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ON THE UPPER AND LOWER CLASS FOR GAUSSIAN PROCESSES WITH SEVERAL PARAMETERS

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1. In the study on Hölder-continuity of Brownian motion, A.N.Kolmogorov introduced the concept of upper and lower classes and presented a criterion with the integral form to test whether some function belongs to upper or lower class; the so-called Kolmogorov's test (I.Petrovesky [10]). P.Lévy considered the upper and lower class with regard to the uniform continuity of Brownian motion. We shall recall the definition of the upper and lower classes. We shall call $\varphi(t)$ a function belonging to the upper class with regard to the uniform continuity of Brownian motion $x(t)$ if there exists a positive number $\varepsilon(w)$ such that, for almost all w,

$$|t-t'| \leqq \varepsilon(w) \qquad \text{implies}$$

$$|x(t)-x(t')| \leqq |t-t'|^{1/2}\cdot\varphi(1/|t-t'|). \tag{1.1}$$

On the otherhand, we shall call $\varphi(t)$ a function belonging to the lower class with regard to the uniform continuity of Brownian motion $x(t)$ if, for almost all w and for any positive number δ, there exist a pair (t, t') such that $|t-t'| \leqq \delta$ and (1.1) does not hold.*)

P.Lévy showed that the function

$$\varphi(t) = c\cdot(2\log t)^{1/2}$$

belongs to the upper class if $c>1$ and to the lower class if $c<1$ (P. Lévy [8]). Further, K.L.Chung, P.Erdös and T.Sirao [3] proved that a continuous, non-negative and non-decreasing function $\varphi(t)$ belongs to upper or lower class according as the integral

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*) It turns out that every continuous positive and non-decreasing function belongs to either upper class or lower class.

$$\int^{\infty}\varphi^3(t)\cdot\exp\Big(-\frac{1}{2}\varphi^2(t)\Big)dt$$

is convergent or divergent. T.Sirao [11] extended these results to the case of Brownian motion with several parameters of P.Lévy (P.Lévy [9]). Recently T.Sirao and H.Watanabe [12] have studied Hölder-continuity of a class of Gaussian processes and obtained a similar criterion of upper and lower classes. Their result is an extension of Yu.K. Belayev [1] in some sense.

In this paper we shall try to extend the result of T.Sirao and H.Watanabe [12] to the Gaussian processes with several parameters:
Consider the Gaussian processes $\{x(A,w);\ A\in D\}$ such that

$$E\{x(A)\}=0,$$

$$E\{x(A)\cdot x(B)\}=\frac{1}{2}\{d^{\alpha}(0,A)+d^{\alpha}(0,B)-d^{\alpha}(A,B)\}, \tag{1. 2}$$

where

$$D=\{A=(a_1,a_2,\cdots,a_N);\ a_i\in R,\ |a_i|\leqq 1,\ i=1,2,\cdots,N$$

$$0<\alpha\leqq 1,\ d^2(A,B)=\sum_{i=1}^{N}(a_i-b_i)^2 \text{ and } 0=(0,\cdots,0).$$

The right-hand side of (1. 2) becomes a positive definite kernel (R. Gangolli [6]). From (1. 2) we have

$$E\{(x(A)-x(B))^2\}=d^{\alpha}(A,B) \tag{1. 3}$$

and this condition implies the continuity of almost all sample paths (X. Fernique [5], R.M.Dudley [4]). We define, after P.Lévy and T.Sirao, the upper and lower class for the above Gaussian processes. Set

$$\sigma^2(A,B)=E\{(x(A)-x(B))^2\}.$$

If there exists a positive number $\varepsilon(w)$ such that, for almost all w, $d(A,B)\leqq\varepsilon(w)$ implies

$$|x(A)-x(B)|\leqq\sigma(A,B)\cdot\varphi(1/d(A,B)), \tag{1. 4}$$

then $\varphi(t)$ is called the function belonging to upper class with regard to the uniform continuity of this process.

On the otherhand if, for almost all w and for any positive number δ, there exists a pair (A,B) such that $d(A,B)\leqq\delta$ and (1. 4) does not hold, then

$\varphi(t)$ is defined as the function belonging to lower class with regard to the uniform continuity of the process. Denoting the upper and lower classes with regard to uniform continuity as U^u and L^u respectively, we have;

THEOREM. *Let $\varphi(t)$ be a positive, continuous and non-decreasing function of $t \in [e, \infty)$. Set*

$$K[x] = x^{\frac{4N}{\alpha}-1} \cdot \exp(-x^2/2) \tag{1.5}$$

and

$$I(\varphi) = \int_e^\infty t^{N-1} \cdot K[\varphi(t)]dt. \tag{1.6}$$

Then $\qquad \varphi(t) \in U^u \quad$ *if* $\quad I(\varphi) < \infty$

and $\qquad \varphi(t) \in L^u \quad$ *if* $\quad I(\varphi) = \infty$.

As the consequence, we have the following:

COROLLARY. *If we set for any positive number ε,*

$$\varphi_i(t, \varepsilon) = \left\{2N \log t + \left(\frac{4N}{\alpha} + 1 + (-1)^i \cdot \varepsilon\right) \cdot \log \log t\right\}^{1/2}, \quad i = 1, 2,$$

then $\qquad \varphi_1(t, \varepsilon) \in L^u$, $\varphi_2(t, \varepsilon) \in U^u$ *and* $\varphi_1(t, 0) \in L^u$.

If we set for any positive number ε and for any integer $n \geqq 3$,

$$\varphi_j(t, \varepsilon) =$$

$$\left\{2N \log t + \left(\frac{4N}{\alpha} + 1\right) \cdot \log_{(2)} t + 2(\log_{(3)} t + \cdots + \log_{(n-1)} t) + (2 + (-1)^j \cdot \varepsilon) \log_{(n)} t\right\}^{1/2},$$

$$j = 3, 4,$$

where $\qquad \log t = \log_{(1)} t, \quad \log_{(n)} t = \log(\log_{(n-1)} t),$

then $\qquad \varphi_3(t, \varepsilon) \in L^u$, $\varphi_4(t, \varepsilon) \in U^u$ *and* $\varphi_3(t, 0) \in L^u$.

The contents of the paper are as follows; In section 2, we shall show that it is enough to prove the theorem only for some restricted class of functions, which have, roughly speaking, the same order with $(\log t)^{1/2}$. The first half of section 3 is devoted to define the sequence of events and to order them with a numbering. This device is convenient throughout the proof, in particular, in referring Borel-Cantelli lemma and K.L. Chung-P. Erdös lemma. In the second half of section 3, Lemma 2 is stated, which is the key lemma in the proof of the theorem. Since many other lemmas are necessary for the proof of Lemma 2, it is postponed in section 5. The proof of theorem will be completed in section 4.

The author is greatly indebted to Professors T. Sirao and H. Watanabe who proposed the problem with several suggestions and communicated their result [12].

2. Let F be the class of functions:

$$F = \{f(t);\ f_1(t) \leqq f(t) \leqq f_2(t),\ \text{for}\ t \geqq e\}$$

where
$$f_1(t) = \left\{2N \log t + \left(\frac{4N}{\alpha} - 2\right) \log_{(2)} t\right\}^{1/2}$$

and
$$f_2(t) = \left\{2N \log t + \left(\frac{4N}{\alpha} + 2\right) \log_{(2)} t\right\}^{1/2}.$$

A computation shows that

$$I(f_1) = \infty \quad \text{and} \quad I(f_2) < \infty. \tag{2.1}$$

Then we have

LEMMA 1. *The theorem holds under the general situation if it is valid only for $f(t) \in F$,* i.e. *it suffices to prove it only for $f(t) \in F$.*

Proof. For a function $\varphi(t)$ cited in theorem, set

$$\hat{\varphi}(t) = (\varphi(t) \vee f_1(t)) \wedge f_2(t)^{*)}.$$

We see easily that $\hat{\varphi}(t) \in F$.

