GREEN POTENTIAL OF EVANS TYPE ON ROYDEN'S COMPACTIFICATION OF A RIEMANN SURFACE

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Introduction

1. Let R be a hyperbolic Riemann surface and $g^w(z)$ be the Green function on R with its pole w in R. We denote by $\mathscr{F}(R)$ the totality of sequences $(z_n)_{n=1}^{\infty}$ of points in R not accumulating in R and

$$\liminf_{n\to\infty} g^w(z_n) > 0.$$

Clearly the family $\mathscr{F}(R)$ is independent of the special choice of the pole w and so $\mathscr{F}(R)$ is determined completely by the structure of R. We say that R is *regular* (resp. *irregular*) if $\mathscr{F}(R) = \emptyset$ (resp. $\mathscr{F}(R) \neq \emptyset$). It is well recognized that for many problems, regular hyperbolic Riemann surfaces are more manageable than irregular ones. Hence it is important to provide tools to eliminate the irregularity in some sense. The main pourpose of this paper is to show the following[1)]

THEOREM 1. *On any irregular hyperbolic Riemann surface R, there exists a positive harmonic function $u(z)$ satisfying the following three properties:*

(1) *$u(z)$ is an Evans function on R, i.e.* $\lim_{n\to\infty} u(z_n) = \infty$ *for any sequence $(z_n)_{n=1}^{\infty}$ belonging to the class $\mathscr{F}(R)$;*

(2) *$u(z)$ is quasi Dirichlet finite of the first order, i.e. there exists a finite positive constant K such that $D_R(\min(u(z), c)) \leq Kc$ for any poitive number c;*

(3) *$u(z)$ is singular, i.e. the greatest harmonic minorant of* $\min(u(z), c)$ *on R is identically zero for any positive number c.*

Here we make a remark to the property (1). Let $(R_n)_{n=1}^{\infty}$ be a normal exhaustion of R and set $R_0 = \emptyset$. For a positive number a and a point w in R and a non-negative integer n, we set

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1) We published an outline of a part of this paper in [9]. We also published some results closely related to this paper in [8] and [10].

$$V(w, a, n) = (z \in R;\ g^w(z) > a) - \overline{R}_n \qquad (n = 0, 1, 2, \ldots).$$

Then the property (1) is equivalent to the following property:

(1′) $\lim_{n\to\infty} \inf(u(z);\ z \in V(w, a, n)) = \infty$ *for any* (w, a) *such that* $V(w, a, 0)$ *is not a compact set in* R.

In fact, let $(z_n)_{n=1}^\infty$ be in $\mathscr{F}(R)$. Then there exists a positive integer $n(m)$ such that $(z_n)_{n=n(m)}^\infty$ is contained in $V(w, 2^{-1} \liminf_{k\to\infty} g^w(z_k), m)$ for any positive integer m. From this, it follows that (1′) implies (1). Convrsely, assume that (1) holds. If (1′) is not true for some (w, a) with non-compact $V(w, a, 0)$, then there exists a sequence $(z_n)_{n=1}^\infty$ of points in R such that there exists an increasing sequence $(k_n)_{n=1}^\infty$ of positive integers with $z_n \in V(w, a, k_n)$ and $(u(z_n))_{n=1}^\infty$ is bounded. Clearly $(z_n)_{n=1}^\infty$ belongs to $\mathscr{F}(R)$ and so the boundedness of $(u(z_n))_{n=1}^\infty$ contradicts (1).

2. Let R' be a parabolic Riemann surface and $(R_n')_{n=0}^\infty$ be a normal exhaustion of R' such that R_0' is a disc $(z;\ |z| < 1)$. Applying Theorem 1 to the hyperbolic Riemann surface $R = R' - R_0'$ which is clearly irregular, we get a positive harmonic function $u(z)$ on R satisfying (1). Let $g^w(z)$ be Green's function on $R = R' - \overline{R}_0'$ with its pole w in R. Then, since R' is parabolic, $\inf(g^w(z);\ z \in R' - R_1') > 0$. Hence from (1′), it follows that

$$\text{lin}_{n\to\infty} \inf(u(z);\ z \in R' - R_n') = \infty.$$

This is equivalent to that $u = \infty$ continuously at the Alexandroff point of R'. Modifying u, it may be assumed that

(3′) $\qquad u(z) = 0$ continuously at each point of $\partial R_0'$,

From this (3) follows, since R' is parabolic. We must remark that in the present case, without assuming (2), properties (1) and (3′) implies the more precise properties than (2). In fact, for any $c > 0$, since $(z \in R;\ u(z) < c)$ is closure compact in R' and $\int_{\partial R_0'} {}^*du = \int_{u=c} {}^*du$, we get

$$D_R(\min(u, c)) = \int_{\partial R_0' \cup (u=c)} u\,{}^*du = \int_{u=c} c\,{}^*du = \left(\int_{\partial R_0'} {}^*du\right)c.$$

Now by multiplying a suitable positive constant, we may assume $\int_{\partial R_0'} {}^*du = 2\pi$. Let

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$$s(z) = \begin{cases} u(z) & \text{on } R' - R'_0; \\ -g_0^w(z) & \text{on } R'_0, \end{cases}$$

where $g_0^w(z)$ is the Green function in R'_0 with its pole w in R'_0. Then since

$$\int_{\partial R_0 + \partial(R - R_0)} {}^*ds = 0,$$

the equation $L(h-s) = h-s$, where L is a normal linear operator of Sario [13], has a solution on R' which is harmonic in $R-(w)$ and has the same singularity as $s(z)$ at w and the Alexandroff point of R'. Hence $h(z)$ has the negative logarithmic pole at w and $h = \infty$ continuously at the Alexandorff point of R'. This $h(z)$ is the so called *Evans-Selberg's potential* on R'. Hence *Theorem 1 contains generalized Evans-Selberg's theorem* (see Evans [2], Selberg [14], Noshiro [12], Kuramochi [3], Nakai [8]).

3. Theorem 1 is a consequence of a more precise facts as mentioned below. Let R be a hyperbolic Riemann surface and R^* be its Royden's compactification. We denote by $\Gamma = R^* - R$, which is called Royden's boundary of R. We denote by Δ the totality of regular points in Γ with respect to the Dirichlet problem, which is called (Royden's) harmonic boundary of R and $\Delta \neq \emptyset$ if and only if R is hyperbolic. It will be seen that the Green function $g^w(z)$ on R can be extended to the Green kernel $g(p, q)$ on R^* such that $g(z, p)$ is finitely continuous in (z, p) of $R \times \Gamma$ and $g(z, w) = g^w(z)$ and as the function of z, $g(z, p)$ $(p \in R^*)$ is a non-negative singular harmonic function on $R-(p)$ and continuous on R^*. We set

$$\Gamma_0 = (p \in \Gamma;\ g(z, p) > 0 \text{ on } R),$$

which is an F_σ-set in Γ and $\Gamma_0 \neq \emptyset$ if and only if R is irregular.

THEOREM 2. *Assume that R is an irregular hyperbolic Riemann surface. Then there exists a unit positive regular Borel measure μ on R^* satisfying the following six properties:*

(4) *there exists a sequence $(q_n)_{n=1}^{\infty}$ of points in Γ_0 such that $\mu(R^* - (q_n)_{n=1}^{\infty}) = 0$, i.e. $\mu = \sum_{i=1}^{\infty} t_i\, \varepsilon_{q_i}$, where $t_i > 0$ and $\sum_{i=1}^{\infty} t_i = 1$ and ε_{q_i} is a unit point measure at q_i;*

(5) *$g_\mu(z) = \int g(z, q)\, d\mu(q)$ is a positive harmonic function on R;*

(6) *$D_R(\min(g_\mu(z), c)) \leq 2\pi c$ for any positive number c;*

(7) $g_\mu(z)$ *is continuous on* R^*,

(8) $g_\mu(p)=\infty$ on Γ_0;

(9) $g_\mu(p)=0$ *on* Δ.

4. In Chapter 1, we explain Royden's compactification and some of its fundamental properties. In Chapter 2, we define Green kernel on Royden's compactification and discuss its fundametal properties. In Chapter 3, we treat transfinite diameters and modified Tychebycheff's constants for subsets of R^* with respect to Green kernel. In Chapter 4, we complete the proofs of Theorems 1 and 2.

5. Here we explain some notations and terminologies used in this paper. Functions (resp. continuous functions) on a space (resp. a topological space) considered in this paper are assumed to be mappings (resp. continuous mappings) of the space into the completed real line $[-\infty, \infty]$. For two numbers or functions a and b, we denote

$$a \cap b = \min(a, b) \text{ and } a \cup b = \max(a, b).$$

Let R be an open Riemann surface and R^* be its Royden's compactification (see Section 1.1). For a subset A in R^*, we denote by $\mathring{A}$ (resp. $\overline{A}$) the totality of inner point of A (resp. the closure of A) considered in R^*. For a set A in R^*, we denote by ∂A the boundary of A relative to R (and not to R^*), i.e. $\partial A=(\overline{A}-A)\cap R$. Hence $\partial(A\cap R)=\partial A$. A normal exhaustion $(R_n)_{n=0}^{\infty}$ of R is a sequence of closure compact subdomains R_n of R such that

$$R_{n+1}\supset \overline{R}_n \text{ and } R=\bigcup_{n=1}^{\infty} R_n$$

and $R-\overline{R}_n$ $(n=0, 1, 2, \ldots)$ have no component which is closure compact in R and $R-\overline{R}_0$ is connected and each ∂R_n consists of a finite number of mutually disjoint analytic closed Jordan curves. If ∂R_n consists of a finite number of mutually disjont piece-wise analytic closed Jordan curves, then we say that $(R_n)_{n=1}^{\infty}$ is a normal exhaustion of R with piece-wise analytic boundary. Finally, for two a.c.T functions f and g on an open set G in R (see Section 1.1), we set Dirichlet inner product and Dirichlet integral by

$$D_G(f, g)=\iint_G df\wedge {}^*dg \text{ and } D_G(f)=\iint_G df\wedge {}^*df=\iint_G |\operatorname{grad} f(z)|^2 dxdy,$$

respectively, where $z=x+iy$ is a local parameter on R. If there is no afraid

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of confusion, we simply write $D(f, g)$ and $D(f)$ instead of $D_G(f, g)$ and $D_G(f)$ respectively.

1. Royden's compactification and some fundamental properties

1.1. A real vlaued function $F(x, y)$ defined in $(a, b)\times(c, d)$ is said to be *absolutely continuous in the sense of Tonelli* (abreviated as a.c.T) if F is continuous and if for any fixed y in (c, d) except a set of measure zero the function $x\to F(x, y)$ is absolutely continuous in the usual sense and the same is true if x and y are interchanged and, further, $\frac{\partial}{\partial x}F(x, y)$ and $\frac{\partial}{\partial y}F(x, y)$ are integrable in any compact subset of $(a, b)\times(c, d)$. Since this notion is conformally invariant, this notion can be easily carried over Riemann surfaces using local parameters.

A (real) *Royden's algebra* $M(R)$ associated with a Riemann surface R is the totality of real valued bounded continuous a.c.T functions f on R with finite Dirichlet integrals $D_R(f)$. This $M(R)$ is a Banach algebra with the usual algebraic operations and the norm $\|f\|=\sup_R|f|+\sqrt{D_R(f)}$ ([5]).

A sequence $(f_n)_{n=1}^\infty$ of functions on R is said to converge to a function f on R in C- (or D-) topology if f_n converges to f uniformly on each compact subset of R (or $D_R(f_n-f)\to 0$), in notation $f=C\text{-}\lim_n f_n$ (or $f=D\text{-}\lim_n f_n$). We also say that $(f_n)_{n=1}^\infty$ converges to f in B- (or BD-) topology if $(f_n)_{n=1}^\infty$ is bounded and converges to f in C-topology (or $(f_n)_{n=1}^\infty$ converges to f in B- and D-topology), in notation $f=B\text{-}\lim_n f_n$ (or $f=BD\text{-}\lim_n f_n$). We remark that $M(R)$ is BD-complete ([5]).

We denote by $M_0(R)$ the totality of functions in $M(R)$ with compact support in R. We also denote by $M_\Delta(R)$ the BD-closure of $M_0(R)$ in $M(R)$. Clearly $M_0(R)$ and $M_\Delta(R)$ are ideals of $M(R)$. We also remark that $M_\Delta(R)$ is BD-complete.

