Here $t^{-\delta}\phi(t) \in BV(0, \pi)$, for each $\delta \in (-\infty, \infty)$, but

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)} \in |R, \exp\{w/\log w\}, 1|,$$

for otherwise, by virtue of a summability factor theorem (Dikshit (1958)), we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)\log(2n+1)} \notin |R, \exp(w), 1|,$$

i.e. the series is absolutely convergent.

This example also shows that $\eta > 0$ of Corollary 3 may not be dropped even if '$\phi(t) \in BV(0, \pi)$' is replaced by

$$`t^{-\delta}\phi(t) \in BV(0, \pi), \quad \delta > 0'.$$

References

M. T. Cheng (1948), 'Summability factors of Fourier series at a given point', *Duke Math. Jour.* **15**, 29–36.

G. D. Dikshit (1958), 'On the absolute Riesz summability factors of infinite series I', *Indian J. Math.* **1**, 33–40.

G. D. Dikshit (1960), 'A note on absolute Riesz summability of infinite series', *Proc. Nat. Inst. Sc. India* **26**A, 541–544.

G. D. Dikshit (1961), 'A summability factor theorem on the absolute Riesz summability of Fourier series', *Indian J. Math.* **3**, 7–26.

G. D. Dikshit (1965), 'Localization relating to the summability $/R, \lambda_n, 1/$ of Fourier series', *Indian J. Math.* **7**, 31–39.

G. D. Dikshit (1965), 'On inclusion relation between Riesz and Nörlund means', *Indian J. Math.* **7**, 73–81.

R. Mohanty (1951), 'On the absolute Riesz summability of Fourier series and allied series', *Proc. London Math. Soc.* (2) **52**, 295–320.

N. Obrechkoff (1928), 'Sur la sommation absolue des séries de Dirichlet', *Comptes Rendus* **186**, 215–217.

N. Obrechkoff (1929), 'Über die absolute Summierung der Dirichletchen Reihen', *Math. Z.* **30**, 375–386.

J. B. Tatchell (1953), 'On some integral transformations', *Proc. London Math. Soc.* (3) **3**, 257–266.

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