Case 1: $I(\varphi) < \infty$. For any monotone increasing sequence $\{t_m\}$ it holds for all sufficiently large m

$$\varphi(t_m) > f_1(t_m). \tag{2.2}$$

In fact, if there exists a monotone increasing sequence $\{t_m\}$ such that for any m $\varphi(t_m) \leqq f_1(t_m)$, it yields a contradiction as follows. Set

$$t_e = \min\left\{t;\ \varphi(t) \geqq \left(\frac{4N}{\alpha} - 1\right)^{1/2}\right\},$$
$$t_n \geqq (2t_e \vee e^e).$$

Evaluate the integral

$$\int_{t_e}^{t_n} t^{N-1} \cdot K[\varphi(t)] dt,$$

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*) $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$

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which is not greater than $I(\varphi)$. Since $K[x]$ is monotone decreasing for $x \geqq \left(\frac{4N}{\alpha}-1\right)^{1/2}$, we have by the assumption that

$$I(\varphi) \geqq \frac{1}{N}(t_n^N - t_e^N) \cdot K[f_1(t_n)]$$

$$\geqq \frac{1}{N}(2N)^{\frac{1}{2}\left(\frac{4N}{\alpha}-1\right)} \cdot \left(1-\frac{1}{2^N}\right) \cdot (\log t_n)^{1/2} \to \infty, \quad n \to \infty$$

which contradicts the assumption, $I(\varphi) < \infty$. Hence, for large t, we obtain (2. 2), which implies

$$\hat{\varphi}(t) \leqq \varphi(t). \tag{2. 3}$$

Moreover, we have

$$I(\hat{\varphi}) < \infty. \tag{2. 4}$$

Really, separating the domain of integration of $I(\hat{\varphi})$ into two parts; $\{t;\ \varphi < f_2\}$ and $\{t;\ \varphi \geqq f_2\}$, it holds

$$I(\hat{\varphi}) \leqq I(\varphi) + I(f_2),$$

where both in the right-hand side are finite. Therefore, from (2. 3) we can conclude $\varphi(t) \in U^u$ if we can show that (2. 4) implies $\hat{\varphi}(t) \in U^u$.

Case 2: $I(\varphi) = \infty$. In this case $I(\hat{\varphi}) = \infty$. Actually, if there exists a monotone increasing sequence $\{t_n\}$ for which $\varphi(t_n) < f_1(t_n)$ holds for all sufficiently large n, the fact $I(\hat{\varphi}) \to \infty$ is similarly shown as well as in the Case 1. On the other hand, if $f_1(t) < \varphi(t)$ for large t, then we have $\hat{\varphi}(t) \leqq \varphi(t)$. Using the monotone property of the function $K[x]$ for $x \geqq \left(\frac{4N}{\alpha}-1\right)^{1/2}$, we evaluate by Lemma 5 (a) below

$$\begin{aligned} I(\hat{\varphi}) &= \int_e^\infty t^{N-1} \cdot K[\hat{\varphi}(t)]dt \\ &\geqq \int_{e^{2/\alpha}}^\infty t^{N-1} \cdot K[\varphi(t)]dt \\ &= I(\varphi) - \int_e^{e^{2/\alpha}} t^{N-1} \cdot K[\varphi(t)]dt \\ &= \infty. \end{aligned}$$

Thus we have $I(\hat{\varphi}) = \infty$ in either case. In the sequel it suffices to prove that $\hat{\varphi}(t) \in L^u$ implies $\varphi(t) \in L^u$. The assumption $\hat{\varphi}(t) \in L^u$ implies that for almost all w and every positive ε_1, a pair (A, B) can be found such that $d(A, B) < \varepsilon_1$ and

$$|x(A, w) - x(B, w)| > \sigma(A, B) \cdot \hat{\varphi}(1/d(A, B)) \tag{2. 5}$$

happens. On the other hand, since $I(f_2) < \infty$ and $f_2(t) \in F$, we know $f_2(t) \in U^u$ in view of the assumption of Lemma 1. This means that for almost all w, there exists $\varepsilon_2(w) > 0$ such that if $d(A, B) < \varepsilon_2(w)$, then

$$|x(A, w) - x(B, w)| \leqq \sigma(A, B) \cdot f_2(1/d(A, B)). \tag{2. 6}$$

From (2. 5) and (2. 6) we obtain that for $d(A_n, B_n) < \varepsilon(w)$ for all large n it holds

$$\hat{\varphi}(1/d(A_n, B_n)) < f_2(1/d(A_n, B_n)).$$

Relying on the definition of $\hat{\varphi}(t)$ we have for $d(A, B) \leqq \varepsilon(w)$

$$\hat{\varphi}(1/d(A_n, B_n)) \geqq \varphi(1/d(A_n, B_n)),$$

which implies $\qquad \varphi(t) \in L^u$.

3. We define the point-sets as follows, which are really the set of partition-point of D. Using these point-sets, the sequence of events will be defined. Several evaluations for upper and lower classes will be developed by these terms.

Let p take integers; $p - [e^{3/\alpha}]$*) $= 1, 2, 3, \cdots\cdots$. Set the point-sets for each p:

$$\boldsymbol{B}^{(p)} = \{B;\ B = (k_i/2^p) \in D,\ k_i = \pm 1, \cdots \pm 2^p,\ i = 1, 2, \cdots, N\}$$
$$\boldsymbol{L}^{(p)} = \{L;\ L = (l_i/2^p),\ l_i = 0, \pm 1, \pm 2, \cdots.\ i = 1, 2, \cdots, N\}$$
$$\boldsymbol{A}^{(p)} = \{A;\ A = B + L \in D,\ B \in B^{(p)},\ L \in \boldsymbol{L}^{(p)},\ 1/3v_p \leqq d(A, B) \leqq 1/v_p\},$$
$$\boldsymbol{Q}^{(p,d)} = \{Q;\ Q = (m_i^{(d)} \cdot e^{-dc}/2^p),\ m_i^{(d)} = 0, \pm 1, \cdots, \pm e^{dc},\ i = 1, 2, \cdots, N\},$$
$$d = 1, 2, \cdots,$$

where $v_p = 2^p/p^{1/\alpha}$ and c denotes some large number which makes e^c an integer (c.f. Lemma 9 below) and (a_i) denotes $(a_1, a_2, \cdots, a_n)$.

$$\boldsymbol{A}^{(p,d)} = \{A';\ A' = A + Q \in D,\ A \in \boldsymbol{A}^{(p)},\ Q \in \boldsymbol{Q}^{(p,d)}\},$$
$$\boldsymbol{B}^{(p,d)} = \{B';\ B' = B + Q \in D,\ B \in \boldsymbol{B}^{(p)},\ Q \in \boldsymbol{Q}^{(p,d)}\},$$

*) $[x]$ indicates the integral part of x.

$$\boldsymbol{X}^{(p)}(A) = \{X;\ X = (x_i),\ (k_i + l_i - 1)/2^p \leqq x_i \leqq (k_i + l_i - 1)/2^p,\ A = (k_i + l_i/2^p) \in \boldsymbol{A}^{(p)}\},$$

$$\dot{\boldsymbol{Y}}^{(p)}(B) = \{Y;\ Y = (y_i),\ (k_i - 1)/2^p \leqq y_i \leqq (k_i + 1)/2^p,\ B = (k_i/2^p) \in \boldsymbol{B}^{(p)}\}.$$

For a function $\varphi(t)$, define the sequence of function $\{\lambda_{(d)}(\varphi)\}$ by

$$\lambda_{(d)}(\varphi)(t) = \varphi(t) + \frac{2N\cdot c}{\varphi(t)}\cdot\sum_{l=0}^{d-1}(1/2^\alpha)^l,\ d = 1, 2, \cdots,$$

$$\lambda_{(\infty)}(\varphi)(t) = \varphi(t) + \frac{2N\cdot c'}{\varphi(t)},\qquad c' = 2^\alpha\cdot c/(2^\alpha - 1).$$

Further, we define the three types of events by

(1) for $P \in \boldsymbol{A}^{(p)}$ and $Q \in \boldsymbol{B}^{(p)}$

$$E(P,Q) = \{w;\ x(P,w) - x(Q,w) > \sigma(P,Q)\cdot\varphi(1/d(P,Q))\},$$

(2) for $P = A + Q_1 \in \boldsymbol{A}^{(p,d)}$ and $Q = B + Q_2 \in \boldsymbol{B}^{(p,d)}$

$$F^{(d)}(P,Q) = \{w;\ x(P,w) - x(Q,w) > \sigma(P,Q)\cdot\lambda_{(d-1)}(\varphi)(1/d(A,B))\},$$

