SOME THEOREMS ON OPEN RIEMANN SURFACES

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1

Let F be an open Riemann surface spread over the z-plane. We say that F is of positive or null boundary, according as there exists a Green's function on F or not. Let $u(z)$ be a harmonic function on F and

$$D(u)=\iint_F\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy \qquad (z=x+iy)$$

be its Dirichlet integral. As R. Nevanlinna[1] proved, if F is of null boundary, there exists no one-valued non-constant harmonic function on F, whose Dirichlet integral is finite. This Nevanlinna's theorem was proved very simply by Kuroda.[2] By this mehod, we will prove

THEOREM 1. *Let F be an open Riemann surface with null boundary and Δ be a non-compact domain on F, whose houndary Λ consists of (compact or non-compact) analytic curves. Let $u(z)$ be a one-valued harmonic function in Δ, such that $u(z)=0$ on Λ and its Dirichlet integral in Δ*

$$D(u)=\iint_\Delta\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy$$

is finite. Then $u(z)\equiv 0$.

This theorem was proved by R. Nevanlinna[3] under the condition that $u(z)$ is harmonic outside a compact domain F_0 and its Dirichlet integral in $F-F_0$ is finite.

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1) (a) R. Nevanlinna: Quadratisch integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit. Annales Acad. Sci. Fenn. Series A, Mathematica-Physica **1** (1941).
(b) Über das Anwachsen des Dirichletintegrals einer analytischen Funktion auf einer offenen Riemannschen Fläche. Annales Acad. Sci. Fenn. Series A, Mathematica-Physica **45** (1948).

2) T. Kuroda: Some remarks on an open Riemann surface. To appear in the Tohoku Math. Journ.

3) R. Nevanlinna. l.c. 1) (a).

Proof. We choose a schlicht disc F_0 in $\varDelta$, whose boundary is $\varGamma_0$. We appoximate F by a sequence of compact Riemann surfaces F_n, $\bar{F}_n \subset F_{n+1}$ $(n=0, 1, 2, \ldots)$, $F_n \to F$, whose boundary $\varGamma_0 + \varGamma_n$ consists of a finite number of analytic Jordan curves.

Let

$$\omega_n(z) = \omega(z, \varGamma_n, F_n - F_0) \tag{1}$$

be the harmonic measure of $\varGamma_n$ with respect to $F_n - F_0$, such that $\omega_n(z)$ is harmonic in $F_n - F_0$ and $\omega_n(z) = 0$ on $\varGamma_0$, $\omega_n(z) = 1$ on $\varGamma_n$.

Let $\bar{\omega}_n(z)$ be its conjugate harmonic function, then

$$D(\omega_n) = \iint_{F_n - F_0} \left(\left(\frac{\partial \omega_n}{\partial x} \right)^2 + \left(\frac{\partial \omega_n}{\partial y} \right)^2 \right) dxdy = \int_{\varGamma_n} \omega_n d\bar{\omega}_n = \int_{\varGamma_n} d\bar{\omega}_n = \int_{\varGamma_0} d\bar{\omega}_n.$$

As Nevanlinna[4] proved, $D(\omega_n) \to 0$ as $n \to \infty$, so that

$$\mu_n = \frac{2\pi}{D(\omega_n)} \to \infty \quad \text{as} \quad n \to \infty. \tag{2}$$

We put

$$u_n(z) = \mu_n \omega_n(z), \qquad v_n(z) = \mu_n \bar{\omega}_n(z), \tag{3}$$

then $u_n(z) = 0$ on $\varGamma_0$, $u_n(z) = \mu_n$ on $\varGamma_n$ and $v_n(z)$ is its conjugate harmonic function, such that

$$\int_{\varGamma_0} dv_n(z) = 2\pi. \tag{4}$$

Let D_λ $(0 \leqq \lambda \leqq \mu_n)$ be the domain, such that $0 \leqq u_n(z) \leqq \lambda$ and $\varDelta_\lambda$ be the common part of $\varDelta$ and $D_\lambda + F_0$. Let $\varGamma_\lambda$ be the niveau curve $u_n(z) = \lambda$ and $\varGamma_\lambda(\varDelta)$ be its part contained in $\varDelta$.

To prove $u(z) \equiv 0$, we assume that $u(z) \not\equiv 0$ and let $v(z)$ be its conjugate harmonic function.

Since $u(z) = 0$ on $\varLambda$,

$$D(\lambda) = \iint_{\varDelta_\lambda} \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right) dxdy = \int_{\varGamma_\lambda(\varDelta)} u \frac{\partial u}{\partial \nu} ds = \int_{\varGamma_\lambda(\varDelta)} u \frac{\partial u}{\partial u_n} dv_n, \tag{5}$$

where ν is the inner normal and ds is the arc element on $\varGamma_\lambda$.