The *Royden's compactification* R^* of R is a unique compact Hausdorff space containing R as its open and dense subset and each function can be continuously extended to R^* and $M(R)$ separates points in R^*. The set $\Gamma=R^*-R$ is called *Royden's boundary* of R ([5], [1]). A part Δ of R^* defined by

$$\Delta=(p\in R^*;\ f(p)=0 \text{ for any } f\in M_\Delta(R))$$

is called the harmonic boundary of R. This set Δ is a compact subset of Γ and $\Delta\neq\emptyset$ if and only if R is hyperbolic ([6]). This set Δ is the totality of regular points in Γ with respect to Dirichlet problem ([7]). It holds the following duality ([6])

$$M_\Delta(R) = (f \in M(R);\ f \text{ vanishes on } \Delta).$$

1.2. For the application of Royden's compactification, the following four facts play the fundamental role. From the definition of a.c.T function, we have the following ([6], p. 69 in [1]):

LEMMA 1.2.1. *The algebra $M(R)$ forms a vector lattice with lattice operations $f \cup g$ and $f \cap g$.*

Since $M(R)$ is a subalgebra of total algebra $B(R^*)$ of bounded continuous functions on R^* with $M(R) \ni 1$ and $M(R)$ separates points in R^*, by Weierstrass-Stone's approximation theorem, we get:

LEMMA 1.2.2. *The algebra $M(R)$ is dense in $B(R^*)$ with respect to the uniform convergence topology on R^*.*

LEMMA 1.2.3 (Kusunoki-Mori [5]). *Let U be a subdomain of R sunch that $\overline{U} \cap \Delta = \emptyset$ and ∂U consists of at most countable number of mutually disjoint analytic Jordan curves not ending and not accumulating in R. Then the double $\hat{U}$ of U along ∂U is a compact or parabolic Riemann surface.*

LEMMA 1.2.4 (Nakai [6]). *Suppose that $u(z)$ is a harmonic function bounded from below (or above) defined on a subdomain U of R such that ∂U (which may be empty) consists of at most a countable number of Jordan curves not accumulating in R. If u satisfies*

$$\liminf_{U \ni z \to \zeta} u(z) \geq m \quad (\text{or } \limsup_{U \ni z \to \zeta} u(z) \leq M)$$

at any point ζ in $\partial U \cup (\overline{U} \cap \Delta)$, then $u \geq m$ (or $u \leq M$) on U.

1.3. Since R is dense in R^*, we may say that a function f defined on R is continuous on R^* if f is continuously extended to R^*. We say that a non-negative function f on R is *quasi Dirichlet finite* on R if $f \cap c$ is a.c.T and $D_R(f \cap c) < \infty$ for any positive number c. It is easy to see that $\alpha f_1 + \beta f_2$ $(\alpha, \beta > 0)$ is quasi Dirichlet finite on R along with f_1 and f_2.

LEMMA 1.3.1. *If f is quasi Dirichlet finite continuous function on R, then f is continuous on R^*.*

Proof. For any $n = 1, 2, \ldots, f \cap n$ can be continuously extended to R^*. We denote by f_n the extended function. Let $h(p) = \lim_{n \to \infty} f_n(p)$, whose existence is clear, since $(f_n)_{n=1}^{\infty}$ is non-decreasing on R^*. Assume that $h(p_0) < \infty$ $(p_0 \in R^*)$. Then there exists an n such that $h(p_0) < n$. Since f_n is continuous at p_0, there exists a neighborhood U of p_0 in R^* such that $f_n(p) < n$ $(p \in U)$. If $m > n$, then $(f \cap m) \cap n = f \cap n$ on R. Hence $f_m \cap n = f_n$ on R^*. Thus $f_m(p) \cap n = f_n(p) < n$ $(p \in U)$ implies that $f_m(p) = f_n(p)$ in U and so by making $m \nearrow \infty$, we get $h(p) = f_n(p)$ in U. Thus h is continuous at p_0. Next suppose that $h(p_0) = \infty$. Then for any $c > 0$, there exists an f_n such that $f_n(p_0) > c$. Since f_n is continuous on R^*, there exists a neighborhood V of p_0 such that $f_n(p) > c$ on V. As $h \geq f_n$ on R^*, so $h(p) > c$ $(p \in V)$. This shows that h is continuous at p_0. Hence h is continuous on R^*. Clearly $h(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} f(z) \cap n = f(z)$ on R. Thus h is a continuous extention of f. In other words, f is continuous on R^*.

1.4. We prove three more lemmas which plays an important role in our paper.

LEMMA 1.4.1. *Let $(\varphi_n)_{n=1}$ be a sequence in $M_\Delta(R)$ such that $\varphi = B\text{-}\lim_n \varphi_n$ on R and φ is a.c.T on R and $\lim_n D_K(\varphi_n - \varphi) = 0$ for each compact subset K of R and $D_R(\varphi_n) \leq A < \infty$ $(n = 1, 2, \ldots)$. Then φ belongs to $M_\Delta(R)$ and $D_R(\varphi, f) = \lim_n D(\varphi_n, f)$ for each f in $M(R)$*[2].

Proof. Let $(R_n)_{n=0}^{\infty}$ be a normal exhaustion of R. We take a continuous function ϕ_n on R such that

$$\phi_n = \begin{cases} \varphi_n & \text{on } R_0; \\ \text{harmonic} & \text{in } R_1 - R_0; \\ 0 & \text{on } R - R_1, \end{cases}$$

$n = 1, 2, \ldots, \infty$, where we set $\varphi_\infty = \varphi$. Then clearly $\phi_n \in M_0(R) \subset M_\Delta(R)$. Let $\varphi'_n = \varphi_n - \phi_n$ and $\varphi' = \varphi - \phi_\infty$. Then φ'_n and φ' satisfy the assumption of Lemma 1.4.1. If the conclusion of Lemma 1.4.1 is valid for φ'_n and φ', then the same is true for φ_n and φ. Hence to prove our lemma, we may assume without loss of generality that $\varphi = \varphi_n = 0$ on R_0 $(n = 1, 2, \ldots)$.

Let $\Gamma^2(R)$ be the real Hilbert space of all real first order measurable differentials α such that $\|\alpha\|^2 = \iint_R \alpha \wedge {}^*\alpha < \infty$. We denote $(\alpha, \beta) = \iint_R \alpha \wedge {}^*\beta$. Notice

[2] Mr. M. Kawamura proved that this Lemma is true without assuming that φ is a.c. T and $\lim_n D_K(\varphi_n - \varphi) = 0$.

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that $df \in \Gamma^2(R)$ if $f \in M(R)$. Then $(d\varphi_n)_{n=1}^{\infty} \subset \Gamma^2(R)$ and $\|d\varphi_n\|^2 \leq A$ $(n=1, 2, \ldots)$. Hence by the weak compactness of bounded set in the Hilbert space, any subsequence of $(d\varphi_n)_{n=1}^{\infty}$ possesses a waekly convergent subsequence. We show that the weak limit of any weakly convergent subsequence is $d\varphi$. If this can be shown, then we can conclude that $(d\varphi_n)_{n=1}^{\infty}$ itself converges weakly to $d\varphi$.

Assume that $(d\varphi_{n_k})_{k=1}^{\infty}$ is wearkly convergent. Then $\beta \rightarrow \lim_k(\beta, d\varphi_{n_k})$ is a bounded linear functional on $\Gamma^2(R)$. By the reflexivity of the Hilbert space, there exist an α in $\Gamma^2(R)$ such that for any $\beta \in \Gamma^2(R)$, $(\beta, \alpha) = \lim_k(\beta, d\varphi_{n_k})$. Assume that $\beta = 0$ outside R_m. Then

$$|(\beta, d\varphi - d\varphi_{n_k})| \leq \|\beta\| \cdot \|d\varphi - d\varphi_{n_k}\|_{R_m} = \|\beta\| \sqrt{D_{R_m}(\varphi - \varphi_{n_k})} \rightarrow 0$$

as $k \nearrow \infty$. Thus $(\beta, d\varphi) = \lim_k(\beta, d\varphi_{n_k})$. Hence $(\beta, d\varphi - \alpha) = 0$ for any $\beta \in \Gamma^2(R)$ with $\beta = 0$ outside R_m. This shows that $\alpha = d\varphi$ on R_m. Since R_m is arbitrary, $\alpha = d\varphi$ on R and $\lim_k d\varphi_{n_k} = d\varphi$ (weakly).

Hence we obtain that $\lim_n d\varphi_n = d\varphi$ (weakly) and in particular, $\lim_n D_R(\varphi_n, f) = D_R(\varphi, f)$ for any f in $M(R)$ and $D_R(\varphi) < \infty$. As $\varphi = B\text{-}\lim \varphi_n$, so φ is continuous on R and so $\varphi \in M(R)$.

Finally we prove that $\varphi \in M_\Delta(R)$. Let u_m be continuous on R such that

$$u_m = \begin{cases} \varphi = 0 & \text{on } R_0; \\ \text{harmonic} & \text{in } R_m - R_0; \\ \varphi & \text{on } R - R_m. \end{cases}$$

As we have

$$D_R(u_{m+p} - u_m, u_{m+p}) = \int_{\partial R_0 \cup \partial R_{m+p}} (u_{m+p} - u_m)^* du_{m+p} = 0,$$

so $D_R(u_{m+p} - u_m) = D_R(u_m) - D_R(u_{m+p})$. Hence $(u_m)_{m=1}^{\infty}$ is D-convergent. Since $u_m = 0$ on R_0, $(u_m)_{m=1}^{\infty}$ converges to a function u in C-topology, where $u = 0$ on R_0 and harmonic in $R - R_0$. Moreover, since $|u_m| \leq \sup_R |\varphi|$, $u = B\text{-}\lim_m u_m$ on R. Hence

$$u = BD\text{-}\lim_m u_m \quad \text{on } R.$$

Let $\phi_m = \varphi - u_m$ and $\phi = \varphi - u$. Then

$$\phi = BD\text{-}\lim_m \phi_m$$

and as $\phi_m \in M_0(R)$, so $\phi \in M_\Delta(R)$. Since

$$D_R(u_m, \phi_m) = \int_{\partial R_0 \cup \partial R_m} \phi_m{}^* du_m = 0,$$

$D_R(u, \phi) = 0$. Next, since $\varphi_n \in M_\Delta(R)$, we can find $\varphi_{n,k} \in M_0(R)$ such that $\varphi_n = BD\text{-}\lim_k \varphi_{n,k}$. Suppose that the support of $\varphi_{n,k}$ is contained in $R_n(k)$. Then

$$|D(u, \varphi_{n,k})| = \left| \int_{\partial R_0 \cup \partial R_n(k)} \varphi_{n,k}{}^* du \right| \leq \sup_{\partial R_0} |\varphi_{n,k}| \cdot \int_{\partial R_0} {}^* du \to 0.$$

as $k \nearrow \infty$. Hence $D(u, \varphi_n) = 0$ $(n = 1, 2, \ldots)$. Thus $D(u, \varphi) = \lim_n D(u, \varphi_n) = 0$. Hence from

$$D(\varphi, u) = D(u, u) + D(\phi, u),$$

we conclude that $D(u) = 0$. As $u = 0$ on R_0, so $u \equiv 0$ on R and so $\varphi = \phi \in M_\Delta(R)$.

Q.E.D.

LEMMA 1.4.2 [Generalized Dirichlet principle]. *Let K be a compact set in R^* (which may be empty) such that $K \cap \Delta = \emptyset$ and $\overline{K \cap R} = K$ and ∂K consists of at most a countable number of piece-wise analytic curves not accumulating in R. Assume that $f \in M(R)$. Then there exists a unique u in $M(R)$ such that*

$$u = \begin{cases} f & \text{on } \Delta \cup K; \\ \text{harmonic} & \text{in } R - K \end{cases}$$

and $D_R(f) = D_R(u) + D_R(f - u)$ and

$$D_R(u) = \min(D_R(h)\,;\ h \in M(R) \text{ and } h = f \text{ on } \Delta \cup K).$$

Proof. Let $(R_n)_{n=1}^\infty$ be a normal exhaustion of R and u_n be continuous on R such that

$$u_n = \begin{cases} f & \text{on } R - (R_n - K)\,; \\ \text{harmonic} & \text{in } R_n - K. \end{cases}$$

Clearly $u_n \in M(R)$ and $|u_n| \leq \sup_R |f|$ on R $(n = 1, 2, \ldots)$. Hence by choosing a suitable subsequence, we may assume that $(u_n)_{n=1}^\infty$ is B-convergent on R. We set $u = B\text{-}\lim_n u_n$ on R. Then $u = f$ on $K \cap R$ and harmonic in $R - K$. Clearly $f - u_n \in M_0(R)$ and $f - u_n = 0$ on $R - (R_n - K)$. Hence

$$D_R(f - u_n, u_n) = \int_{\partial(R_n - K)} (f - u_n)^* du_n = 0$$

and so $D_R(f) = D_R(u_n) + D_R(f - u_n)$ and in particular, $D_R(u_n) \leq D_R(f)$. Similarly,

$D_R(u_n - u_{n+p}, u_{n+p}) = 0$ and so $D_R(u_n - u_{n+p}) = D_R(u_n) - D_R(u_{n+p})$. Thus $(u_n)_{n=1}^\infty$ is D-convergent and so $u = D\text{-}\lim_n u_n$ on R. Hence

$$u = BD\text{-}\lim_n u_n \quad \text{on } R.$$

Thus $u \in M(R)$ and u is continuous on R^*. From this, $u = f$ on $K \cap R$ implies $u = f$ on $K = \overline{K \cap R}$. Moreover, $f - u = BD\text{-}\lim_n (f - u_n)$ and $f - u_n \in M_0(R)$ implies that $f - u \in M_\Delta(R)$ or $u = f$ on Δ. From $D_R(f) = D_R(u_n) + D_R(f - u_n)$, we get $D_R(f) = D_R(u) + D_R(f - u)$ and $D_R(u) \leq D_R(f)$. The unicity of such a u follows from Lemma 1.2.4.