(3) for $A = B + L \in \boldsymbol{A}^{(p)}$ and $B \in \boldsymbol{B}^{(p)}$

$$\tilde{E}(A,B) = \{w;\ \max_{\substack{P\in X^{(p)}(A)\\ Q\in Y^{(p)}(B)}}(x(P,w) - x(Q,w))/\sigma(P,Q) \geqq \lambda_{(\infty)}(\varphi)(1/d(A,B))\}.$$

For the collection of events; $\mathscr{E} = \{E(P,Q);\ P \in \boldsymbol{A}^{(p)},\ Q \in \boldsymbol{B}^{(p)},\ P = Q + L,\ L \in \boldsymbol{L}^{(p)}\}$, we shall order as follows where $\mathscr{E}$ is defined: If $E_n = E(P,Q) \in \mathscr{E}$ where $P = (k_i + l_i/2^p) \in \boldsymbol{A}^{(p)}$, $Q = (k_i/2^p) \in \boldsymbol{B}^{(p)}$ and if $E_m = E(P', Q',) \in \mathscr{E}$ where $P' = (k'_i + l'_i/2^p) \in \boldsymbol{A}^{(p')}$, $Q' = (k'_i/2^p) \in \boldsymbol{B}^{(p')}$, then $n < m$ holds if and only if

(a) $p < p'$

or

(b) $\|l'\| < \|l\|$ when $p = p'$

or

(c) $k_i < k'_i$ $(i \leqq N)$ when $p = p'$, $\|l'\| = \|l\|$ and $k_j = k'_j$ $(j = 1, \cdots, i-1)$

or

(d) $l_i < l'_i$ $(i \leqq N)$ when $p = p'$, $\|l'\| = \|l\|$, $k_i = k'_i$ for all i and $l_j = l'_j$ $(j = 1, 2, \cdots, i-1)$,

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where $\|l\|$ denotes $(\sum_{i=1}^{N} l_i^2)^{1/2}$.

Also we give the same numbering for event in $\tilde{\mathscr{E}} = \{\tilde{E}(A,B);\ A \in \boldsymbol{A}^{(p)}, B \in \boldsymbol{B}^{(p)},\ A = B + L,$ for all $p\}$, i.e. when $E_n = E(A,B)$, then $\tilde{E}_n = \tilde{E}(A,B)$.

Under these preparations Lemma 2 can be stated as follows;

LEMMA 2.

[2. 1] *If* $I(\varphi) < \infty$, *then* $\sum_{n=1}^{\infty} P(\tilde{E}_n) < \infty$.

[2. 2] *If* $I(\varphi) = \infty$, *then* $\sum_{n=1}^{\infty} P(E_n) = \infty$.

[2. 3] *For each* n, $\lim_{m\to\infty} \rho(u_n, u_m) = 0$,

where
$$u_k = (x(A_k) - x(B_k))/\sigma(A_k, B_k)$$
for $A_k = P$, $B_k = Q$, *respectively if* $E_k = E(P,Q) \in \mathscr{E}$, *and* $\rho(u_n, u_m)$ *denotes the correlation-coefficient between* u_n *and* u_m.

[2. 4] *There exist two absolute constants* k_1 *and* k_2 *with the following property: to each* E_j *there corresponds a finite set of events* $\boldsymbol{E}_j = \{E_{j1}, E_{j2}, \cdots, E_{js(j)}\}$ $(E_{ji} \in \mathscr{E})$ *such that*

$$\sum_{i=1}^{s(j)} P(E_j \cap E_{ji}) < k_1 \cdot P(E_j) \tag{3. 1}$$

and if E_k *is not in* $\boldsymbol{E}_j$ $(k > j)$, *then*

$$P(E_j \cap E_k) < k_2 \cdot P(E_j) \cdot P(E_k). \tag{3. 2}$$

4. (*Proof of theorem*). In view of Lemma 1 and symmetric property of the process $\{x(A,w);\ A \in D\}$ it is enough to prove only for $\varphi(t) \in F$ and for the events defined in **3**.

Case 1: $I(\varphi) < \infty$. From Lemma 2, [2,1] and by Borel-Cantelli lemma, for almost all w there exists the number $n_0(w)$, namely $p_0(w)$ such that for any $p > p_0(w)$, $\tilde{E}(A,B)$ $(A \in \boldsymbol{A}^{(p)},\ B \in \boldsymbol{B}^{(p)})$ can not occur. Set

$$p_1(w) = p_0(w) \vee ([6\sqrt{N}\,] + 1) \vee \left(\left[\frac{10c'}{\alpha \log 2} + 2\sqrt{N}\right] + 1\right).$$

Take any points A and B satisfying the relation

$$d(A,B) < (p_1^{1/\alpha}(w) - 2\sqrt{N}\,)/2^{p_1(w)}. \tag{4. 1}$$

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Then, since $(x^{1/\alpha}-2\sqrt{N})/2^x$ is monotone decreasing for $x>\frac{1}{\alpha\log 2}+2\sqrt{N}$, there exists a integer $p(\geqq p_1)$ such that

$$((p+1)^{1/\alpha}-2\sqrt{N})/2^{p+1}\leqq d(A,B)\leqq(p^{1/\alpha}-2\sqrt{N})/2^p \tag{4. 2}$$

holds. For this p, choose points $A'\in \boldsymbol{A}^{(p)}$ and $B'\in \boldsymbol{B}^{(p)}$ satisfying the following both conditions;

$$d(A,B)\leqq d(A',B'), \tag{4. 3}$$

$$\{d(A,A')+d(B,B')\}=\min_{\substack{X\in \boldsymbol{A}^{(p)}\\ Y\in \boldsymbol{B}^{(p)}}}\{d(A,X)+d(B,Y)\}.$$

Then it holds clearly that

$$d(A,A')\leqq\sqrt{N}/2^p \quad\text{and}\quad d(B,B')\leqq\sqrt{N}/2^p.$$

Hence by (4. 2) we have

$$1/3v_p\leqq d(A',B')\leqq 1/v_p.$$

Since this implies $E(A',B')\in\mathscr{E}$, we can see that the event $\tilde{E}(A',B')$ belongs to $\tilde{\mathscr{E}}$. We have, therefore, because of (4. 1) and (4. 3)

$$x(A,W)-x(B,W)\leqq\sigma(A,B)\cdot\lambda_{(\infty)}(\varphi)(1/d(A',B')).$$

Since $\lambda_{(\infty)}(\varphi)(v_p)$ is monotone decreasing for $p>10c'/\log 2$, we obtain by (4. 3)

$$x(A,W)-x(B,W)\leqq\sigma(A,B)\lambda_{(\infty)}(\varphi)(1/d(A,B)),$$

which for (4. 1) implies

$$\varphi(t)+\frac{2N\cdot c'}{\varphi(t)}\in U^u.$$

In order to assure that $\varphi(t)$ itself belongs to U^u, we set

$$\eta(t)=\varphi(t)-\frac{10Nc'}{\varphi(t)}.$$

Then a computation show that $I(\eta)\leqq e^{10Nc'}\cdot I(\varphi)$. Accordingly $I(\eta)<\infty$ holds under the condition $I(\varphi)<\infty$.

This implies from the above argument that

$$\eta(t)+\frac{2Nc'}{\eta(t)}\in U^u.$$

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Since for large t

$$\eta(t)+\frac{2Nc'}{\eta(t)}\leqq\varphi(t),$$

we have $\varphi(t)\in U^u$. This completes the proof of theorem in the case $I(\varphi)<\infty$.