From

$$D(\lambda) = \int_0^\lambda d\lambda \int_{\varGamma_\lambda(\varDelta)} \left(\left(\frac{\partial u}{\partial u_n} \right)^2 + \left(\frac{\partial u}{\partial v_n} \right)^2 \right) dv_n + \text{const.}, \tag{6}$$

we have

$$D'(\lambda) = \int_{\varGamma_\lambda(\varDelta)} \left(\left(\frac{\partial u}{\partial u_n} \right)^2 + \left(\frac{\partial u}{\partial v_n} \right)^2 \right) dv_n,$$

so that from (5),

[4] R. Nevanlinna. l.c. [1] (a).

$$D^2(\lambda)\leqq\int_{\Gamma_\lambda(\Delta)}u^2dv_n\int_{\Gamma_\lambda(\Delta)}\left(\frac{\partial u}{\partial u_n}\right)^2dv_n\leqq D'(\lambda)\int_{\Gamma_\lambda(\Delta)}u^2dv_n.$$

We put

(7) $$m(\lambda)=\int_{\Gamma_\lambda(\Delta)}u^2dv_n,$$

then

(8) $$D^2(\lambda)\leqq m(\lambda)D'(\lambda).$$

Since $u=0$ on Δ,

(9) $$m'(\lambda)=2\int_{\Gamma_\lambda(\Delta)}u\frac{\partial u}{\partial u_n}dv_n=2\,D(\lambda)>0,$$

(10) $$m''(\lambda)=2\,D'(\lambda).$$

Since by (9), $m'(\lambda)$ (>0) is an increasing function of λ, $m(\lambda)$ is an increasing convex function of λ.

Since $m'(\lambda)>0$, we have from (8), (9), (10),

$$\frac{m'(\lambda)}{m(\lambda)}\leqq 2\frac{m''(\lambda)}{m'(\lambda)}.$$

Hence integrating, we have

$$m(\lambda)\leqq k[m'(\lambda)]^2,\qquad k=\frac{m(0)}{(m'(0))^2}=\frac{m(0)}{4\,(D(0))^2},$$

so that from (8), (9),

$$\frac{D^2(\lambda)}{D'(\lambda)}\leqq 4kD^2(\lambda),\quad\text{or}\quad d\lambda\leqq 4kdD(\lambda).$$

Hence integrating on $[0,\ \mu_n]$, we have

$$\mu_n\leqq 4k(D(\mu_n)-D(0))\leqq 4kD(\mu_n).$$

Since by the hypothesis, $D(\mu_n)$ is bounded and $\mu_n\to\infty$ as $n\to\infty$, this is absurd. Hence $u(z)\equiv 0$. q.e.d.

THEOREM 2. *Let F be an open Riemann surface with null boundary and Δ be a non-compact domain on F, whose boundary Γ consists of (compact or non-compact) analytic curves. Let $u(z)$ be a one-valued harmonic function in Δ, whose Dirichlet integral in Δ is finite. If $u(z)$ is bounded on Γ, then $u(z)$ is bounded in Δ, such that*

$$m\leqq u(z)\leqq M\quad\textit{in}\quad\Delta,$$

where

$$m=\inf_{\Gamma}\,u(z),\qquad M=\sup_{\Gamma}\,u(z).^{5)}$$

[5] c. f. R. Nevanlinna. l.c. [1] (a).

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Proof. Suppose that there exists a point z_0 in Δ, such that $u(z_0) > M$. We choose K, such that $u(z_0) > K > M$ and let $\Delta(K)$ be the sub-domain of Δ, such that $u(z) \geqq K$ and Λ be its boundary. Then $u(z) = K$ on Λ, so that Λ has no common points with Γ, hence $\Delta(K)$ is non-compact. Since $u(z) = K$ on Λ and its Dirichlet integral in $\Delta(K)$ is finite, we have by Theorem 1, $u(z) \equiv K$, which is absurd, since $u(z_0) > K$. Hence $u(z) \leqq M$ in Δ. Similarly $u(z) \geqq m$ in Δ. q.e.d.