Next let $h \in M(R)$ with $h = f$ on $\Delta \cup K$. Construct $v \in M(R)$ for h such that $v = h$ on $\Delta \cup K$ and harmonic in $R - K$. Then $D_R(v) \leq D_R(h)$. As $v = h = f = u$ on $\Delta \cup K$, the unicity assures that $v = u$ on R. Thus $D_R(u) \leq D_R(h)$.

Q.E.D.

LEMMA 1.4.3. *Let K and K' be compact sets in R^* such that $\mathring{K}' \supset K$ and $K' \cap \Delta = \phi$ and $\overline{K \cap R} = K$ and $\overline{K' \cap R} = K'$ and relative boundaries ∂K and $\partial K'$ consist of at most a counsable number of disjoint analytic Jordan curves not ending and not accumulating in R. Then*

(1.4.1) *there exists a unique u in $M(R)$ such that $u = 1$ on K and $u = 0$ on Δ and u is harmonic in $R - K$;*

(1.4.2) $D_R(u) = \int_{\partial K} {}^*du$;

(1.4.3) *if* $\int_{\partial K'} |{}^*du| < \infty$, *then* $\int_{\partial K'} {}^*du = \int_{\partial K} {}^*du.$

Proof. Let $(R_n)_{n=0}^\infty$ be a normal exhaustion of R and F be a compact set in R^* such that $\mathring{F} \supset \Delta$ and $\overline{F \cap R} = F$ and $F \cap K = \emptyset$ and ∂F consists of at most a countable number of piece-wise analytic Jordan curves not ending and not accumulating in R. We put $F_k = F - R_k$. By Lemmas 1.2.1 and 1.2.2, we can find a function f in $M(R)$ such that $f = 0$ on F and $f = 1$ on K. By Lemma 1.4.2, there exists a function u_k in $M(R)$ such that $u_k = f$ on $K \cup \overline{\partial F_k} \cup \Delta$ and u is harmonic in $R - K \cup \partial F_k$. By Lemma 1.2.4, $u_k = 0$ on $\mathring{F}_k \cap R$. Hence

$$u_k = \begin{cases} 1 & \text{on } K \\ \text{harmonic} & \text{in } R - K \cup F_k; \\ 0 & \text{on } F_k \end{cases}$$

and by Lemma 1.4.2,

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$$D_R(u_k) \leq D(f).$$

Next let $u_{k,n}$ be continuous on $\overline{R_n - K - F_k}$ $(n > k)$ and harmonic in $R_n - K - F_k$ such that $u_{k,n} = 1$ on $\partial K \cap R_n$ and $u_{k,n} = 0$ on $\partial F_k \cap R_n$ and $\frac{\partial}{\partial \nu} u_{k,n} = 0$ on $\partial R_n - K - F_k$. We set $u_{k,n} = 0$ on F_k and $u_{k,n} = 1$ on K. Then, since

$$D_{R_n}(u_{k,n+p} - u_{k,n}, u_{k,n}) = \int_{\partial(R_n - K - F_k)} (u_{k,n+p} - u_{k,n})^* du_{k,n} = 0,$$

we get

$$D_{R_n}(u_{k,n+p} - u_{k,n}) = D_{R_n}(u_{k,n+p}) - D_{R_n}(u_{k,n}) \leq D_{R_{n+p}}(u_{k,n+p}) - D_{R_n}(u_{k,n}).$$

On the other hand,

$$D_{R_n}(u_k - u_{k,n}, u_{k,n}) = \int_{\partial(R_n - K - F_k)} (u_k - u_{k,n})^* du_{k,n} = 0$$

implies that $D_{R_n}(u_{k,n}) = D_{R_n}(u_k, u_{k,n}) \leq \sqrt{D_{R_n}(u_k)} \cdot \sqrt{D_{R_n}(u_{k,n})}$, or

$$D_{R_n}(u_{k,n}) \leq D_{R_n}(u_k) \leq D_R(f).$$

As $(D_{R_n}(u_{k,n}))_{n>k}$ is non-decreasing and bounded, so

$$\lim_n D_{R_n}(u_{k,n+p} - u_{k,n}) = 0.$$

Since $(u_{k,n})_{n>k}$ is bounded and $u_{k,n}$ is harmonic in $R_n - K - F_k$ and 0 on ∂F_k and 1 on ∂K, $\lim_n u_{k,n}$ is a harmonic function in $R - K - F_k$ which equals 0 on ∂F_k and 1 on ∂K. Hence by Lemma 1.2.4, $u_k = \lim_n u_{k,n}$ in $R - K - F_k$ and so on R. Hence by Fatou's Lemma,

$$D_{R_n}(u_k - u_{k,n}) \leq \lim\inf_p D_{R_n}(u_{k,p} - u_{k,n})$$

and so

$$\lim_n D_{R_n}(u_k - u_{k,n}) \leq \lim_n(\lim\inf_p D_{R_n}(u_{k,p} - u_{k,n})) = 0$$

and also

$$\lim_n D_{R_n}(u_{k,n}) = D_R(u_k).$$

Similarly as before, since

$$D_{R_n}(u_{k+q,n} - u_{k,n}, u_{k+q,n}) = \int_{\partial(R_n - F_{k+q} - K)} (u_{k+q,n} - u_{k,n})^* du_{k+q,n} = 0 \quad (n > k+q),$$

we have

$$D_{R_m}(u_{k+q,n} - u_{k,n}) \leq D_{R_n}(u_{k+q,n} - u_{k,n}) = D_{R_n}(u_{k,n}) - D_{R_n}(u_{k+q,n}) \qquad (n > m)$$

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and so by making $n \nearrow \infty$ and then $m \nearrow \infty$, we get

$$D_R(u_{k+q} - u_k) \leq D_R(u_k) - D_R(u_{k+q}).$$

As $D_R(u_k) = \lim_n D_{R_n}(u_{k,n}) \leq D_R(f)$, so

$$\lim_k D_R(u_{k+q} - u_k) = 0.$$

Since $u_k = 1$ on K and $(u_k)_{k=1}^{\infty}$ is bounded and u_k is harmonic on $R - K - F_k$, $(u_k)_{k=1}^{\infty}$ converges to a function u on R in B-topology and u is harmonic on $R - K$ and $u = 1$ on K. Moreover

$$u = BD\text{-}\lim_k u_k$$

and $u_k \in M_\Delta(R)$ implies $u = 0$ on Δ. This u is the desired in (1.4.1) and the unicity follows from Lemma 1.2.4.

Next we prove (1.4.2). From (1.4.1), we can find a function e in $M_\Delta(R)$ such that

$$e = \begin{cases} 1 & \text{on } \overline{R - F_k}\,; \\ \text{harmonic} & \text{in } \mathring{F}_k\,; \\ 0 & \text{on } \Delta. \end{cases}$$

Clearly e is superharmonic on R. Let h_m be continuous on R such that

$$h_m = \begin{cases} e & \text{on } R - R_m\,; \\ \text{harmonic} & \text{in } R_m. \end{cases}$$

Then it is easily seen that $0 \leq h_{m+p} \leq h_m \leq e$ on R and since

$$D_R(h_m - h_{m+p}, h_{m+p}) = \int_{\partial R_{m+p}} (h_m - h_{m+p})^* dh_{m+p} = 0,$$

we get $D_R(h_m - h_{m+p}) = D_R(h_m) - D_R(h_{m+p})$. Hence if we put $h = \lim_m h_m$ on R, then $h = BD\text{-}\lim_m h_m$ on R and so $h \in HBD(R) \subset M(R)$ and as $0 \leq h \leq e$, so $h = 0$ on Δ. Hence by Lemma 1.2.4, $h \equiv 0$. Hence if we put

$$\varphi_m = e - h_m,$$

then $0 \leq \varphi_m \leq \varphi_{m+p} \leq e \leq 1$ on R and $e = BD\text{-}\lim_m \varphi_m$ and the support of φ_m is contained in $\overline{R}_m$. Now we have

$$\begin{aligned} D_R(u_k \varphi_m, u_k) &= \lim_n D_{R_n}(u_{k,n}\varphi_m, u_{k,n}) \\ &= \lim_n \int_{\partial(R_n - K - F_k)} u_{k,n} \varphi_m{}^* du_{k,n} \end{aligned}$$

$$= \lim_n \int_{\partial K \cap R_m} \varphi_m {}^* du_{k,n}.$$

As $u_{k,n} \to u_k$ $(n \to \infty)$ uniformly on R_m, so ${}^*du_{k,n} \to {}^*du_k$ on $\partial K \cap R_m$ uniformly and so

$$D(u_k \varphi_m, u_k) = \int_{\partial K \cap R_m} \varphi_m {}^* du_k.$$

We can easily show that

$$u_k = eu_k = BD\text{-}\lim_m \varphi_m u_k$$

on R. Since ${}^*du_k > 0$ on ∂K and $0 \leq \varphi_m \leq 1$, we get $\varphi_m {}^*du_k \leq {}^*du_k$ on ∂K. Hence $D(u_k \varphi_m, u_k) \leq \int_{\partial K} {}^*du_k$ and so

$$D(u_k) \leq \int_{\partial K} {}^*du_k.$$

On the other hand, by Fatou's lemma,

$$\int_{\partial K} {}^*du_k \leq \lim\inf_m \int_{\partial K \cap R_m} \varphi_m {}^*du_k = \lim_m D(u_k \varphi_m, u_k) = D(u_k).$$

This shows that $D(u_k) = \int_{\partial K} {}^*du_k$. By Lemma 1.2.4, $u_k \leq u_{k+p}$ on R and as $u_k = u_{k+p} = 1$ on ∂K, so ${}^*du_k \geq {}^*du_{k+p} \geq 0$ on ∂K. Hence ${}^*du_k \to {}^*du$ and by Lebesgue's convergence theorem,

$$\int_{\partial K} {}^*du = \lim_k \int_{\partial K} {}^*du_k = \lim_k D(u_k) = D(u),$$

which proves (1.4.2).

Finally we prove (1.4.3). *Let* $R \cap (\mathring{K}' - K) = \bigcup_n S_n$ be the decomposition into connected components. If we have $\int_{\bar{S}_n \cap \partial K} {}^*du = \int_{\bar{S}_n \cap \partial K'} {}^*du$, then since $\int_{\partial K} |{}^*du|$ *and* $\int_{\partial K'} |{}^*du|$ are finite, we get

$$\int_{\partial K} {}^*du = \sum_n \int_{\bar{S}_n \cap \partial K} {}^*du = \sum_n \int_{S_n \cap \partial K'} {}^*du = \int_{\partial K'} {}^*du.$$

Hence we may assume without loss of generality that $(\mathring{K}' - K) \cap R$ is a domain. Let T be the double of $(\mathring{K}' - K) \cap R$ with respect to ∂K and $\partial K'$ and $(T_n)_{n=1}^\infty$ be a normal exhaustion of T and T_0 is a disc in $(\mathring{K}' - K) \subset R$ such that $\bar{T}_0 \subset (\mathring{K}' - K) \cap R \cap T_1$. For convinience, we set $\gamma_n = \partial T_n \cap (K' - \mathring{K}) \cap R$. We take a continuous function v_n on T such that

$$v_n = \begin{cases} 1 & \text{on } T_0; \\ \text{harmonic} & \text{in } T_n - T_0; \\ 0 & \text{on } T - T_n. \end{cases}$$

Since $\overline{(\mathring{K}' - K) \cap R} \cap \Delta = \phi$, T is parabolic by Lemma 1.2.3 and so

$$BD\text{-}\lim_n v_n = 1$$

on T. Since ${}^*dv_n = 0$ on $\partial K \cup \partial K'$ and $\int_{\gamma_0} {}^*du = 0$, we get

$$D_{(\mathring{K}' \cap R - K)}(v_n, u) = \int_{(\partial K - \partial K') \cup \gamma_0 \cup \gamma_n} v_n {}^*du = \int_{\partial K - \partial K'} v_n {}^*du.$$

As $v_n {}^*du \to {}^*du$ and $|v_n {}^*du| \le {}^*du$ on $\partial K \cap \partial K'$ and $|{}^*du|$ is integrable on $\partial K \cup \partial K'$, so by Lebesgue's convergence theorem,

$$\int_{\partial K - \partial K'} {}^*du = \lim_n \int_{\partial K - \partial K'} v_n {}^*du = \lim_n D_{(\mathring{K}' - K) \cap R}(v_n, u) = D_{(\mathring{K} - K) \cap R}(1, u) = 0,$$

which shows $\int_{\partial K} {}^*du = \int_{\partial K'} {}^*du$, i.e. (1.4.3). This completes the proof.