Case 2: $I(\varphi)=\infty$. First of all we shall show that for elements of $\mathscr{E}$; $E_h, E_{h+1}, \cdots, E_{h+(n-h)}\,(m\geqq n\geqq h)$ it holds for large m,

$$P(E_m/E'_h\cap\cdots\cap E'_{h+(n-h)})>\frac{1}{8}\cdot P(E_m), \tag{4. 4}$$

where E'_k denotes the complement of E_k and $P(E_m/C)$ the conditional probability of E_m under the condition C. Corresponding to $E_m(=E(A,B))$ and $E_l(=E(P,Q))\,(l=h,\cdots,n)$, we define their subsets E^*_m and $\dot{E}_l$ by

$$E^*_m=\{w;\,\sigma(A,B)\cdot\varphi(1/d(A,B))<x(A,w)-x(B,w)<2\sigma(A,B)\cdot\varphi(1/d(A,B))\},$$

and

$$\dot{E}_l=\{w;\,x(P,w)-x(Q,w)\in B_l\},\quad l=h,\cdots,n,$$

where B_l is any bounded Borel set. If we set $\varepsilon(m)$ by

$$P(E^*_m/\dot{E}_h\cap\cdots\cap\dot{E}_n)=(1+\varepsilon(m))\cdot P(E^*_m),$$

then $\varepsilon(m)$ is a function of $\rho(u_l,u_m)$, $(l=h,\cdots,n)$ and of $\sigma(A,B)\cdot\varphi(1/d(A,B))$. Then we obtain from Lemma 2, [2,3]

$$\varepsilon(m)\longrightarrow 0\quad\text{as}\quad\max_l\rho(u_l,u_m)\longrightarrow 0$$

(T. Sirao [11]). Therefore, we have for large m,

$$P(E^*_m/\dot{E}_h\cap\cdots\cap\dot{E}_n)>\frac{1}{2}\cdot P(E^*_m). \tag{4. 5}$$

On the otherhand, we see easily

$$P(E_m)<2P(E^*_m) \tag{4. 6}$$

for all large m. Further, if we denote the event that u_l+a is positive by $E_l(a)$, it is clear that

$$P(E_l(a))\longrightarrow 1\quad\text{as}\quad a\longrightarrow\infty.$$

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Choosing, therefore, $a_{h,n}$ for each pair of positive integers h and n $(h \leqq n)$ such that

$$P\{\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))\} \geqq P(\bigcap_{l=h}^{n} E'_l)/2,$$

we have

$$P(E_m/E'_h \cap \cdots \cap E'_n) \geqq P(E_m/\bigcap_{l=h}^{n}(E'_l \cap E_l(a_{h,n}))/2. \tag{4. 7}$$

Therefore taking $\dot{E}_l = E'_l \cap E_l(a_{h,n})$, $(l = h, \cdots, n)$ in (4. 5), it yields (4. 4) (T. Sirao [11]). (4. 4) together with Lemma 2, [2. 2] and Lemma 2, [2. 4] implies that

$$P(E_n \text{ occur infinitely often}) = 1,$$

(K.L.Chung and P.Erdös [2]), which implies $\varphi(t) \in L^u$. This establishes the proof of theorem in the Case 2, $I(\varphi) = \infty$.

5. (*Proof of Lemma* 2.) This section is devoted to prove Lemma 2. In the first place Lemma 2, [2. 1] is verified after several preparations. In the second place Lemma 2, [2. 2], [2. 3] and [2. 4] will be proved, which were required to refer K.L.Chung-P.Erdös lemma for the proof of lower class.

Following Lemma 3 will be cited so often from now on, but we shall omit the proof, because it is easy.

LEMMA 3.

(a) *For* $x \geqq 1$,

$$\frac{1}{2x}\exp(-x^2/2) \leqq \int_x^\infty \exp(-t^2/2)dt \leqq \frac{1}{x}\exp(-x^2/2).$$

(b) *At* $x = e$, $\log x/x$ *attains its maximum and for* $x > e$ *it is monotone decreasing.*

(c) $\log_{(2)}x/\log x$ *attains its maximum at* $x = e^e$ *and monotone decreasing for* $x > e^e$.

(d) *For sufficiently large* x (e.g. $x \geqq e^{3/\alpha}$) *and for* $0 < \alpha \leqq 1$,

$$(\alpha \cdot \log 2) \cdot (\log x/x) < 1$$

LEMMA 4. *For* $\varphi(t) \in F$, *it yields;*

(a) *For* $t > e$,

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$$(2N\log t)^{1/2}\leqq \varphi(t)\leqq c_1\cdot(2N\log t)^{1/2},$$

where
$$c_1=\left\{1+\left(\frac{4}{\alpha}+\frac{2}{N}\right)\frac{1}{2e}\right\}^{1/2}.$$

(b) *For sufficiently large* p (*e.g.* $p>e^{3/\alpha}$),

$$\varphi(v_p)>1 \quad \textit{and} \quad \varphi(3v_p)>1.$$

(c) *For the same* p *in* (b),

$$c_2\cdot p\leqq\varphi^2(v_p)\leqq p\cdot\{c_1^2 2N\log 2\},$$
$$c_2\cdot p\leqq\varphi^2(3v_p)\leqq c_3\cdot p$$

where
$$c_2=2N\cdot\log 2\cdot\left\{1-\frac{3}{\alpha^2\cdot\log 2\cdot e^{3/\alpha}}\right\},$$

and
$$c_3=c_1^2\cdot 2N\log 2\cdot\left(1+\frac{\log 3}{2\log 2}\right).$$

Proof. (a). Since $\varphi(t)\in F$ implies that

$$(2N\log t)^{1/2}\cdot\left\{1+\left(\frac{4}{\alpha}-\frac{2}{N}\right)\cdot\frac{\log_{(2)}t}{2\log t}\right\}^{1/2}\leqq\varphi(t$$
$$\leqq(2N\cdot\log t)^{1/2}\cdot\left\{1+\left(\frac{4}{\alpha}+\frac{2}{N}\right)\cdot\frac{\log_{(2)}t}{2\log t}\right\}^{1/2},$$

we obtain (a) due to Lemma 3, (c).

(b). From Lemma 4, (a) it is sufficient to choose so large p that

$$v_p>e, \quad \text{i.e.}$$
$$p\log 2-\frac{1}{\alpha}\log p>1.$$

Since from Lemma 3, (b) we have

$$p\log 2\left\{1-\frac{\log p}{p}\cdot\frac{1}{\alpha\log 2}\right\}>p\cdot\log 2\left\{1-\frac{3}{\alpha^2\cdot\log 2\cdot e^{3/\alpha}}\right\}$$

for $p>e^{3/\alpha}>e$, we obtains (b). The latter of (b) could be checked similarly.

(c). From the above results (a) and (b), we have for $p>e^{3/\alpha}$,

$$2N\cdot\log v_p\leqq\varphi^2(v_p)\leqq c_1^2 2N\log v_p.$$

If we set c_2 as cited above, we find (c). The latter part of (c) is similarly derived.

LEMMA 5. *For* $\varphi(t) \in F$ *we have;*

(a) *For sufficiently large* t *(e.g.* $t \geqq e^{2/\alpha}$*),* $K[\varphi(t)]$ *is monotone decreasing in* t*.*

(b) *For* $p > 2 \cdot e^{3/\alpha}/\alpha$*,* $K[\varphi(v_p)]$ *is monotone decreasing in* p *and so does* $K[\varphi(3v_p)]$*.*

Proof. (a). Since $K[x]$ is monotone decreasing for $x \geqq \left(\frac{4N}{\alpha} - 1\right)^{1/2}$, it suffices to find t such that $\varphi(t) \geqq \left(\frac{4N}{\alpha} - 1\right)^{1/2}$.

By Lemma 4, (a) we can find it as follows;

$$2N \log t \geqq \frac{4N}{\alpha} \geqq \frac{4N}{\alpha} - 1, \quad \text{i.e.} \quad t \geqq e^{2/\alpha}$$

(b). In view of the above (a), it is enough to choose p such that

$$1/v_p \geqq e^{2/\alpha}, \quad \text{i.e.}$$

$$p \cdot \log 2 \cdot \left\{1 - \frac{\log p}{p} \cdot \frac{1}{\alpha \log 2}\right\} \geqq 2/\alpha\,.$$

Using the same argument as well as in the proof of Lemma 4, (b), we have (b) for $p \geqq 2 \cdot e^{3/\alpha}/\alpha$. The remaining part is obtained similarly.