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We will prove

THEOREM 3. *Let F be an open Riemann surface and $z=0$ be contained in F. We approximate F by a sequence of compact Riemann surfaces F_n, $\bar{F}_n \subset F_{n+1}$ $(n = 0, 1, 2. \ldots)$, $F_n \to F$, whose boundary Γ_n consists of a finite number of analytic Jordan curves and F_0 contains $z = 0$. Let $g_n(z, 0)$ be the Green's function of F_n, with $z = 0$ as its pole and let at $z = 0$,*

$$g_n(z, 0) = \log \frac{1}{|z|} + \gamma_n + \varepsilon_n(z) \qquad (\varepsilon_n(0) = 0),$$

where γ_n is the Robin's constant. Then

$$g_n(z, 0) - \gamma_n \qquad (n = 0, 1, 2, \ldots)$$

is uniformly bounded in any compact domain of F, which does not contain $z = 0$.

Proof. Let $F_0 : |z| \leqq \rho < 1$ be contained in F and $\Gamma_0 : |z| = \rho$.
We put

$$M_n = \operatorname*{Max.}_{\Gamma_0} g_n(z, 0). \tag{1}$$

Then Heins[6] proved that

$$u_n(z) = M_n - g_n(z, 0) \tag{2}$$

is uniformly bounded in any compact domain of F, which does not contain $z = 0$. For the sake of completeness, we will reproduce his proof. Now

$$u_n(z) > 0 \quad \text{in} \quad F_n - F_0 \tag{3}$$

and since $u_1(z) - u_n(z)$ is harmonic in F_1 and at some point z_0 on Γ_0, $u_n(z_0) = 0$, $u_1(z_0) \geqq 0$, by the maximum principle, we have

$$\operatorname*{Max.}_{\Gamma_1} (u_1(z) - u_n(z)) \geqq 0.$$

Since $u_1(z) = M_1$ on Γ_1, we have

[6] M. Heins: The conformal mapping of simply connected Riemann surfaces. Annals of Math. **50** (1949).

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$$\text{(4)} \qquad \underset{\Gamma_1}{\mathrm{Min.}}\, u_n(z) \leqq M_1,$$

Let $|z| \leqq \rho_1$ $(\rho < \rho_1 < 1)$ be contained in F_1, then from (3), (4) and Harnack's theorem on positive harmonic functions, we conclude that for any compact domain Δ of F, which contains $|z| \leqq \rho_1$, there exists a constant $K = K(\Delta)$, such that for $n \geqq n_0$,

$$\text{(5)} \qquad |g_n(z,\ 0) - M_n| \leqq K \quad \text{in} \quad \Delta \quad \text{outside} \quad |z| = \rho_1.$$

Hence

$$|v_n(z)| = |g_n(z,\ 0) - M_n - \log\frac{1}{|z|}| \leqq K + \log\frac{1}{\rho_1} \quad \text{on} \quad |z| = \rho_1.$$

Since $v_n(z)$ is harmonic in $|z| \leqq \rho_1$,

$$|v_n(0)| = |\gamma_n - M_n| \leqq K + \log\frac{1}{\rho_1},$$

so that from (5),

$$\text{(6)} \qquad |g_n(z,\ 0) - \gamma_n| \leqq K + |\gamma_n - M_n| \leqq 2K + \log\frac{1}{\rho_1} \quad \text{in} \quad \Delta \quad \text{outside} \quad |z| = \rho_1.$$

Hence

$$\text{(7)} \qquad |g_n(z,\ 0) - \log\frac{1}{|z|} - \gamma_n| \leqq 2K + 2\log\frac{1}{\rho_1} \quad \text{on} \quad |z| = \rho_1.$$

Since the left hand side of (7) is harmonic in $|z| \leqq \rho_1$, (7) holds in $|z| \leqq \rho_1$, so that

$$\text{(8)} \qquad |g_n(z,\ 0) - \gamma_n| \leqq 2K + 2\log\frac{1}{\rho_1} + \log\frac{1}{|z|} \quad \text{in} \quad |z| \leqq \rho_1.$$

From (6), (8), we have the theorem. q.e.d.

By Theorem 3, we can find a partial sequence n_κ, such that

$$\text{(9)} \qquad \lim_{\kappa}\,(g_{n_\kappa}(z,\ 0) - \gamma_{n_\kappa}) = u(z,\ 0)$$

uniformly in any compact domain of F, which does not contain $z = 0$. $u(z,\ 0)$ is harmonic on F, except at $z = 0$, where it has a logarithmic singularity. Hence we have

THEOREM 4. *Let F be an open Riemann surface and z_0 be any point of F. then there exists a potential function $u(z,\ z_0)$, which is harmonic on F, except at z_0, where it has a logarithmic singularity, such that*

$$u(z,\ z_0) - \log\frac{1}{|z - z_0|}$$

is harmonic at z_0.

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