II. Green kernel on Royden's compactification

2.1. Let R be a hyperbolic Riemann surface and $g^w(z)$ be the Green function on R with its pole w in R. Let $(R_n)_{n=0}^{\infty}$ be a normal exhaustion of R with w in R_0 and $g_n^w(z)$ be the Green function on R_n. We set $g_n^w(z) = 0$ for z in $R - R_n$. By definition, $g_n^w(z) \nearrow g^w(z)$ on R and $u_n(z) = g^w(z) - g_n^w(z)$ is bounded and $B\text{-}\lim_n u_n(z) = 0$ on R. Since u_n is harmonic in R_n and $u_n = g^w$ on $R - R_n$,

$$D_R(u_n - u_{n+p}, u_{n+p}) = \int_{\partial R_{n+p}} (u_n - u_{n+p}) {}^*du_{n+p} = 0.$$

Hence $D_R(u_n - u_{n+p}) = D_R(u_n) - D_R(u_{n+p})$. From these, we conclude that $BD\text{-}\lim_n u_n = 0$. In other words, $g_n^w(z) \nearrow g^w(z)$ on R and $D_R(g_n^w(z) - g^w(z)) \to 0$ as $n \nearrow \infty$. Hence

$$g^w(z) \cap c = BD\text{-}\lim_n g_n^w(z) \cap c$$

on R for any $c > 0$ (cf. p. 78, Satz 7.4 in [1]). As $g_n^w(z) \cap c \in M_0(R)$ and $D_R(g_n^w(z) \cap c) = 2\pi c$, so we get

$$g^w(z) \cap c \in M_\Delta(R) \qquad \text{and} \qquad D_R(g^w(z) \cap c) = 2\pi c$$

for any $c>0$. Hence $g^w(z)$ is quasi Dirichlet finite and so by Lemma 1.3.1, g^w is continuous on R^*. As $g^w(p) \cap c = 0$ on Δ for any $c>0$, so $g^w(p)=0$ on Δ. Since $g^w(z)=g^z(w)$ on $R\times R$, we may define $g^p(z)$ as $g^z(p)$ for p in Γ. We set

$$\Gamma_0 = (p\in\Gamma;\ g^w(p)>0)$$

for a fixed w in R. By Harnack's inequality, we can easily see that Γ_0 is independent of the special choice of w in R. Hence $\Gamma_0 = (p\in\Gamma;\ g^w(p)>0$ for any $w\in R) = (p\in\Gamma;\ g^p(z)>0$ on R as the function of z). Since $g^w(p)=0$ $(p\in\Delta)$, we see that

$$\Gamma_0 \subset \Gamma - \Delta.$$

Thus, if $p\in\Gamma-\Gamma_0$ (resp. $p\in\Gamma_0$), then $g^p(z)\equiv 0$ (resp. $g^p(z)>0$) on R.

LEMMA 2.1.1. *The function $g^w(p)=g^p(w)$ is continuous in (p, w) on $R^*\times R$ and finitely continuous in (p, w) on $\Gamma\times R$.*

Proof. As $g^w(z)=g^z(w)$ is continuous in (z, w) on $R\times R$, so we have only to show the finite continuity of $g^w(p)=g^p(w)$ at (p_0, w_0) in $\Gamma\times R$. Let ε be an arbitrary positive number. Since $g^{w_0}(p)=g^p(w_0)$ is continuous in p on R^*, we can find a neighborhood W of p_0 such that $w_0 \notin W$ and

$$|g^p(w_0)-g^{p_0}(w_0)|<\varepsilon$$

for any p in W. There exists a closure compact open neighborhood U of w_0 in R such that $R^*-\overline{U}\supset\overline{W}$ and $g^{w_0}(z)=g^z(w_0)\le N<\infty$ on $R-\overline{U}$. By Harnack's inequality, there exists a neighborhood V of w_0 such that $\overline{V}\subset U$ and

$$c^{-1}g^z(w_0)\le g^z(w)\le c g^z(w_0)$$

for any w in V and z in $R-U$, where $c=1+\varepsilon/N$. Hence

$$|g^z(w)-g^z(w_0)|<\varepsilon$$

for any w in V and z in $R-U$. Since $g^p(w)-g^p(w_0)$ is continuous in p on R^* for any fixed w in V and $R-U$ is dense in R^*-U, we have

$$|g^p(w)-g^p(w_0)|\le\varepsilon$$

for any w in V and p in R^*-U. Thus for any (p, w) in $W\times V$,

$$|g^w(p)-g^{w_0}(p_0)|\le|g^p(w)-g^p(w_0)|+|g^p(w_0)-g^{p_0}(w_0)|<2\varepsilon,$$

which shows that $g^w(p)=g^p(w)$ is continuous in (p, w) at (p_0, w_0). Q.E.D.

LEMMA 2.1.2. *Given a point p in Γ_0 (resp. $\Gamma-\Gamma_0$) and an arbitrary neighborhood U of p. Then there exists a decreasing sequence $(V_n)_{n=1}^{\infty}$ of neighborhoods of p such that $\bar{V}_n \subset U$ and $\cap_{n=1}^{\infty}(V_n \cap R)=\phi$ and* $\lim_n \sup_{w \in V_n \cap R}|g^p(z)-g^w(z)| =0$ *uniformly in z on each compact subset of R. Hence in particular, $g^p(z)$ is positive harmonic (resp. identically zero) on R.*

Proof. Take a countable dense subset $(z_m)_{m=1}^{\infty}$ of R. By induction, we can find sequences $(U_{m,n})_{n=1}^{\infty}$ $(m=1, 2, \ldots)$ of neighborhoods of p such that

$$U \supset U_{m,n} \supset U_{m,n+1},\ U_{m+1,n}$$

and $\cap_{n=1}^{\infty}(R \cap U_{1,n})=\phi$ and $\lim_n \sup_{w \in U_{m,n} \cap R}|g^{z_m}(w)-g^{z_m}(p)|=0$. This is possible, since $g^{z_m}(q)$ is continuous in q at p for each $m=1, 2, \ldots$. Set $V_n = U_{n,n}$. Then $\lim_n \sup_{w \in V_n \cap R}|g^z(w)-g^z(p)|=0$ for any $z=z_m$ $(m=1, 2, \ldots)$. Since $(z_m)_{m=1}^{\infty}$ is dense in R, by Lemma 2.1.2, $\lim_n \sup_{w \in V_n \cap R}|g^z(w)-g^z(p)| = 0$ holds for any z in R. As $g^z(w)=g^w(z)$ is harmonic in z on R except w, so $\lim_n \sup_{w \in V_n \cap R}|g^w(z)-g^p(z)|=0$ holds for z uniformly on each compact subset of R.

LEMMA 2.1.3. *For any fixed p in R^*, $D_R(g^p(z) \cap c) \leq 2\pi c$ $(c>0)$.*

Proof. This is clear if p belongs to $R^*-\Gamma_0$. Hence we have only to treat the case where p belongs to Γ_0. By Lemma 2.1.2, we can find a sequence $(w_n)_{n=1}^{\infty}$ of points in R which do not accumulate in R such that

$$g^p(z) = C\text{-}\lim_n g^{w_n}(z)$$

on R. Then

$$\lim_n |\operatorname{grad}(g^{w_n}(z) \cap c)|^2 = |\operatorname{grad}(g^p(z) \cap c)|^2$$

at each point of R except the set $(z \in R\ ;\ g^p(z)=c)$ for each fixed local parameter. Hence by Fatou's lemma and $D(g^{w_n}(z) \cap c)=2\pi c$, we get

$$D_R(g^p(z) \cap c) \leq \liminf_n D_R(g^{w_n}(z) \cap c) = 2\pi c \quad (c>0).$$

2.2. From Lemma 2.1.3, it follows that $g^p(z)$ is quasi Dirichlet finite on R for any fixed p in R^* and so $g^p(z)$ is continuous on R^* by Lemma 1.3.1. Hence we can give the following

DEFINITION. *The Green kernel $g(p, q)$ on R^* is given as the function of (p, q) in $R^* \times R^*$ by the following double limit:*

$$g(p, q) = \lim_{R \ni z \to p}(\lim_{R \ni w \to q} g^w(z)).$$

PROPOSITION 1. *The Green kernel $g(p, q)$ on R^* possesses the following properties:*

(g.1) $g(z, w) = g^w(z)$ *for* (z, w) *in* $R \times R$;

(g.2) $g(z, p) = g(p, z)$ *for* z *in* R *and* p *in* R^*;

(g.3) $g(p, q)$ *is continuous in* p *on* R^* *for fixed* q *in* R^*;

(g.4) $g(z, p)$ *is continuous in* (z, p) *on* $R \times R^*$ *and finitely continuous in* (z, p) *on* $R \times \Gamma$;

(g.5) $g(z, p)$ *is harmonic in* z *on* $R - (p)$ *for fixed* p *in* R^*;

(g.6) $g(z, p) > 0$ (*resp.* $\equiv 0$) *on* R *if* $p \in R \cup \Gamma_0$ (*resp.* $\Gamma - \Gamma_0$);

(g.7) $D_R(g(z, p) \cap c) \leq 2\pi c$ *for any fixed* p *in* R^* *and* $c > 0$;

(g.8) *if* q *is fixed in* $\Gamma_0 \cup R$, *then* $g(p, q) > 0$ *for any* p *in* $\Gamma_0 \cup R$;

(g.9) *if* q *is fixed in* R^*, *then* $g(p, q) = 0$ *for* p *in* Δ.

Proof. The properties (g.1)-(g.7) are easy consequences of the definition of $g(p, q)$ and Lemmas 2.1.1, 2, 3. To prove (g.8), we have only to treat the case where $q \in \Gamma_0$ and $p \in \Gamma_0$. We take a normal exhaustion $(R_n)_{n=0}^{\infty}$ of R and we fix a point w in R_0. Since $g(z, q)$ $(q \in \Gamma_0)$ is a positive harmonic function in z on R by (g.5) and (g.6), we can find a positive number a such that

$$ag(z, q) \geq g^w(z)$$

on ∂R_0. As $ag(z, q) \geq g_n^w(z)$ on $R_n - R_0$, so $ag(z, q) \geq g^w(z)$ on $R - R_0$. By letting $z \to p \in \Gamma_0$, we have

$$ag(p, q) \geq g^w(p) > 0.$$

Finally, we prove (g.9). To avoid the trivial case, we assume that $q \in \Gamma_0$. By Lemma 2.1.2, we can find a sequence $(w_n)_{n=1}^{\infty}$ of points in R which do not accumulate in R such that $g(z, w_n)$ converges to $g(z, q)$ in C-topology on R. Hence for any $c > 0$,

$$g(z, q) \cap c = B\text{-}\lim_n g(z, w_n) \cap c$$

and since $g(z, q)$ and $g(z, w_n)$ are harmoinc on any compact subset K of R for sufficiently large n, we get (cf. p. 73, Satz 7.4 in [1])

$$\lim_n D_K(g(z, q) \cap c - g(z, w_n) \cap c) = 0$$

and by (g.7),

$$D(g(z, w_n) \cap c) \leq 2\pi c \qquad (n = 1, 2, \ldots).$$

As $g(z, w_n)\cap c\in M_\Delta(R)$, so by Lemma 1.4.1, $g(z, q)\cap c\in M_\Delta(R)$, or $g(p, q)\cap c = 0$ in p on Δ. Thus $g(p, q)=0$ for p in Δ. Q.E.D.