LEMMA 6. *For* $E_n = E(A, B) \in \mathscr{E}$*, we have*

$$\exp\left\{-\frac{1}{2}\varphi^2(1/d(A, B))\right\} \Big/ 2\sqrt{2\pi} \cdot \varphi(1/d(A, B)) \leqq p(E_n) \leqq$$

$$\leqq \exp\left\{-\frac{1}{2}\varphi^2(1/d(A, B))\right\} \Big/ \sqrt{2\pi} \cdot \varphi(1/d(A, B))\,.$$

Proof. It is evident from Lemma 4, (b), Lemma 3, (a) and from the fact;

$$\sqrt{2\pi} \cdot P(E_n) = \int_{\varphi(1/d(A, B))}^{\infty} \exp(-t^2/2)\,dt\,.$$

LEMMA 7. *If* $I(\varphi) < \infty$*, then we have*

$$\sum_{n=1}^{\infty} P(E_n) < \infty.$$

Proof. Set the number corresponding to $p_0 = [2e^{3/\alpha}/\alpha] + 1$ by n_0. It is enough to check only the following;

$$\sum_{n=n_0}^{\infty} P(E_n) \leqq \sum_{\substack{E(A,B)\in\mathscr{E} \\ \text{for } p=p_0}} P(E(A,B))$$

$$\leqq \frac{1}{\sqrt{2\pi}} \cdot \sum_{p=p_0}^{\infty} (2.\ 2^p)^N \cdot (p^{1/\alpha})^N \cdot \exp\left\{-\frac{1}{2}\varphi^2(1/d(A,B)\right\} \Big/ \varphi(1/d(A,B)).$$

By Lemma 5, (b) and Lemma 4, (c)

$$\sum_{n=n_0}^{\infty} P(E_n) \leqq c_4 \sum_{p=p_0}^{\infty} v_p^{N-1} \cdot (v_p - v_{p-1}) p^{2N/\alpha} \exp\left(-\frac{1}{2}\varphi^2(v_p)\right) \Big/ \varphi(v_p)$$

$$\leqq c_4 \cdot c_2^{-2N/\alpha} \cdot \sum_{p=p_0}^{\infty} \int_{v_{p-1}}^{v_p} v_p^{N-1} K[\varphi(v_p)] dt$$

$$\leqq 2^{N-1} \cdot c_4 \cdot c_2^{-2N/\alpha} \sum_{p=p_0}^{\infty} \int_{v_{p-1}}^{v_p} t^{N-1} K[\varphi(t)] dt$$

where
$$c_4 = \frac{2^N}{\sqrt{2\pi}} \cdot \left\{1 - \frac{1}{2}\left(\frac{p_0}{p_0 - 1}\right)^{1/\alpha}\right\}^{-1}.$$

As a consequence, we obtain

$$P(E_n) \leqq 2^{N-1} c_4 \cdot c_2^{-2N/\alpha} \cdot \int_{v_{p_0}-1}^{\infty} t^{N-1}\, K[\varphi(t)] dt$$

$$\leqq 2^{N-1} \cdot c_4 \cdot c_2^{-2N/\alpha} \cdot I(\varphi),$$

which completes the proof of Lemma 7.

The above Lemma 7 plays an essential role to prove Lemma 2, [2. 1].

LEMMA 8. *For any pair of points* (A, B), *where* $A \in \boldsymbol{A}^{(p,d)}$, $B \in \boldsymbol{B}^{(p,d)}$ *and* $A - B = L + Q_1 + Q_2$, $L \in \boldsymbol{L}^{(p)}$, Q_1, $Q_2 \in \boldsymbol{Q}^{(p,d)}$, *if we choose properly a pair* (A', B') *such that* $A' \in \boldsymbol{A}^{(p,d-1)}$ *and* $B' \in \boldsymbol{B}^{(p,d-1)}$, *and* $A' - B' = L + Q_1' + Q_2'$ (Q_1' *and* $Q_2' \in \boldsymbol{Q}^{(p,d-1)}$), *we have*

$$\rho(u, u') \geqq 1 - (1+\alpha)\left\{3\sqrt{N} \Big/ \sqrt{1 - \frac{12\sqrt{N}}{p^{1/\alpha}}} \cdot p^{1/\alpha} e^{(d-1)c}\right\}^{\alpha}, \tag{5. 1}$$

where $u = x(A) - x(B)$, $u' = x(A') - x(B')$ *and* $p \geqq [12\sqrt{N}\, e^{3/\alpha}] + 1$.

Proof. Choose A' and B' as follows: If i-th coordinate of point A is not smaller than i-th coordinate of point B, that is, if $k_i + l_i + m_i^{(d)} e^{-dc} \geqq k_i + n_i^{(d)} e^{-dc}$, then we choose the point $A' \in \boldsymbol{A}^{(p,d-1)}$ whose i-th coordinate is greater than i-th coordinate of A and has the minimum distance, i.e.

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$$A' = ((k_i+l_i+m_{io}^{(d-1)}\cdot e^{-(d-1)c})/2^p), \text{ for which}$$

(5. 2)
$$m_{io}^{(d-1)}e^{-(d-1)c} > m_i^{(d)}e^{-dc},$$

$$m_{io}^{(d-1)}\cdot e^{-(d-1)c} - m_i^{(d)}e^{-dc} \leqq m_i^{(d-1)}\cdot e^{-(d-1)c} - m_i^{(d)}e^{-dc}$$

for any $A = ((k_i+l_i + m_i^{(d-1)}\cdot e^{-(d-1)c})/2^p) \in \boldsymbol{A}^{(p,d-1)}$.

Further choose the point $B' \in \boldsymbol{B}^{(p,d-1)}$, whose i-th coordinate is smaller than i-th coordinate of B and has the minimum distance, i.e.

$$B' = \{(k_i + n_{io}^{(d-1)}\cdot e^{-(d-1)c})/2^p\}$$

for which

$$n_{io}^{(d-1)}\cdot e^{-(d-1)c} < n_i^{(d)}\cdot e^{-dc},$$

(5. 3)
$$n_i^{(d)}\cdot e^{-dc} - n_{io}^{(d-1)}\cdot e^{-(d-1)c} < n_i^{(d)}\cdot e^{-dc} - n_i^{(d-1)}\cdot e^{-(d-1)c}$$

for any $B = \{(k_i + n_i^{(d-1)}\cdot e^{-(d-1)c})/2^p\} \in B^{(p,d-1)}$.

In the other case, i.e. if i-th coordinate of A is smaller than that of B, then choose the points A' and B' whose coordinates satisfy the reversed inequality respected to (5. 2) and (5. 3), respectively. For points chosen as above, we see

(5. 4)
$$d(A,A') \leqq \sqrt{N}\ e^{-(d-1)c}/2^p, \quad d(B,B') \leqq \sqrt{N}\ e^{-(d-1)c}/2^p,$$

$$d(A,B) \leqq d(A',B) \quad \text{and} \quad d(A,B) \leqq d(A,B').$$

Then we have

$$\rho(u,u') \geqq \frac{\{2\cdot d^\alpha(A,B) - d^\alpha(A,A') - d^\alpha(B,B')\}}{2\{d^\alpha(A,B)\cdot d^\alpha(A'B')\}^{1/2}}$$
$$\geqq 1-(1+\alpha)m^\alpha,$$

where
$$m = d(A,A')/d(A,B) \vee d(B,B')/d(A,B).$$

In order to estimate m, it requires to evaluate $d(A,B)$:

$$(2^p)^2\cdot d(A,B)^2 = \sum_{i=1}^N \{l_i+(m_i^{(d)} - n_i^{(d)})e^{-dc}\}^2$$

$$\geqq (\sum_{i=1}^N l_i^2)\cdot\left\{1 - \frac{2\cdot\sum_{i=1}^N l_i(m_i^{(d)} - n_i^{(d)})e^{-dc}}{(\sum_{i=1}^N l_i^2)}\right\}$$

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$$\geqq (\sum_{i=1}^{N} l_i^2)\cdot\left\{1-\frac{2\{\sum_{i=1}^{N}(m_i^{(d)}-n_i^{(d)})^2 e^{-2dc}\}^{1/2}}{(\sum_{i=1}^{N} l_i^2)^{1/2}}\right\}$$

$$\geqq (\sum_{i=1}^{N} l_i^2)\cdot\left\{1-\frac{4\sqrt{N}}{(\sum_{i=1}^{N} l_i^2)^{1/2}}\right\}$$

$$\geqq \left(\frac{1}{3}p^{1/\alpha}\right)^2\left(1-\frac{4\sqrt{N}}{\frac{1}{3}p^{1/\alpha}}\right).$$

From this and (5. 4) we have Lemma 8.

