1.6.7 Fin Optimization

It is useful to go through the exercise of optimizing a fin in order achieve the highest rate of heat transfer per volume of fin material; the result of this optimization provides general guidelines relative to the dimensionless characteristics of a well-designed fin. From Table 1-4, the rate of heat transfer to a constant cross-section fin with an adiabatic tip is given by:

$$\dot{q}_{fin} = \sqrt{k\,A_c\,\overline{h}\,per}\left(T_b - T_\infty\right)\tanh\left(m\,L\right) \tag{1-254}$$

where mL is:

$$mL = \sqrt{\frac{per\,\overline{h}}{k\,A_c}} \tag{1-255}$$

For a rectangular fin with width W and thickness th (shown in Figure 1-31), the cross-sectional area is $th\ W$ and the perimeter is $2\ W$, assuming that the fin thickness is small compared to its width ($th << W$). Therefore, Eq. (1-254) can be written as:

$$\dot{q}_{fin} = W\sqrt{k\,th\,\overline{h}\,2}\left(T_b - T_\infty\right)\tanh\left(m\,L\right) \tag{1-256}$$

where

$$m\,L = \sqrt{\frac{\overline{h}\,2}{k\,th}}\,L \tag{1-257}$$

The rate of heat transfer per width of surface is:

$$\frac{\dot{q}_{fin}}{W} = \sqrt{k\,th\,\overline{h}\,2}\left(T_b - T_\infty\right)\tanh\left(m\,L\right) \tag{1-258}$$

The volume of the fin is:

$$V = W\,th\,L \tag{1-259}$$

For this optimization, the volume of the fin material per width of surface (V/W) will be held constant.

$$\frac{V}{W} = th\,L \tag{1-260}$$

Therefore, the fin parameter, mL, can be written as:

$$m\,L = \sqrt{\frac{\overline{h}\,2\,W}{k\,V}}\,L^{3/2} \tag{1-261}$$

and the heat transfer per width can be written as:

$$\frac{\dot{q}_{fin}}{W} = \sqrt{\frac{k\,V\,\overline{h}\,2}{W\,L}}\left(T_b - T_\infty\right)\tanh\left(m\,L\right) \tag{1-262}$$

Solving Eq. (1-261) for L allows the length to be expressed in terms of the fin parameter (mL) and the volume to width ratio:

$$L = \frac{\left(m\,L\right)^{2/3}}{\left(\dfrac{\overline{h}\,2\,W}{k\,V}\right)^{1/3}} \tag{1-263}$$

Substituting Eq. (1-263) into Eq. (1-262) allows the heat transfer per width to be expressed in terms of the volume per width and the fin constant.

$$\frac{\dot{q}_{fin}}{W} = \sqrt{\frac{k\,V\,\overline{h}\,2}{W}\frac{\left(\dfrac{\overline{h}\,2\,W}{k\,V}\right)^{1/6}}{\left(m\,L\right)^{1/3}}}\left(T_b - T_\infty\right)\tanh\left(m\,L\right) \tag{1-264}$$

which can be simplified to:

$$\frac{\dot{q}_{fin}}{W} = \left(\frac{V}{W}\right)^{1/3} k^{1/3}\,\overline{h}^{2/3}\left(T_b - T_\infty\right)2^{2/3}\,\frac{\tanh\left(m\,L\right)}{\left(m\,L\right)^{1/3}} \tag{1-265}$$

Equation (1-265) provides a useful result. In a typical design, the objective will be to maximize the rate of heat transfer per unit width ($\dot{q}_{fin}/W$) that can be obtained for a given volume per unit width (V/W) given constraints related to the conductivity, heat transfer coefficient, and driving temperature difference. Equation (1-265) shows that the performance of the fin will become larger as any of these parameters (k, $\overline{h}$, or T_b-T_∞) are increased. The only free parameter on the right hand side of Eq. (1-265) is the fin parameter, mL. The dimensionless fin performance ($\tilde{\dot{q}}_{fin}$) is defined as:

$$\tilde{\dot{q}}_{fin} = \frac{\left(\dfrac{\dot{q}_{fin}}{W}\right)}{\left(\dfrac{V}{W}\right)^{1/3} k^{1/3}\,\bar{h}^{2/3}\left(T_b - T_\infty\right)} = 2^{2/3}\frac{\tanh\left(m\,L\right)}{\left(m\,L\right)^{1/3}} \tag{1-266}$$

Note that the denominator of the dimensionless fin performance definition has units of heat transfer rate per unit width and represents, approximately, the highest performance that is possible for a well-designed fin. The dimensionless fin performance is a function only of the fin parameter, as shown in Figure 1-44.