Remark. Notice that we do not claim the general symmetricity of g: $g(p, q)=g(q, p)$. We also do not claim the continuity of $g(p, q)$ in q at Γ_0 for fixed p in Γ_0.

It follow from (g.5) and (g.9) that $g(z, p)$ $(p\in\Gamma_0)$ is a singular positive harmonic function, i.e. the greates harmonic minorant of $g(z, p)\cap c$ is identically zero. In fact, let $u(z)$ be harmonic on R with $0\le u(z)\le g(z, p)\cap c$ on R. Then by (g.9), $0\le \limsup_{R\ni z\to q} u(z)\le \lim_{R\ni z\to q} g(z, p)\cap c = g(q, p)\cap c = 0$ for any q in Δ. Hence by Lemma 1.2.4, $u(z)\equiv 0$ on R.

III. Quantities concerning Green kernel

3.1. Let Ω be an arbitrary non-empty set and K be a mapping of $\Omega\times\Omega$ into $[c, \infty]$ $(c>-\infty)$. For each set X in Ω, we set

$$\binom{n}{2}\tau_n(X;\ \Omega, K)=\inf_{p_1,\ \dots,\ p_n\in X}\sum_{i<j}^{1,\dots,n} K(p_i, p_j)$$

when $X\neq\emptyset$ and $\tau_n(X;\ \Omega, K)=\infty$ when $X=\emptyset$. Let $p_1, \dots, p_{n+1}$ be arbitrary points in X. Then

$$\sum_{i<j}^{1,\dots,n+1} K(p_i, p_j)=\sum_{i=1}^{k-1}K(p_i, p_k)+\sum_{j=k+1}^{n+1}K(p_k, p_j)+\sum_{i<j;\ i,\ j\neq k}^{1,\dots,n+1}K(p_i, p_j)$$

and so

$$\sum_{i<j}^{1,\dots,n+1} K(p_i, p_j)\ge\sum_{i=1}^{k-1}K(p_i, p_k)+\sum_{j=k+1}^{n+1}K(p_k, p_j)+\binom{n}{2}\tau_n(X;\ \Omega, K)$$
$$(k=1, 2, \dots, n+1).$$

Summing up these $n+1$ inequality, we get

$$(n+1)\sum_{i<j}^{1,\dots,n+1}K(p_i, p_j)\ge 2\sum_{i<j}^{1,\dots,n+1}K(p_i, p_j)+(n+1)\binom{n}{2}\tau_n(X;\ \Omega, K)$$

or

$$(n-1)\sum_{i<j}^{1,\dots,n+1}K(p_i, p_j)\ge(n+1)\binom{n}{2}\tau_n(X;\ \Omega, K).$$

Hence we get $(n-1)\binom{n+1}{2}\tau_{n+1}(X;\ \Omega, K)\ge(n+1)\binom{n}{2}\tau_n(X;\ \Omega, K)$, or

$$\tau_{n+1}(X;\ \Omega, K)\ge\tau_n(X;\ \Omega, K)\qquad(n=1, 2, \dots).$$

Hence we can define

$$\tau(X;\ \Omega, K) = \lim_{n\to\infty} \tau_n(X;\ \Omega, K).$$

This is called the *transfinite diameter of* X *with respect to* (Ω, K). Similarly we set

$$n\rho_n(X;\ \Omega, K) = \sup_{p_1,\dots,p_n\in X}(\inf_{p\in X}\sum_{i=1}^{n} K(p, p_i))$$

when $X \neq \emptyset$ aud $\rho_n(X;\ \Omega, K) = \infty$ when $X = \emptyset$. Let $p_1, \dots, p_{n+m}$ be arbitrary points in X. Then

$$\sum_{i=1}^{n+m} K(p, p_i) = \sum_{i=1}^{n} K(p, p_i) + \sum_{i=n+1}^{m} K(p, p_i)$$

and so

$$\inf_{p\in X}\sum_{i=1}^{n} K(p, p_i) \geq \inf_{p\in X}\sum_{i=1}^{n} K(p, p_i) + \inf_{p\in X}\sum_{i=n+1}^{m} K(p, p_i).$$

Hence

$$(n+m)\rho_{n+m}(X;\ \Omega, K) \geq n\rho_n(X;\ \Omega, K) + m\rho_m(X;\ \Omega, K).$$

It is well known that for a sequence $(a_n)_{n=1}^{\infty}$ of points in $(-\infty, \infty)$ such that $a_{n+m} \geq a_n + a_m$, $\lim_n n^{-1}a_n$ exists. Hence we can define

$$\rho(X;\ \Omega, K) = \lim_{n\to\infty}\rho_n(X;\ \Omega, K).$$

This is called the *modified Tchebycheff's constant of* X *with respect to* (Ω, K). Concerning these two quautities τ and ρ, we have

PROPOSITION 2. $\rho(X;\ \Omega, K) \geq \tau(X;\ \Omega, K)$.

Proof. Let $n > 1$ be an arbitrary positive integer. We set $r = (n-1)^{-1}$ and choose n points $p_n, p_{n-1}, \dots, p_2, p_1$ in X satisfying

$$\sum_{j=n-i+1}^{n} K(p_{n-i}, p_j) \leq \inf_{p\in K} \sum_{j=n-i+1}^{n} K(p, p_j) + r \quad (i = 1, 2, \dots, n-1). \tag{3.1.1}$$

We choose these n points inductively as follows. Let p_n be an arbitrary point in X. Assume that $p_n, p_{n-1}, \dots, p_{n-i+1}$ $(i \leq n-1)$ have been already chosen. Consider

$$f(p) = \sum_{j=n-i+1}^{n} K(p, p_j)$$

on X. Then, since $\inf_{p\in X} f(p) \geq ic > -\infty$, we can find a point p_{n-i} in X such that $f(p_{n-i}) \leq \inf_{p\in X} f(p) + r$. This is nothing but (3.1.1) and the induction is completed. By the definition of $\rho_i(X;\ \Omega, K)$, we get

$$\inf_{p\in X} \sum_{j=n-i+1}^{n} K(p, p_j) \leq i\rho_i(X;\ \Omega, K).$$

Hence by (3.1.1), we get

$$\sum_{j=n-i+1}^{n} K(p_{n-i}, p_j) \leq i\rho_i(X;\ \Omega, K) + r \qquad (i = 1, 2, \ldots, n-1).$$

Summing up these $n-1$ inequalities, we get

$$\sum_{i<j}^{1,\ldots,n} K(p_i, p_j) \leq \sum_{i=1}^{n-1} i\rho_i(X;\ \Omega, K) + (n-1)r$$

and so by the definition of $\tau_n(X;\ \Omega, K)$, we get

$$\binom{n}{2} \tau_n(X;\ \Omega, K) \leq \sum_{i=1}^{n-1} i\rho_i(X;\ \Omega, K) + 1$$

or

$$\tau_n(X;\ \Omega, K) \leq \frac{\sum_{i=1}^{n-1} i\rho_i(X;\ \Omega, K)}{\binom{n}{2}} + \frac{1}{\binom{n}{2}}. \tag{3.1.2}$$

As $\lim_{i\to\infty} \rho_i(X;\ \Omega, K) = \rho(X;\ \Omega, K)$, so we can easily see that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} i\rho_i(X;\ \Omega, K)}{\binom{n}{2}} = \rho(X;\ \Omega, K).$$

Hence by making $n \nearrow \infty$ in (3.1.2), we get $\tau(X;\ \Omega, K) \leq \rho(X;\ \Omega, K)$.

3.2. Let R be a hyperbolic Riemann surface and R^* be its Royden's compactification and $g(p, q)$ be the Green kernel on R^*. For the sake of simplicity, we set, for X in R^*,

$$\tau_n(X) = \tau_n(X;\ R^*, g), \qquad \tau(X) = \tau(X;\ R^*, g)$$

and similarly

$$\rho_n(X) = \rho_n(X;\ R^*, g), \qquad \rho(X) = \rho(X;\ R^*. g)$$

and we simply say that $\tau(X)$ (resp. $\rho(X)$) the *transfinite diameter* (resp. *modi-*

fied Tchebycheff's constant) of X in R^*. From the considerations in Section 3.1, we see that

$$\tau(X) = \lim_{n\to\infty} \tau_n(X), \qquad \rho(X) = \lim_{n\to\infty} \rho_n(X)$$

and

$$\infty \geq \rho(X) \geq \tau(X) \geq 0.$$

3.3. In this section, we state two lemmas concerning Green potentials on hyperbolic Riemann surface. They are well known and contained in the general potential theory on compact metrizable space with positive symmetric kernel (see for example, Ninomiya [11]). But for the sake of completeness, we give proofs for two lemmas following Constantinescu-Cornea's book [1].

Let μ be a *positive regular Borel measure* on R and S_μ be the *support* of μ. The *Green potential* $g_\mu(z)$ is defined by

$$g_\mu(z) = \int g(z, p)\, d\mu(p) \qquad (z \in R).$$

If $g_\mu(z) < \infty$ for a point z in R, then $g_\mu(z) > 0$ and $g_\mu(z)$ is harmonic in $R - S_\mu$ and $g_\mu(z)$ is superharmonic in R (p. 34 in [1]).

A set A in R with $\bar{A} \subset R$ is said to be *polar* if there exists a positive superharmonic function on R which is infinite on A. A property is said to hold *quasi everywhere* if it holds except a polar set.

The *energy* $\|\mu\|^2$ (resp. *mutual energy* $\langle\mu, \nu\rangle$) of a measure μ (resp. measures μ and ν) is defined by

$$\|\mu\|^2 = \iint g(z, w)\, d\mu(z) d\mu(w) \text{ resp. } \langle\mu, \nu\rangle = \iint g(z, w) d\mu(z) d\nu(w)).$$

For a set X in R, we denote by m_X the totality of unit positive regular Borel measures μ on R with $S_\mu \subset X$. We put

$$\gamma(X) = \inf_{\mu \in m_X} \|\mu\|^2$$

when $X \neq \emptyset$ and $\gamma(X) = \infty$ when $X = \emptyset$. We say that $1/\gamma(X)$ is the *capacity* of X induced by energy integral when X is compact. Then we have

LEMMA 3.3.1. *Let F be a compact set in R consisting of a finite number of analytic arcs. Then there exists a unique measure μ_0 in m_F such that $\gamma(F) = \|\mu_0\|^2 < \infty$ and $g_{\mu_0}(z) \leq \gamma(F)$ on R and $g_\mu(z) = \gamma(F)$ on F. Moreover g_{μ_0} belongs* *to* $M_\Delta(R)$ *and* $D_R(g_{\mu_0}(z)) = 2\pi\gamma(F)$.

Proof. Let $(R_n)_{n=0}^\infty$ be a normal exhaustion of R with $R_0 \supset F$. By Lemma 1.4.3, there exists a function w in $M(R)$ such that $w=1$ on F and $w=0$ on $\varDelta$ and w is harmonic in $R-F$. Hence $w \in M_\Delta(R)$ and so we can find $\varphi_n \in M_0(R)$ such that $w = BD\text{-}\lim_n \varphi_n$ on R. If we put $\tilde{F}$ the sum of arcs F with positive direction and arcs F with negative direction, then we get

$$D(\varphi_n, w) = \int_{\tilde{F}} \varphi_n {}^*dw$$

and it implies, by making $n \nearrow \infty$,

$$D(w) = \int_{\tilde{F}} {}^*dw = \int_{\partial R_0} {}^*dw. \tag{3.3.1}$$

By Frostman's theorem (p. 40 in [1]), there exists a unique positive regular Borel measure μ_1 with $S\mu_1 \subset F$ such that $g_{\mu_1}(z) \leq 1$ on R and $g_{\mu_1}(z) = 1$ quasi everywhere on F and $g_{\mu_1}(z) \leq w(z)$ on R. Let $s(z)$ be a positive superharmonic function on R with $s(z) = \infty$ on $(z \in F;\ g_{\mu_1}(z) < 1)$. Then since

$$\liminf_{R \ni z \to \xi} (g_{\mu_1}(z) + \varepsilon s(z) - w(z)) \geq 0$$

for any ξ in $F \cup \varDelta$ and $\varepsilon > 0$, by Lemma 1.2.4, $g_{\mu_1}(z) + \varepsilon s(z) \geq w(z)$ on R. Hence $g_{\mu_1}(z) \geq w(z)$ on R quasi everywhere. As a polar set is measure zero (i.e. the Lebesgue measure of the intersection of polar set with any parameter neighborhood is zero) (p. 31 in [1]), so $g_{\mu_1}(z) \geq w(z)$ almost everywhere on R and so $g_{\mu_1}(z) \geq w(z)$ everywhere on R (p. 13 in [1]). Hence

$$w(z) = g_{\mu_1}(z)$$

on, R, i.e. $g_{\mu_1}(z) \leq 1$ on R and $g_{\mu_1}(z) = 1$ on F. From this, it is clear that $\mu_1(F) > 0$. Let $\mu \in m_F$. Then by energy principle (p. 46 in [1]),

$$1 = \left(\int d\mu\right)^2 = \left(\int g_{\mu_1} d\mu\right)^2 = \langle \mu_1, \mu \rangle^2 \leq \|\mu_1\|^2 \cdot \|\mu\|^2$$

or

$$\|\mu\|^2 \geq 1/\|\mu_1\|^2.$$

On the other hand,

$$\mu_1(F) = \int d\mu_1 = \int g_{\mu_1} d\mu_1 = \|\mu_1\|^2.$$

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Hence, if we put $\mu_0 = \mu_1/\mu_1(F)$, then $\mu_0 \in m_F$ and

$$\|\mu_0\|^2 = \|\mu_1\|^2/\mu_1(F)^2 = \|\mu_1\|^2/\|\mu_1\|^4 = 1/\|\mu_1\|^2.$$

Thus $\|\mu\|^2 \geq \|\mu_0\|^2$ for any $\mu \in m_F$ and $\gamma(F) = \|\mu_0\|^2 = 1/\mu_1(F) < \infty$. Hence $g_{\mu_0}(z) = \gamma(F) g_{\mu_1}(z)$ fulfils the properties that $g_{\mu_0}(z) \leq \gamma(F)$ on R and $g_{\mu_0}(z) = \gamma(F)$ on F. The unicity follows again from Frostman's theorem (p. 40 in [1]).