In the following Lemma 9, notation $A^{(p,d)}$ and $B^{(p,d)}$ are used for the fixed points A and B in their definition, respectively.

LEMMA 9. *For some fixed points A and B*

$$P(\bigcup_{\substack{P\in A^{(p,d)}\\ Q\in B^{(p,d)}}} F^{(d)}(P,Q))\leqq C_5 P(E(A,B))$$

where $p=[12Ne^{3/\alpha}]+1$ and c_5 is an absolute constant.

Proof. We shall prove by induction on d. By de Morgan's law, we see

$$P(\bigcup_{\substack{P\in A^{(p,d)}\\ Q\in B^{(p,d)}}} F^{(d)}(P,Q))\leqq P(\bigcup_{\substack{X\in A^{(p,d-1)}\\ Y\in B^{(p,d-1)}}} F^{(d-1)}(X,Y))+ \tag{5. 5}$$

$$+\sum_{\substack{P\in A^{(p,d)}\\ Q\in B^{(p,d)}}} P(\bigcap_{\substack{X\in A^{(p,d-1)}\\ Y\in B^{(p,d-1)}}} F'^{(d-1)}(X,Y)\cap F^{(d)}(P,Q)).$$

Set $s=1/d(A,B)$. For the case $d=1$ we can estimate as follows: For $P\in A^{(p,1)}$ and $Q\in B^{(p,1)}$, we have by Lemma 4, (b) and Lemma 3

$$P(F^{(1)}(P,Q))\leqq\frac{1}{\sqrt{2\pi}}\left(\varphi(s)+\frac{2Nc}{\varphi(s)}\right)^{-1}\exp\left\{-\frac{1}{2}\left(\varphi(s)+\frac{2Nc}{\varphi(s)}\right)^2\right\}$$

$$\leqq\frac{e^{-2Nc}}{\sqrt{2\pi}}\cdot\frac{1}{\varphi(s)}\exp\left\{-\frac{1}{2}\varphi^2(s)\right\}$$

$$\leqq 2e^{-2Nc}P(E(A,B)).$$

Since the number of all combinations of $P\in A^{(p,1)}$ and $Q\in B^{(p,1)}$ does not exceed $(2e^c)^N\times(2e^c)^N$, we obtains

(5. 6) $$P\Big(\bigcup_{\substack{P\in A^{(p,1)}\\ Q\in B^{(p,1)}}} F^{(1)}(P,Q)\Big) \leqq 2\cdot 4^N P(E(A,B).\ )$$

Next we consider the second term in (5. 5). If we choose the pair (P_0, Q_0), $P_0 \in A^{(p,d-1)}$, $Q_0 \in B^{(p,d-1)}$ as in Lemma 8 corresponding to each pair (P,Q), $P\in A^{(p,d)}$, $Q\in B^{(p,d)}$, which satisfy (5. 1), then we can see that

$$\bigcap_{\substack{X\in A^{(p,d-1)}\\ Y\in B^{(p,d-1)}}} \{F'^{(d-1)}(X,Y)\cap F^{(d)}(P,Q)\} \subset F'^{(d-1)}(P_0,Q_0)\cap F^{(d)}(P,Q)$$

since the set $\{(X,Y);\ X\in A^{(p,d-1)},\ Y\in B^{(p,d-1)}\}$ contains (P_0,Q_0). Hence in order to evaluate second term in (5. 5), it suffices to evaluate the probability of the right-hand side above:

$$P(F'^{(d-1)}(P_0,Q_0)\cap F^{(d)}(P,Q)) = \frac{1}{\sqrt{2\pi}}\int_{\lambda_{(d-1)}}^{\infty} P\Big[W \leqq \frac{\lambda_{(d-2)}-\rho t^2}{\sqrt{1-\rho^2}}\Big]\cdot e^{-\frac{t^2}{2}}dt,$$

where we write $\lambda_{(k)}$ briefly instead of $\lambda_k(\varphi)(1/d(A,B))$ for fixed A and B, and W (and V below) are mutually independent random variables of standard normal distribution. Since the last integral is monotone decreasing in ρ (T. Sirao [11]) and by replacing ρ by ρ_0 which is set by the right-hand side in (5. 1), we find

$$P(F'^{(d-1)}(P_0,Q_0)\cap F^{(d)}(P,Q)) \leqq P(V>\varphi(s))\cdot P(W\geqq -(1-\rho_0^2)^{1/2}\cdot\lambda_{(d-2)} + \\ + \{(1-\rho_0^2)^{-1/2}\cdot\rho_0\cdot 2Nc/\varphi(s)2^{\alpha(d-1)}\})$$

where $s = 1/d(A,B)$.
Further, for $p\geqq 12\sqrt{N}\cdot e^{3/\alpha}$ we estimate using (5. 1)

$$(1-\rho_0^2)^{1/2}\cdot\lambda_{(d-2)}$$

$$\leqq (6\sqrt{N}(1+\alpha))^{1/2}\cdot\Big\{\Big(1-\frac{12\sqrt{N}}{p^{1/\alpha}}\Big)^{1/4}\sqrt{p}\cdot e^{(d-1)c/2}\Big\}^{-1}\cdot\Big\{\varphi(s)+\frac{2Nc}{\varphi(s)}\cdot\sum_{l=0}^{d-2}\Big(\frac{1}{2^\alpha}\Big)^l\Big\}$$

$$\leqq (6\sqrt{N}(1+\alpha))^{1/2}\cdot\Big\{\Big(1-\frac{12\sqrt{N}}{p^{1/\alpha}}\Big)^{1/4}\cdot\sqrt{p}\cdot e^{(d-1)c/2}\Big\}^{-1}\cdot\Big\{\sqrt{p}\cdot c_1\cdot(2N\log 2)^{1/2}+\frac{2Nc'}{\sqrt{c_2 p}}\Big\}$$

$$\leqq (6N(1+\alpha))^{1/2}\cdot\Big\{c_1(2N\log 2)^{1/2}+\frac{2Nc'}{\sqrt{c_2\cdot e^{3/\alpha}}}\Big\}\Big/\Big(1-\frac{1}{e^{3/\alpha}}\Big)^{1/4}\equiv c_6 \qquad \text{(say)}.$$

In view of (5. 1) and Lemma 4, (c) we can estimate similarly;

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$$2Nc(1-\rho_0)^{-1/2}\cdot\rho_0/2^{\alpha(d-1)}\cdot\varphi(s)\geqq c_7\cdot(e^{c/2}/2)^{\alpha(d-1)},$$

where

$$c_7=\frac{2Nc}{(6\sqrt{N}\,(1+\alpha))^{1/2}\cdot c_1\cdot(2N\log 2)^{1/2}}\cdot\left\{1-\frac{(1+\alpha)}{\sqrt{1-\frac{1}{e^{3/\alpha}}\cdot e^{3/\alpha}}}\right\}\left(1-\frac{1}{e^{3/\alpha}}\right)^{1/4}.$$

As a consequence we have

$$P(F'^{(d-1)}(P_0,Q_0)\cap F^{(d)}(P,Q))\leqq P(V\geqq\varphi(s))\cdot P(W\geqq -c_6+c_7(e^{c/2}/2)^{\alpha(d-1)}).$$

In order to apply Lemma 3, (a) to the above, it is required to set c such that

$$-c_6+c_7(e^{c/2}/2)^{\alpha(d-1)}>1.$$

But for this it is enough to take c such that

$$c>\left\{\frac{2}{\alpha}\cdot\log\left(1+\frac{c_6'}{c_7'}\right)+\log 2\right\}+1$$

where c_6' and c_7' are constants dependent on c_6 and c_7, respectively. Choose such c and set c_8 by $c_8=c_7-\dfrac{c_6}{(e^{c/2}/2)^{\alpha(d-1)}}$.