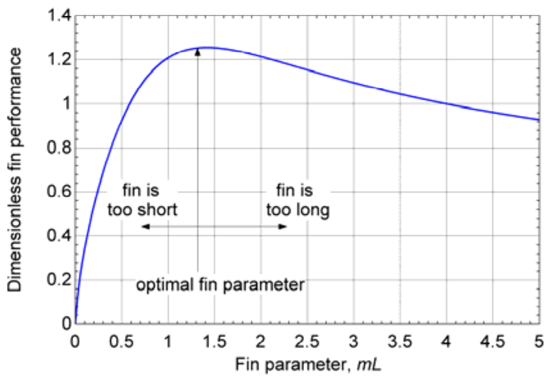

Figure 1-44: **Dimensionless fin performance as a function of the fin parameter.**

Figure 1-44 shows that the optimal value of *mL* is approximately 1.4. It is possible to determine the optimal value of *mL* more exactly using the optimization capability in EES. The dimensionless fin performance is entered in the equation window.

```
Q=2^(2/3)*tanh(mL)/(mL)^(1/3)                    "dimensionless fin heat transfer rate"
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It is not possible to solve this equation because there is one equation with two unknowns. It is possible to maximize or minimize the value of one of the unknowns by varying the other. In this case, it is of interest to find the value of *mL* that maximizes $\tilde{\dot{q}}_{fin}$. Use the Min/Max option from the Calculate menu to bring up the Find Minimum or Maximum dialog, Figure 1-45.

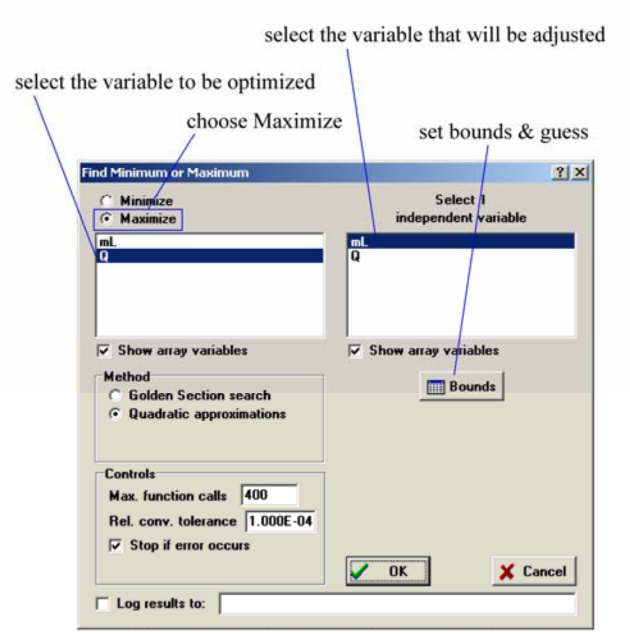

Figure 1-45: **Find Minimum or Maximum dialog**

EES will ask you to identify the parameter to be optimized (in this case, the variable Q, note that you can either maximize or minimize this parameter) and identify the independent variable(s) that will be adjusted to accomplish the optimization (in this case, the variable mL). You need to set bounds and a guess for the independent variable; appropriate bounds for mL are 0.001 to 10 with a guess of 5. You can optimize using several methods; because this is a simple 1-D optimization (as opposed to a multidimensional optimization) only the Golden Section search or Quadratic approximations techniques are available. Either technique will work fine for this problem. Hitting OK sets the optimization in motion and EES will identify that the optimal value of mL is precisely 1.419 and the optimized value of $\dot{Q}$ is 1.256; the optimized solution is retained in the Solution window at the conclusion of the optimization.

The optimal value of mL provides a rule of thumb that can be used to quickly design fins or as a "sanity check" for an existing design. Fins with mL much less than 1.4 are shorter than optimal and therefore have very small temperature gradients due to conduction; additional length will provide a substantial benefit and therefore the available volume of fin material should be stretched, providing additional length at the expense of cross-sectional area. Fins with mL much greater than 1.4 are longer than optimal and therefore have large temperature gradients due to conduction; additional length will not provide much benefit as the tip temperature is approaching the ambient temperature. Therefore, the available volume should be compressed, reducing the length but providing more cross-sectional area for conduction. Lacking compelling constraints related to, for example, manufacturing or structural limitations, a well-designed fin will generally have a value of mL that is close to this optimal value.