Notice that $g_{\mu_0} = \gamma(F) \cdot w \in M_\Delta(R)$. Thus by (3.3.1)

$$D_R(g_{\mu_0}) = \gamma(F)^2 \cdot \int_{\partial R_0} {}^*dw = \gamma(F) \int_{\partial R_0} {}^*dg_{\mu_0}$$
$$= \gamma(F) \int_F \left(\int_{\partial R_0} {}^*dg(z, p) \right) d\mu_0(p),$$

Since p varies in R_0 in the last term of the above, it is easy to see that

$$\int_{\partial R_0} {}^*dg(z, p) = 2\pi.$$

Hence by noticing $\mu_0(F) = 1$, we finally obtain

$$D_R(g_{\mu_0}) = 2\pi\gamma(F).$$ Q.E.D.

LEMMA 3.3.2. *Let F be as in Lemma* 3.3.1. *Then* $\gamma(F) = \tau(F) = \rho(F)$.[3]

Proof. For each n, we can find points $p_1^{(n)}, \ldots, p_n^{(n)}$ in F such that

$$\binom{n}{2} D_n(F) \geq \sum_{i<j}^{1, \ldots, n} g(p_i^{(n)}, p_j^{(n)}) - 1/n.$$

Let μ_n be defined by $\mu_n(p_i^{(n)}) = 1/n$ $(i = 1, \ldots, n)$ and $\mu_n(R - \bigcup_{i=1}^{n} (p_i^{(n)})) = 0$. Then μ_n belongs to m_F. Then there exists a subsequence $(\mu_{n_k})_{k=1}^\infty$ of $(\mu_n)_{n=1}^\infty$ such that

$$\lim_{k\to\infty} \int f d\mu_{n_k} = \int f d\mu$$

for every finitely continuous function f on F (p. 9 in [1]). Clearly $\mu \in m_F$. Let $c > 0$. Then by Stone-Weierstrass' theorem, there exists a function

$$\varphi(z, w) = \sum_{j=1}^{n} a_j f_j(z) h_j(w),$$

where a_j are real numbers and f_j and h_j are finitely continuous functions on F, such that

3) It is well known that this is true for any compact set F in R.

$$|g(z, w) \cap c - \varphi_n(z, w)| < 1/n$$

on F. Then

$$\tau_{n_k}(F) + 1/n_k > \frac{1}{n_k^2} \sum_{i=j}^{1, \ldots, n_k} g(p_j^{(n_k)}, p_j^{(n_k)}) \geq \iint (g \cap c) d\mu_{n_k} d\mu_{n_k} - \frac{c}{n_k}$$

$$\geq \iint \varphi_n d\mu_{n_k} d\mu_{n_k} - \frac{1}{n} - \frac{c}{n_k}.$$

Since $\iint \varphi_n d\mu_{n_k} d\mu_{n_k} = \sum a_j \cdot \int f_j d\mu_{n_k} \cdot \int h_j d\mu_{n_k} \to \sum a_j \int f_j d\mu \cdot \int h_j d\mu = \iint \varphi_n d\mu d\mu$ ($k \to \infty$), we obtain by making $k \to \infty$,

$$\tau(F) \geq \iint \varphi d\mu d\mu - \frac{1}{n} \geq \iint (g \cap c) d\mu d\mu - \frac{2}{n}.$$

By making $n \nearrow \infty$, $\tau(F) \geq \iint (g \cap c) d\mu d\mu$. Again by making $c \nearrow \infty$, $\tau(F) \geq \|\mu\|^2$. Thus we get $\tau(F) \geq \gamma(F)$.

By Proposition 2, $\rho(F) \geq \tau(F)$. Hence if we prove $\gamma(F) \geq \rho(F)$, then the proof of Lemma 2 is completed. Let μ_0 be as in Lemma 3.3.1 and $p_1, \ldots, p_n$ be arbitrary points in F. Then

$$\gamma(F) \geq \frac{1}{n} \sum_{i=1}^{n} g_{\mu_0}(p_i) = \frac{1}{n} \int \sum_{i=1}^{n} g(z, p_i)\, d\mu_0(z) \geq \frac{1}{n} \inf_{p \in F} \sum_{i=1}^{n} g(p, p_i).$$

Hence

$$\gamma(F) \geq \frac{1}{n} \sup_{p_1, \ldots, p_n \in F} (\inf_{p \in F} \sum_{i=1}^{n} g(p, p_i)) = \rho_n(F).$$

Thus by making $n \nearrow \infty$, we get $\gamma(F) \geq \rho(F)$,

3.4. Asume that $\Gamma_0 \neq \emptyset$. Fix a point z_0 in R and $(r_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$r_n > r_{n+1}, \qquad \lim_{n \to \infty} r_n = 0$$

and the level curve $(z \in R;\ g(z, z_0) = r_n)$ consists of a countable number of analytic Jordan curves not ending in R. Moreover we may assume that the set

$$U_n = (z \in R;\ g(z, z_0) > r_n)$$

is not relatively compact in R. We set

$$\Gamma_n = \overline{U}_n \cap \Gamma.$$

Then $\Gamma_0 = \bigcup_{n=1}^{\infty} \Gamma$. Let $(R_n)_{n=1}^{\infty}$ be a normal exhaustion of R such that $z_0 \in R_1$. We set

$$F_{n,m} = \overline{U}_n \cap \partial R_m.$$

LEMMA 3.4.1. *The set U_n is a subdomain in R and $\overline{U} \cap \Delta = \emptyset$.*

Proof. Since $g(p, z_0) = 0$ $(p \in \Delta)$, $\overline{U}_n \cap \Delta = \emptyset$. Assume, contrary to our assertion, that there exists a component G of U_n with $z_0 \notin G$. Then $g(z, z_0)$ is the bounded harmonic function on G with $g(z, z_0) = r_n$ on ∂G. Then by Lemma 1.2.4, since $\overline{G} \cap \Delta = \emptyset$, $g(z, z_0) = r_n$ on G. This is a contradiction. Q.E.D.

LEMMA 3.4.2. *There exists a unique positive harmonic function $w_{n,m}$ on $U_{n+1} - R_m$ such that $w_{n,m} = 0$ on $\partial U_{n+1} - R_m$ and $w_{n,m} = 1$ on $F_{n+1,m}$. Moreover, $w_{n,m}$ is continuous on $\overline{U}_{n+1} - \overline{R}_m$ and there exists a constant $\sigma_n > 0$ such that*

$$w_{n,m}(p) \geq \sigma_n \text{ for any } p \in \Gamma_n \qquad (m = 1, 2, \ldots).$$

Proof. Let $k > m+1$ and u_k be harmonic in $R_k \cap U_{n+1} - \overline{R}_m$ with boundary value $u_k = 1$ on $F_{n+1,m}$ and $u_k = 0$ on $F_{n+1,k} \cup (\partial(R_k \cap U_{n+1}) - \overline{R}_m)$. Since $u_k \leq u_{k+p}$, $w_{n,m} = \lim_k u_k$ is a positive harmonic function on $U_{n+1} - R_m$ with boundary value $w_{n,m} = 1$ on $F_{n+1,m}$ and $w_{n,m} = 0$ on $\partial U_{n+1} - \overline{R}_m$. The unicity of such a $w_{n,m}$ follows from Lemma 1.2.4. We get

$$D_{R_k \cap U_{n+1} - \overline{R}_{m+1}}(u_k) = \int_{R_{m+1} \cap U_{n+1}} u_k{}^* du_k.$$

As u_k is harmonic on $\partial R_{m+1} \cap \overline{U}_{n+1}$ and converges uniformly to $w_{n,m}$ on $\partial R_{m+1} \cap \overline{U}_{n+1}$, so we get

$$0 \leq \lim_k \int_{\partial R_{m+1} \cap U_{n+1}} u_k{}^* du_k = \int_{\partial R_{m+1} \cap U_{n+1}} w_{n,m}{}^* dw_{n,m} < \infty.$$

Hence by Fatou's lemma,

$$D_{U_{n+1} - \overline{R}_{m+1}}(w_{n,m}) \leq \liminf_k D_{R_k \cap U_{n+1} - \overline{R}_{m+1}}(u_k) = \int_{\partial R_{m+1} \cap U_{n+1}} w_{n,m}{}^* dw_{n,m} < \infty.$$

On the other hand, let v be harmonic in $R_{m+1} \cap U_{n+1} - \overline{R}_m$ with boundary value $v = w_{n,m}$ on $F_{n+1,m+1}$ and $v = 0$ on $\partial(R_{m+1} \cap U_{n+1} - \overline{R}_m) - F_{n+1,m+1}$. Then clearly $D_{R_{m+1} \cap U_{n+1} - \overline{R}_m}(v) < \infty$. Let

$$f = \begin{cases} w_{n,m} & \text{on } R \cap U_{n+1} - \overline{R}_{m+1}; \\ v & \text{on } R_{m+1} \cap U_{n+1} - \overline{R}_m; \\ 0 & \text{on } R - (R \cap U_{n+1} - \overline{R}_m). \end{cases}$$

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Then f is bounded continuous a.c.T function on R and $D_R(f) = D_{R_{m+1}\cap U_{n+1}-R_m}(v) + D_{U_{n+1}-\bar{R}_{m+1}}(w_{n,m}) < \infty$. Hence $f \in M(R)$ and so f is continuous on R^*. In particular, $w_{n,m}$ is continuous on $\bar{U}_{n+1} - R_{m+1}$ and so $\bar{U}_{n+1} - R_m$.

Next let $R_0 = (z \in R;\ g(z, z_0) > b)$ be contained in $R_1 \cap U_{n+1}$ with its closure. Put

$$w(z) = \frac{g(z, z_0) - r_{n+1}}{b - r_{n+1}}.$$

Then $w(z) > 0$ on U_{n+1} and $w(z) = 0$ on ∂U_{n+1} and $w(z) < 1$ on $U_{n+1} - \bar{R}_0$, Hence $w_{n,m} - w = 1 - w > 0$ on $F_{n+1,m}$ and $w_{n,m} - w = 0$ on $\partial U_{n+1} - \bar{R}_m$. From this, by Lemma 4, we get

$$w_{n,m}(z) \geq w(z) \qquad \text{on } \bar{U}_{n+1} - R_m.$$

Let p be in Γ_n. Then

$$w_{n,m}(p) \geq \frac{g(p, z_0) - r_{n+1}}{b - r_{n+1}} \geq \frac{r_n - r_{n+1}}{b - r_{n+1}} = \sigma_n > 0 \qquad (m = 1, 2, \ldots).$$

The unicity of such a $w_{n,m}$ follows from Lemma 1.2.4. Q.E.D.

LEMMA 3.4.3. $\tau(\Gamma_n) \geq \sigma_n^2 \cdot \tau(F_{n+1,m}) \qquad (m = 1, 2, \ldots)$.