Then we obtain

$$\begin{aligned}&P(F'^{(d-1)}(P_0,Q_0)\cap F^{(d)}(P,Q))\\ &\leqq P(E(A,B))\cdot(2/e^{c/2})^{\alpha(d-1)}\cdot\exp\left(-\frac{c_8^2}{2}\left(\frac{e^{c/2}}{2}\right)^{2\alpha(d-1)}\right)\Big/c_8\\ &\leqq c_9e^{-4dNc}\cdot P(E(A,B)),\end{aligned}$$

where $c_9=2^{l_0}\cdot l_0!/c_8^{2l_0+1}$ and $l_0=[4Nc/\alpha\log(e^{c/2}/2)]+1$.

Since the number of all combinations of $P\in\boldsymbol{A}^{(p,d)}$ and $Q\in\boldsymbol{B}^{(p,d)}$ does not exceed $(2e^{dc})^{2N}$, we have

$$\sum_{\substack{P\in\boldsymbol{A}^{(p,d)}\\ Q\in\boldsymbol{B}^{(p,d)}}}P\Big(\bigcap_{\substack{X\in\boldsymbol{A}^{(p,d-1)}\\ Y\in\boldsymbol{B}^{(p,d-1)}}}F'^{(d-1)}(X,Y)\cap F^{(d)}(P,Q)\Big)\leqq 2^N\cdot c_9e^{-2dNc}\cdot P(E(A,B)).\tag{5.7}$$

By (5.5), (5.6) and (5.7) we have

$$P\Big(\bigcup_{\substack{P\in\boldsymbol{A}^{(p,d)}\\ Q\in\boldsymbol{B}^{(p,d)}}}F^{(d)}(P,Q)\Big)\leqq 2^{3N+1}c_9\cdot P(E(A,B))\cdot\Big(\sum_{j=0}^{\infty}e^{-2jNc}\Big)$$

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Thus it completes the proof if we set a constant c_5 by $c_5 = 2^{3N+1} \cdot c_9/(1-e^{-2Nc})$ in statement of Lemma 9.

Lemma 2, [2. 1] follows from these preparations.

Proof of Lemma 2, [2. 1]. Continuity of the Gaussian process implies that for $A \in \boldsymbol{A}^{(p)}$ and $B \in \boldsymbol{B}^{(p)}$, $A = B + L (L \in \boldsymbol{L}^{(p)})$

$$\tilde{E}(A, B) \subseteq \bigcup_{h=1}^{\infty} \bigcup_{d=h}^{\infty} \{ \bigcup_{\substack{P \in \boldsymbol{A}^{(p,d)} \\ Q \in \boldsymbol{B}^{(p,d)}}} F^{(d)}(P, Q) \}.$$

Hence

$$P(\tilde{E}(A, B)) \leqq \varliminf_{d \to \infty} P(\bigcup_{\substack{P \in \boldsymbol{A}^{(p,d)} \\ Q \in \boldsymbol{B}^{(p,d)}}} F^{(d)}(P, Q))$$

$$\leqq c_5 P(E(A, B)).$$

Let the pair (A, B) runs over $\boldsymbol{A}^{(p)}$ and $\boldsymbol{B}^{(p)}$, and corresponds the numbering of $E(A, B)$ or $\tilde{E}(A, B)$ to each pair (A, B). Then from the assumption of Lemma 2, [2. 1] and Lemma 7 we have

$$\sum_{n=1}^{\infty} P(\tilde{E}_n) < \infty,$$

which proved the lemma.

Proof of Lemma 2, [2. 2]. We use the similar estimates as employed in Lemma 7. Set the number corresponding to $P_0 = [2e^{3/\alpha}/\alpha] + 1$ by n_0. For proof it is enough to show that if $I(\varphi) = \infty$, then $\sum_{n=n_0}^{\infty} P(E_n) = \infty$. Relying on Lemma 3 and Lemma 5, (b), we underestimate $P(E_n)$ as follows:

$$\sum_{n=n_0}^{\infty} P(E_n) = \sum_{\substack{E(A, B) \in \mathscr{E} \\ \text{for } p \geqq p_0}} P(E(A, B))$$

$$\geqq \frac{1}{2^{N+1}\sqrt{2\pi}} \cdot \sum_{p=p_0}^{\infty} \left\{ 2^{pN} \left(\frac{1}{3} p^{1/\alpha} \right)^N \exp\left(-\frac{1}{2} \varphi^2(3v_p) \right) \Big/ \varphi(3v_p) \right\}$$

$$\geqq \frac{1}{2^{N+1}\sqrt{2\pi}} \cdot \sum_{p=p_0}^{\infty} \left\{ (3v_p)^{N-1} (3v_{p+1} - 3v_p) (p^{1/\alpha}/3)^{2N} \cdot \right.$$

$$\left. \cdot \exp\left(-\frac{1}{2} \varphi^2(3v_p) \right) \Big/ \varphi(3v_p) \right\}$$

$$\geqq \frac{c_3^{-2N/\alpha}}{2^{N+1} 3^{2N} \sqrt{2\pi}} \sum_{p=p_0}^{\infty} \int_{3v_p}^{3v_{p+1}} (3v_p)^{N-1} K[\varphi(3v_p)] dt$$

$$\geqq \frac{c_3^{-2N/\alpha}}{2^{N+1}3^{2N}\sqrt{2\pi}}\left(\frac{1}{2}\left(\frac{2}{3}\right)^{1/\alpha}\right)^{N-1}\cdot\sum_{p=p_0}^{\infty}\int_{3v_p}^{3v_{p+1}} t^{N-1}K[\varphi(t)]dt.$$

Thus we obtain

$$\sum_{n=n_0}^{\infty} P(E_n)\geqq \frac{c_3^{-2N/\alpha}}{2^{N+1}3^{2N}\sqrt{2\pi}}\left(\frac{1}{2}\left(\frac{1}{3}\right)^{1/\alpha}\right)^{N-1}\cdot\left\{I(\varphi)-\int_e^{3v_{p_0}} t^{N-1}K[\varphi(t)]dt\right\}.$$

which verifies Lemma 2, [2. 2].

Proof of Lemma 2, [2. 3]. We recognize easily that it is sufficient to prove only for the subsequence $\{m_k\}$ such that $\rho(u_n, u_{m_k})\geqq 0$. Using the inequality $(a+b)^\alpha \leqq a^\alpha + b^\alpha$, $(0<\alpha\leqq 1,\ a>0,\ b>0)$ (Hardy-Littlewood-Pólya [7]), we have

(5. 8) $$0\leqq \rho(u_n, u_{m_k})\leqq d(A_{m_k}, B_{m_k})^{\alpha/2}/d(A_n, B_n)^{\alpha/2}\longrightarrow 0,\quad m_k\longrightarrow\infty$$

which, for each n, implies Lemma 2, [2. 3].

The following lemmas are prepared to refer in the proof of Lemma 2, [2. 4]. For the function $\varphi(t)$ and each E_j, we define $\boldsymbol{E}_j$ by

$$\boldsymbol{E}_j = \{E_n;\ \rho(u_j, u_n)\geqq 1/\{\varphi(s_j)\cdot\varphi(s_n)\},\ n>j\},$$

where s_k denotes $1/d(A_k, B_k)$ corresponding to $u_k=(x(A_k)-x(B_k))/\sigma(A_k, B_k)$. Then we have

LEMMA 10. *$\boldsymbol{E}_j$ is a finite set for each j.*

Proof. From (5. 8), we find

$$1/\{\varphi^2(s_j)\cdot\varphi^2(s_n)\}\leqq \rho^2(u_j, u_n)\leqq d^\alpha(A_n, B_n)/d^\alpha(A_j, B_j).$$

By Lemma 4, (c), the above becomes as follows in terms of p and p' corresponding to E_j and E_n, respectively;

$$(c_3^2pp')^{-1}\leqq\rho^2(u_j, u_n)\leqq\{3v_p/pp'\}^\alpha,$$

i.e.

$$(c_3^2pp')^{-1}\leqq 3^\alpha\cdot\left(\frac{p'}{p}\right)\Big/2^{\alpha(p'-p)}.$$

This yields

$$p'\leqq p+c_{10}\cdot\log p,$$

where $$c_{10}=2\log 2+\{\log(c_3^2 3^\alpha)/\alpha(\log 2)^2\}.$$

This assures the lemma.