Proof. Let k be an arbitrary positive integer larger than 4 and $p_1, p_2, \ldots, p_k$ be in Γ_n. We choose k points $z_1, z_2, \ldots, z_k$ in $F_{n+1,m}$ inductively as follows. Let

$$u_1(z) = \sum_{i=2}^{k} g(z, p_i)$$

and z_1 be in $F_{n+1,m}$ such that

$$u_1(z_1) = \min_{z \in F_{n+1,m}} u_1(z).$$

Since $u_1(z) > 0$ on R, $u_1(z) - u_1(z_1) w_{n,m}(z) \geq 0$ on $\partial(U_{n+1} - \bar{R}_m)$, where $w_{n,m}$ is as in Lemma 3.4.2, and so by Lemma 1.2.4, $u_1(z) \geq u_1(z) w_{n,m}(z)$ on $U_{n+1} - \bar{R}_m$ and so on $\bar{U}_{n+1} - \bar{R}_m$. Hence in particular,

$$u_1(p_1) \geq u_1(z_1) w_{n,m}(p_1) \geq \sigma_n u_1(z_1)$$

and so

$$\sigma_n \sum_{i=2}^{k} g(z_1, p_i) + \sum_{i<j}^{2,\ldots,k} g(p_i, p_j) \leq \sum_{i=2}^{k} g(p_1, p_i) + \sum_{i<j}^{2,\ldots,k} g(p_i, p_j)$$

and hence by putting $a = \sum_{i<j}^{1,\ldots,n} g(p_i, p_j)$, we get

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(3.4.1) $$\sigma_n \sum_{i=2}^{k} g(z_1, p_i) + \sum_{i<j}^{2,\dots,k} g(p_i, p_j) \le a.$$

Next we choose $z_2, \dots, z_{n-2}$ in $F_{n+1,m}$ satisfying

(3.4.2) $$\sigma_n^2 \sum_{i<j}^{1,\dots,\nu} g(z_i, z_j) + \sigma_n \sum_{i=1}^{\nu} \sum_{j=\nu+1}^{k} g(z_i, p_j) + \sum_{i<j}^{\nu+1,\dots,k} g(p_i, p_j) \le a$$
$$(\nu = 2, 3, \dots, k-2).$$

First let

$$u_2(z) = \sum_{j=3}^{k} g(z, p_j) + \sigma_n g(z_1, z)$$

and z_2 be in $F_{n+1,m}$ such that

$$u_1(z_2) = \min\nolimits_{z \in F_{n+1,m}} u_2(z).$$

Similarly as above, we have $u_2(p) \ge u_2(z_2) w_{n,m}(p) \ge \sigma_n u_2(z_2)$ for p in $\overline{U}_{n+1} - \overline{R}_m$ and so

$$\sum_{j=3}^{k} g(p_2, p_j) + \sigma_n g(z_1, p_2) \ge \sigma_n \sum_{j=3}^{k} g(z_2, p_j) + \sigma_n^2 g(z_1, z_2).$$

From this with (3.4.1), we get

$$\sigma_n^2 g(z_1, z_2) + \sigma_n \sum_{i=1}^{2} \sum_{j=2}^{k} g(z_i, p_j) + \sum_{i<j}^{3,\dots,k} g(p_i, p_j) \le a.$$

This is nothing but (3.4.2) for $\nu = 2$. Next assume that $z_2, \dots, z_\nu$ $(\nu \le k-3)$ have been already chosen in $F_{n+1,m}$ satisfying (3.4.2). Let

$$u_{\nu+1}(z) = \sum_{j=\nu+2}^{k} g(z, p_j) + \sigma_n \sum_{i=1}^{\nu} g(z_i, z)$$

and $z_{\nu+1}$ be in $F_{n+1,m}$ such that

$$u_{\nu+1}(z_{\nu+1}) = \min\nolimits_{z \in F_{n+1,m}} u_{\nu+1}(z).$$

Similarly as before, we have $u_{\nu+1}(p) \ge u_{\nu+1}(z_{\nu+1}) w_{n,m}(p) \ge \sigma_n u_{\nu+1}(z_{\nu+1})$ for p in $\overline{U}_{n+1} - \overline{R}_m$ and so

$$\sum_{j=\nu+2}^{k} g(p_{\nu+1}, p_j) + \sigma_n \sum_{i=1}^{\nu} g(z_i, p_{\nu+1}) \ge \sigma_n \sum_{j=\nu+2}^{k} g(z_{\nu+1}, p_j) + \sigma_n^2 \sum_{i=1}^{\nu} g(z_i, z_{\nu+1}).$$

From this with (3.4.2) for $\nu \le k-3$, we get

$$\sigma_n^2 \sum_{i<j}^{1,\dots,\nu+1} g(z_i, z_j) + \sigma_n \sum_{i=1}^{\nu+1} \sum_{j=\nu+2}^{k} g(z_i, p_j) + \sum_{i<j}^{\nu+2,\dots,k} g(p_i, p_j) \le a.$$

This is (3.4.2) for $\nu+1$. Thus we have constructed the system $z_2, \ldots, z_{k-2}$. Next let

$$u_{k-1}(z)=g(z, p_k)+\sigma_n \sum_{i=1}^{k-2} g(z_i, z)$$

and z_{k-1} be in $F_{n+1,m}$ such that

$$u_{k-1}(z_{k-1})=\min_{z \in F_{n+1,m}} u_{k-1}(z).$$

Similarly as before, we get $u_{k-1}(p) \geq u_{k-1}(z_{k-1}) w_{n,m}(p) \geq \sigma_n u_{k-1}(z_{k-1})$ for p in $\overline{U}_{n+1}-\overline{R}_m$ and so

$$g(p_{k-1}, p_k)+\sigma_n \sum_{i=1}^{k-2} g(z_i, p_{k-1}) \geq \sigma_n g(z_{k-1}, p_k)+\sigma_n^2 \sum_{i=1}^{k-2} g(z_i, z_{k-1}).$$

From this with (3.4.2) for $\nu=k-2$, we get

$$\sigma_n^2 \sum_{i<j}^{1, \ldots, k-1} g(z_i, z_j)+\sigma_n \sum_{i=1}^{k-1} g(z_i, p_k) \leq a \tag{3.4.3}$$

Finally let

$$u_k(z)=\sigma_n \sum_{i=1}^{k-1} g(z_i, z)$$

and z_k be in $F_{n+1,m}$ such that

$$u_k(z_k)=\min_{z \in F_{n+1,m}} u_k(z).$$

Similarly as before, we have $u_k(p) \geq u_k(z_k) w_{n,m}(p) \geq \sigma_n u_k(z_k)$ for p in $\overline{U}_{n+1}-\overline{R}_m$ and so

$$\sigma_n \sum_{i=1}^{k-1} g(z_i, p_k) \geq \sigma_n^2 \sum_{i=1}^{k-1} g(z_i, z_k).$$

From this with (3.4.3), we get

$$\sigma_n^2 \sum_{i<j}^{1, \ldots, k} g(z_i, z_j) \leq a.$$

Hence by the definition of $\tau_k(F_{n+1,m})$, we get

$$\sigma_n^2 \binom{k}{2} \tau_k(F_{n+1,m}) \leq \sum_{i<j}^{1, \ldots, k} g(p_i, p_j).$$

Since $p_1, \ldots, p_k$ are arbitrary in Γ_n, so

$$\sigma_n^2 \binom{k}{2} \tau_k(F_{n+1,m}) \leq \binom{k}{2} \tau_k(\Gamma_n)$$

or $\sigma_n^2\tau_k(F_{n+1,m}) \leq \tau_k(\Gamma_n)$. Hence by making $k \nearrow \infty$, we finally get $\sigma_n^2\tau(F_{n+1,m}) \leq \tau(\Gamma_n)$.

LEMMA 3.4.4. $\lim_{m\to\infty}\gamma(F_{n,m}) = \infty$.

Proof. Let $\mu_{n,m}$ be as in Lemma 3.3.1 for $F_{n,m}$. Then $D(g_{\mu_{n,m}}) = 2\pi\gamma(F_{n,m})$. Put $u_m = g_{\mu_{n,m}}/\gamma(F_{n,m})$. Then $u_m \in M_\Delta(R)$ and $u_m = 1$ on $F_{n,m}$ and harmonic in $R - F_{n,m}$ and

$$D(u_m) = 2\pi/\gamma(F_{n,m}).$$

By Lemmas 1.2.1 and 2, there exists f in $M(R)$ such that $f = 1$ on $\overline{U}_n - R_m$ and 0 on Δ. Hence by Lemma 1.4.2, there exists $v_m \in M(R)$ such that $v_m = 1$ on $\overline{U}_n - R_m$ and $v_m = 0$ on Δ and v_m is harmonic in $R - (\overline{U}_n - R_m)$. Then by applying Lemma 1.4.2 for $K = F_{n,m}$, we get

$$D(u_m) \leq D(v_m).$$

Again by applying Lemma 1.4.2 for $K = F_{n,m+p}$ and v_m and v_{m+p},

$$D_R(v_m - v_{m+p}) = D_R(v_m) - D_R(v_{m+p}).$$

Hence $(v_m)_{m=1}^\infty$ is D-convergent. By Lemma 1.2.4, $v_m \geq v_{m+p}$ on R and so $v = \lim_m v_m$ is a harmonic function on R and so $v = BD\text{-}\lim_m v_m$. Thus $v \in M(R)$ or v is continuous on R^* and $0 \leq v \leq v_m$ implies $v = 0$ on Δ. Hence by Lemma 1.2.4, $v \equiv 0$ on R. Thus $BD\text{-}\lim v_m = 0$. From this,

$$0 \leq \limsup_m D(u_m) \leq \lim_m D(v_m) = 0.$$

Therefore, we obtain

$$\liminf_m \gamma(F_{n,m}) = \lim_m 2\pi/D(u_m) = \infty, \qquad \text{Q.E.D.}$$

PROPOSITION 3. $\rho(\Gamma_n) = \infty \quad (n = 1, 2, \ldots)$.

Proof. By Proposition 2 in Section 3.1 and Lemmas 3.3.2 and 3.4.3, we get the relation

$$\rho(\Gamma_n) \geq \tau(\Gamma_n) \geq \sigma_n^2\tau(F_{n,m}) = \sigma_n^2\gamma(F_{n,m}).$$

Hence by Lemma 3.4.4, we get, by making $m \nearrow \infty$, $\rho(\Gamma_n) = \infty$. Q.E.D.

IV. Proofs of Theorems 1 and 2

4.1. By Proposition 3, $\rho(\Gamma_n) = \infty$. Since $\rho(\Gamma_n) = \lim_m \rho_m(\Gamma_n)$, we can find an increasing sequence $(m_k)_{k=1}^\infty$ of positive integers such that

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$$\rho_{m_k}(\Gamma_n) > 2^k \qquad (k = 1, 2, \ldots).$$

By the definition of $\rho_{m_k}(\Gamma_n)$, we can find m_k points $p_{k,i}^{(n)}$ $(i = 1, 2, \ldots, m_k)$ in Γ_n such that

$$\inf_{p \in \Gamma_n} \sum_{i=1}^{m_k} g(p, p_{k,i}^{(n)}) > 2^k m_k.$$

Then the function

$$e_{n,k}(p) = 2^{-k-1} m_k^{-1} \sum_{i=1}^{m_k} g(p, p_{k,i}^{(n)})$$

is continuous on R^* and harmonic on R and $e_{n,k}(p) > 1/2$ for any p in Γ_n. Since $g(z_0, q)$ is finitely continuous in q on Γ, $g(z_0, q) \le c_0 < \infty$ for any q in Γ. Hence $e_{n,k}(z_0) \le c_0/2^{k+1}$ and so

$$e_n(z) = \sum_{k=1}^{\infty} e_{n,k}(z)$$

is a positive harmonic function on R. Let $p \in \Gamma_n$. As $e_n(z) > \sum_{k=1}^{N} e_{n,k}(z)$ on R for any positive integer N. Hence

$$\liminf_{R \ni z \to p} e_n(z) \ge \sum_{k=1}^{N} e_{n,k}(p) > \frac{N}{2}.$$

Next by making $N \nearrow \infty$, we get

$$\lim_{R \ni z \to p} e_n(z) = \infty \qquad (p \in \Gamma_n).$$

Now we put, by noticing $e_n(z_0) \le c_0$ $(n = 1, 2, \ldots)$,

$$e(z) = \sum_{n=1}^{\infty} 2^{-n} e_n(z)$$

on R. Then $e(z)$ is a positive harmonic function on R. Let $p \in \Gamma_0$. Then $p \in \Gamma_n$ for some n. Since $e(z) > 2^{-n} e_n(z)$ on R, we get

$$\liminf_{R \ni z \to p} e(z) \ge 2^{-n} \lim_{R \ni z \to p} e_n(z) = \infty.$$

Hence we have

(4.1.1) $$\lim_{R \ni z \to p} e(z) = \infty \qquad (p \in \Gamma_0).$$

4.2. we denote by $\varepsilon_p (p \in R^*)$ the unit positive regular Borel measure such that $\varepsilon_p(p) = 1$ and $\varepsilon_p(R^* - (p)) = 0$. We set

$$\mu = \sum_{n=1}^{\infty} \sum_{k=1} \sum_{i=1}^{m_k} 2^{-n-k-1} m_k^{-1} \varepsilon_{p_{k,i}^{(n)}}.$$

Then μ is a unit positive regular Borel measure on R^* with $S_\mu \subset \bar{\Gamma}_0$ and $\mu(R^* - \Gamma_0) = 0$ such that

$$e(z) = \int_{\Gamma_0} g(z, q)\, d\mu(q).$$

Clearly, we can write

$$\mu = \sum_{i=1}^{\infty} t_i \varepsilon_{q_i},$$

where $(q_i)_{i=1}^{\infty}$ is the sequence of points in Γ_0 which is a rearangement of $(p_{k,i}^{(n)})$, and where $(t_i)_{i=1}^{\infty}$ is the sequence of positive numbers such that $\sum_{i=1}^{\infty} t_i = 1$ which is given by $t_i = 2^{-n-k-1} m_k^{-1}$ with $q_i = p_{k,j}^{(n)}$.