By Lemma 10, we can write down for each j

$$\boldsymbol{E}_j = \{E_{j1}, E_{j2}, \cdots, E_{js(j)}\}, \quad ji > j.$$

LEMMA 11 (K.L.Chung, P.Erdös and T.Sirao [3]). *Let U and V be two random variables whose joint distribution is a two-dimensional Gaussian distribution and each of them is subjected to one-dimensional standard Gaussian distribution. Then,*

(a) *If $\rho(U,V) < 1/ab$, there exists a positive absolute constant d_1 such that*

$$P(U > a, \ V > b) \leqq d_1 P(U > a) \cdot P(V > b).$$

(b) *There exists two positive absolute constants d_2 and δ such that for any $a > 0$,*

$$P(U > a, V > a) \leqq d_2 \cdot \exp\{-\delta(1 - \rho^2(U, V)a^2)\} \cdot P(U > a).$$

For the proof of (3. 1) in Lemma 2, [2. 4] it is convenient to separate

$$\sum_{i=1}^{s(j)} P(E_j \cap E_{ji}) = \sum_{(1)} P(E_j \cap E_{ji}) + \sum_{(2)} P(E_j \cap E_{ji}),$$

where $\sum_{(1)}$ and $\sum_{(2)}$ denote summation over i satisfying

$$\rho(u_j, u_{ji}) > \left(1 - \frac{1}{\sqrt{p}}\right)^{1/2} \quad \text{and} \quad \rho(u_j, u_{ji}) \leqq \left(1 - \frac{1}{\sqrt{p}}\right)^{1/2},$$

respectively. For each summation we evaluate the summand and the number appearing in the sum.

LEMMA 12. *There exists an absolute constant c_{11} such that*

$$\sum_{(1)} P(E_j \cap E_{ji}) \leqq c_{11} \cdot P(E_j).$$

Proof. First we estimate $P(E_j \cap E_{ji})$. From the definition of $\sum_{(1)}$, we find a positive integer k such that $k \leqq \sqrt{p}$ and

$$\left(1 - \frac{k}{p}\right)^{1/2} \leqq \rho(u_j, u_{ji}) < \left(1 - \frac{(k-1)}{p}\right)^{1/2}.$$

Hence, by Lemma 11, (b) and Lemma 4, (c) we obtained

$$\begin{aligned} P(E_j \cap E_{ji}) &\leqq d_2 \exp\{-\delta(c_1^2 2N \log 2)p \cdot (k-1)/p\} \cdot P(E_j) \\ &\leqq c_{12} \exp\{-\delta(c_1^2 2N \log 2) \cdot k\} \cdot P(E_j), \end{aligned}$$

where $\quad c_{12} = d_2 \cdot \exp\{\delta(c_1^2 2N \log 2)\}.$

Next we estimate the number of summand in $\sum_{(1)}$.
If we apply (5. 8) to the present case, we have

$$\left(1-\frac{k}{p}\right)\cdot d^{\alpha}(A,B)\leqq d^{\alpha}(A',B')\leqq d^{\alpha}(A,B), \tag{5. 9}$$

where pairs of points (A,B) and (A',B') correspond to E_j and E_{ji}, respectively. Further using (5. 9) to estimate the denominator of $\rho(u_j, u_{ji})$, we have

$$\left(1-\frac{2k}{p}\right) d^{\alpha}(A,B)\leqq d^{\alpha}(A,B') - d^{\alpha}(B,B'),$$

(5. 10)

$$\left(1-\frac{2k}{p}\right)\cdot d^{\alpha}(A,B)\leqq d^{\alpha}(A',B) - d^{\alpha}(A,A').$$

Thus the number in $\sum_{(1)}$ does not exceed the number of point A' and B' which are contained in the region determined by (5. 9) and (5. 10). A computation shows that this region has volume V smaller than $\{2(4k)^{1/\alpha}/2^p\}^N$. On the other hand, p' is estimated by $p'\leqq p+l_0$, where l_0 is an absolute constant integer. In fact, if we set l_0, n_0 and m_0 by $l_0 = 4\vee n_0 \vee m_0$ where $n_0=\min\{n;\ 2(1-(3/16)^{\alpha})-\sqrt{2}\geqq n\cdot 3^{\alpha}/2^{n\alpha}\}$ and $m_0=\max\{n;\ n+2\geqq 2^{n\alpha+1}/3^{\alpha}\}$, respectively. If we consider (5. 9) in term of p and p', we see that n_0 and m_0 are minimum integers to break inequalities in (5. 10). Since the above argument permits us to overestimate the number in $\sum_{(1)}$ as

$$V/(1/2^{p+l_0})^N \leqq (2^{l_0+1}\cdot 4^{1/\alpha})^N k^{N/\alpha},$$

we have

$$\sum_{(1)} P(E_j\cap E_{ji})\leqq c_{12}(2^{l_0+1}\cdot 4^{1/\alpha})^N P(E_j)\cdot\sum_{k=1}^{\infty} k^{N/2}\cdot\exp\{-\delta(c_1^2 2N\log 2)k\},$$

which completes the proof if we set c_{11} by the coefficient of $P(E_j)$ in the right-hand side.

LEMMA 13. *There exists an absolute constant c'_{11} such that*

$$\sum_{(2)} P(E_j\cap E_{ji})\leqq c'_{11}\cdot P(E_j).$$

Proof. First we estimate the summand and next the number in $\sum_{(2)}$. Since we have $s_j > s_{ji}$ for $ji > j$ $(s_k = 1/d(A_k, B_k))$, it holds by Lemma 4, (c) and Lemma 11, (b)

$$P(E_j\cap E_{ji})\leqq P(u_j>\varphi(s_j),\ u_{ji}>\varphi(s_j))$$

$$\leqq d_2 \exp\{-\delta(1-\rho^2(u_j, u_{ji})\cdot\varphi(s_j)\}\cdot P(u_j > \varphi(s_j))$$
$$\leqq d_2 \exp(-\delta c_3\sqrt{p})\cdot P(E_j).$$

Set

$$l_0' = m_0' \vee n_0' \vee 2,$$
$$m_0' = \min\{l;\ 2(e+c_{10}) > e\cdot 2^l\},$$
$$n_0' = \min\left\{l;\ \log(1/c_{10}) < l\left(\frac{1}{\alpha}-1\right)\right\}.$$

For (A, B) corresponding to E_j and (A', B') corresponding to E_{ji}, if we have inequality, $d(A', B)$ or $d(A, B') \geqq (p^{l_0'/\alpha})/2^p$, then $\rho(u_j, u_{ji})$ does not satisfy the condition of $\boldsymbol{E}_j$. This implies that for all $E_{ji} \in \boldsymbol{E}_j$, $d(A', B)$ and $d(A, B') \leqq (p^{l_0'}/\alpha)/2^p$. Since A' and B' are contained in a cube with volume $(2p^{l_0'}/2^p)^N$, the number $\sharp$ of such points A' or B' is dominated as

$$\sharp \leqq (2p^{l_0'/\alpha}/2^p)/(1/2p')^N$$
$$\leqq 2^N\cdot p^{N\left(\frac{l_0'}{\alpha}+c_{10}\log 2\right)}$$

Thus we have estimate of $\sum_{(2)}$;

$$\sum_{(2)} P(E_j \cap E_{ji}) \leqq d_2 2^{2N} p^{2N\left(\frac{l_0'}{\alpha}+c_{10}\log 2\right)}\cdot \exp\{-\delta c_3\sqrt{p}\}\cdot P(E_j),$$

which verifies the lemma if we set c'_{11} by

$$c'_{11} = (\delta\cdot c_3)^M\cdot M!/2 \text{ and } M = \left\{N\left(\frac{l_0'}{\alpha}+c_{10}\log 2\right)\right\}+1.$$

Proof of Lemma 2, [2. 4]. If we set $k_1 = c_{11} \vee c'_{11}$, it yields (3. 1). From Lemma 11, (a) we find for $E_k \notin \boldsymbol{E}_j$, $(k > j)$,

$$P(u_j > \varphi(s_j),\ u_k > \varphi(s_k)) < d_1\cdot P(u_j > \varphi(s_j))\cdot P(u_k > \varphi(s_k)),$$

since $\rho(u_j, u_k) < 1/\{\varphi(s_j)\cdot\varphi(s_k)\}$. This implies (3. 2).
We established the proof of theorem completely.

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