We shall prove that this μ is required measure in Theorem 2. Clearly μ satisfies (4) and (5). Notice that

$$e(z) = \sum_{i=1}^{\infty} t_i g(z, q_i).$$

4.3. Now we show that $D(e(z) \cap c) \leq 2\pi c$ $(c > 0)$ and $e(z)$ is continuous on R^* and $e(p) = 0$ on Δ and $e(p) = \infty$ on Γ_0, i.e. μ satisfies (6), (7), (8) and (9) in Theorem 2.

Let n be an arbitrary but fixed positive integer. We set

$$x(z) = \sum_{i=1}^{n} t_i g(z, q_i)$$

and

$$dy(z) = {}^*dx(z).$$

Then we can use $x + iy$ as local parameter at each point of R except at most a countable number of isolated points in R where $dx(z) = 0$. We put

$$L(\alpha) = \sum_{i=1}^{n} \int_{x=\alpha} |{}^*dg(z, q_i)|,$$

where $\alpha > 0$ and we assume that $dx \neq 0$ on $(z\,;\, x(z) = \alpha)$. As we have

$$L(\alpha) = \int_{x=\alpha} \Big(\sum_{i=1}^{n} \Big| \frac{\partial}{\partial x} g(z, q_i) \Big| \Big) dy,$$

so by Schwarz's inequality, we obtain

$$(L(\alpha))^2 \leq \Big(\int_{x=\alpha} \Big(\sum_{i=1}^{n} \Big| \frac{\partial}{\partial x} g(z, q_i) \Big|^2 \Big) dy \Big) \cdot n \int_{x=\alpha} dy$$

$$\leq \Big(\int_{x=\alpha} \Big(\sum_{i=1}^{n} |\operatorname{grad} g(z, q_i)|^2 dy \Big) \cdot n \int_{x=\alpha} {}^*dx.$$

Notice that except at most a countable number of $\alpha>0$, $dx \neq 0$ on $(z;\ x(z)=\alpha)$. By (g.3), (g.7) and (g.9) in Proposition 1, $x(z)$ is contiunous on R^* and vanishes on Δ and $x(z)\cap c \in M(R)$ $(c>0)$. Applying (1.4.2) in Lemma 1.4.3 for $K=(p\in R^*;\ x(p)\geq\alpha)$ and $(x(z)\cap\alpha)/\alpha$, we get

$$\alpha^{-2}D_R(x\cap\alpha)=D_R((x\cap\alpha)/\alpha)=\int_{x=\alpha}{}^*d(x/\alpha)<\infty.$$

Hence we have

$$D_R(x\cap\alpha)=\alpha\int_{x=\alpha}{}^*dx<\infty. \tag{4.3.1}$$

Thus if $c\leq\alpha\leq c'$, where $dx\neq 0$ on $(z;\ x(z)=c')$, then

$$\int_{x=\alpha}{}^*dx=\alpha^{-1}D_R(x\cap\alpha)\leq c^{-1}D_R(x\cap c')<\infty.$$

Therefore, we get

$$(L(\alpha))^2\leq nc^{-1}D_R(x\cap c')\int_{x=\alpha}\sum_{i=1}^{n}|\operatorname{grad} g(z,\ q_i)|^2dy,$$

if $c\leq\alpha\leq c'$. Hence

$$\int_{c\leq x\leq c'}(L(x))^2dx\leq nc^{-1}D_R(x\cap c')\cdot\sum_{i=1}^{n}D_R(g(z,\ q_i)\cap c't_i^{-1})<\infty. \tag{4.3.2}$$

Here we used (g.7) in Proposition 1 and the fact that

$$\begin{aligned}\int_c^{c'}d\alpha\int_{x=\alpha}\Big(\sum_{i=1}^{n}|\operatorname{grad} g(z,\ q_i)|^2\Big)dy &= \sum_{i=1}^{n}\iint_{c\leq x\leq c'}|\operatorname{grad} g(z,\ q_i)|^2dxdy\\ &\leq\sum_{i=1}^{n}\iint_{x\leq c'}|\operatorname{grad} g(z,\ q_i)|^2dxdy.\end{aligned}$$

As $g(z,\ q_i)\leq c't_i^{-1}$ on $(z\in R;\ x(z)\leq c')$ $(i=1,\ 2,\ \ldots,\ n)$, so

$$\iint_{x\leq c'}|\operatorname{grad} g(z,\ q_i)|^2dxdy\leq D_R(g(z,\ q_i)\cap c't_i^{-1})\leq 2\pi c't_i^{-1}.$$

Hence we get (4.3.2). From (4.3.2), we get $L(\alpha)<\infty$ or

$$\int_{x=\alpha}|d^*g(z,\ q_i)|<\infty \qquad (i=1,2,\ \ldots,\ n) \tag{4.3.3}$$

for almost every $c<\alpha<c'$ and so for almost every $\alpha>0$. For the sake of simplicity, we say that $\alpha>0$ is regular for $x(z)$ if $dx\neq 0$ on $(z\in R;\ x(z)=\alpha)$

and (4.3.3) holds. We notice that the totality of regular positive numbers are dense in the totality of positive numbers.

Now let $c>0$ be regular for $x(z)$. Then by (4.3.1) and (4.3.3), we get

$$D_R(x\cap c)=c\int_{x=c}{}^*dx=c\sum_{i=1}^{n}t_i\int_{x=c}{}^*dg(z,q_i). \tag{4.3.4}$$

Let $\alpha>ct_i^{-1}$ and $g(z,q_i)\neq 0$ on $(z\in R;\ g(z,q_i)=\alpha)$. Then the interior of $K'=(p\in R^*;\ x(p)\geq c)$ contains $K=(p\in R^*;\ g(p,q_i)\geq\alpha)$. Since

$$\int_{x=c}|{}^*dg(z,q_i)|<\infty$$

by (4.3.3), we can apply (1.4.2) and (1.4.3) in Lemma 1.4.3 for K and K' and $(g(z,q_i)\cap\alpha/\alpha)$ and so we get

$$\begin{aligned}\int_{x=c}{}^*d(\alpha^{-1}g(z,q_i))&=\int_{\partial K'}{}^*d(\alpha^{-1}g(z,q_i))=\int_{\partial K}{}^*d(\alpha^{-1}g(z,q_i))\\&=D_{R-K}(\alpha^{-1}g(z,q_i))=D_R(\alpha^{-1}(g(z,q_i)\cap\alpha))\end{aligned}$$

or

$$\int_{x=c}{}^*dg(z,q_i)=\alpha^{-1}D_R(g(z,q_i)\cap\alpha)\leq\alpha^{-1}\cdot 2\pi\alpha=2\pi.$$

Hence by (4.3.4), we obtain

$$D_R(x\cap c)=c\sum_{i=1}^{n}t_i\int_{x=c}{}^*dg(z,q_i)=2\pi c\sum_{i=1}^{n}t_i<2\pi c,$$

i.e.

$$D_R\Big(\Big(\sum_{i=1}^{n}t_ig(z,q_i)\Big)\cap c\Big)<2\pi c\qquad(n=1,2,\ldots) \tag{4.3.5}$$

As $\big(d\big(\big(\sum_{i=1}^{n}t_ig(z,q_i)\big)\cap c\big)\big)_{n=1}^{\infty}$ converges to $d(e(z)\cap c)$ on R except the set $(z;\ e(z)=c)$ for each local parameter z on R, so by Fatou's lemma,

$$D_R(e(z)\cap c)\leq 2\pi c. \tag{4.3.6}$$

If $c>0$ is not regular for $x(z)$, we choose regular $c_n>0$ for $x(z)$ such that $c_n\searrow c$, then $D_R(e(z)\cap c)\leq D_R(e(z)\cap c_n)\leq 2\pi c_n$. Hence by making $n\nearrow\infty$, we get (4.3.6) for any $c>0$. This is (6) in Theorem 2. From this, by Lemma 1.3.1, $e(z)$ is continuous on R^*, which is (7) in Theorem 2. From (4.1.1), $e(p)=\infty$ on Γ_0, which is (8). Clearly

$$D_K((\sum_{i=1}^{n} t_i g(z, q_i)) \cap c - e(z) \cap c) \to 0 \qquad (n \to \infty)$$

for any compact K in R (cf. p. 78, Satz 7.4 in [1]). Hence by Lemma 1.4.1 with (4.3.6), $e(z) \in M_\Delta(R)$, or $e(p) = 0$ on Δ, which is (9) in Theorem 2. Thus we proved Theorem 2 completely.

4.4. Theorem 1 follows immeadiately from Theorem 2. In fact, assume that $(z_n)_{n=1}^{\infty} \in \mathscr{S}(R)$. We show that $\lim_{n\to\infty} g_\mu(z_n) = \infty$. If this is not true, then we can find a subsequence $(w_n)_{n=1}^{\infty}$ of $(z_n)_{n=1}^{\infty}$ such that

$$0 \leq \lim_{n\to\infty} g_\mu(w_n) = b < \infty.$$

Clearly $(w_n)_{n=1}^{\infty} \in \mathscr{S}(R)$. Let p_0 be an accumulation point of $(w_n)_{n=1}^{\infty}$. Then since

$$\lim\inf_{n\to\infty} g(w_n, z_0) > 0,$$

we conclude that $p_0 \in \Gamma_0$. Let $\Lambda = (\lambda)$ be the totality of neighborhoods of p_0 in R^*. Then $T = \Lambda \times (1, 2, 3, \ldots)$ is a directed set if we define that $t = (\lambda, n) \geq t' = (\lambda', n')$ if $\lambda \subset \lambda'$ and $n \geq n'$. For each $t = (\lambda, n)$, $(w_\nu)_{\nu=1}^{\infty} \cap (\lambda - \overline{R}_n) \neq \emptyset$. We choose a point w_t in $(w_\nu)_{\nu=1}^{\infty} \cap (\lambda - \overline{R}_n)$. Then clearly $\lim_{t\in T} w_t = p_0$. Moreover, let $[t] = n$ if $t = (\lambda, n)$. Then $\lim_{t\in T}[t] = \infty$. Hence by $\lim_{n\to\infty} g_\mu(w_n) = b$, we get $\lim_{t\in T} g_\mu(w_t) = b < \infty$. On the other hand, since g_μ is continuous on R^* and $g_\mu = \infty$ on Γ_0, we get

$$\infty = g_\mu(p_0) = \lim_{t\in T} g_\mu(w_t) = b < \infty,$$

which is clearly a contradiction and so g_μ satisfies (1) in Theorem 1. Since $D(g_\mu \cap c) \leq 2\pi c$, g_μ also satisfies (2) in Theorem 1.

Finally we show that g_μ is singular, which is (3) in Theorem 1. Let $c > 0$ and h be a non-negative harmonic function on R with $h \leq g_\mu \cap c$ on R. Since $g_\mu = 0$ on Δ, we get

$$\lim_{R \ni z \to q} h(z) = 0$$

for any q in Δ. Hence by Lemma 1.2.4, $h(z) \equiv 0$ on R. Thus the greatest harmonic minorant of $g_\mu \cap c$ is identically zero and so g_μ is singular.

Thus $u = g_\mu$ is the required function in Theorem 1. This completes the proof of Theorem 1.

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