Ackermann's formula

It is not trivial to prove Ackermann's formula and most introductory texts do not derive it. We take the compromise approach of Ogata (1997, Chapter 12) and illustrate how one may obtain the formula with a simplified illustration.

The starting point is the system matrix Eq. (9-18), and to make the algebra a bit cleaner later, we define the notation

$$\mathbf{H} = \mathbf{A} - \mathbf{BK} \tag{1}$$

As in the text, the characteristic equation is $|s\mathbf{I} - \mathbf{H}| = |s\mathbf{I} - \mathbf{A} + \mathbf{KB}| = 0$.

Next comes the important step. We invoke the Cayley-Hamilton theorem (from your days of linear algebra), which briefly, states that a matrix must satisfy its own characteristic equation. Hence we can write Eq. (9-20) for the matrix, substituting $\mathbf{H}$ for s,

$$\alpha_c(\mathbf{H}) = \mathbf{H}^n + \alpha_{n-1}\mathbf{H}^{n-1} + \ldots + \alpha_1\mathbf{H} + \alpha_o\mathbf{I} = 0 \tag{2}$$

where we also have adopted the shorthand notation in (9-23) to refer to this characteristic relation.

We now look at the simplified case of when $n = 3$ to see how the Ackermann's formula may arise. Hence, (2) becomes

$$\alpha_c(\mathbf{H}) = \mathbf{H}^3 + \alpha_2\mathbf{H}^2 + \ldots + \alpha_1\mathbf{H} + \alpha_o\mathbf{I} = 0, \quad n = 3 \tag{3}$$

The next task is to substitute for $\mathbf{H}$ in (3). With $\mathbf{H}$ defined in (1), its square is

$$\mathbf{H}^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKA} - (\mathbf{BK})^2$$

We further factor and substitute with $\mathbf{H}$ on the RHS to obtain a more compact look:

$$\mathbf{H}^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKH} \tag{4}$$

If we do the same with $\mathbf{H}^3$, and after some algebraic work, we should find

$$\mathbf{H}^3 = (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABKH} - \mathbf{BKH}^2 \tag{5}$$

Now we substitute (1), (4) and (5) into (3):

$$(\mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABKH} - \mathbf{BKH}^2) + \alpha_2(\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKH}) + \alpha_1(\mathbf{A} - \mathbf{BK}) + \alpha_o\mathbf{I} = 0$$

and factor it to become

$$(\mathbf{A}^3 + \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_o\mathbf{I}) - \alpha_2(\mathbf{ABK} + \mathbf{BKH}) - \alpha_1\mathbf{BK} - (\mathbf{A}^2\mathbf{BK} + \mathbf{ABKH} + \mathbf{BKH}^2) = 0$$

The terms in the first set of parenthesis on the left can be shorthanded as

$$\alpha_c(\mathbf{A}) = \mathbf{A}^3 + \alpha_2\mathbf{A}^2 + \alpha_1\mathbf{A} + \alpha_o\mathbf{I} \tag{6}$$

Keep in mind that $\alpha_c(\mathbf{A}) \neq 0$ because the coefficients α_i are for the characteristic equation of $\mathbf{H}$, not $\mathbf{A}$. Going back to the last intermediate step, we substitute the terms for $\alpha_c(\mathbf{A})$, move the other terms to the RHS, and factor them to have leading terms of $\mathbf{B}$, $\mathbf{AB}$, etc.:

$$\alpha_c(\mathbf{A}) = \mathbf{B}\,(\alpha_1\mathbf{K} + \alpha_2\mathbf{KH} + \mathbf{KH}^2) + \mathbf{AB}\,(\alpha_2\mathbf{K} + \mathbf{KH}) + \mathbf{A}^2\mathbf{BK}$$

This equation is put in matrix form;

$$\alpha_c(\mathbf{A}) = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \begin{bmatrix} \alpha_1\mathbf{K} + \alpha_2\mathbf{KH} + \mathbf{KH}^2 \\ \alpha_2\mathbf{K} + \mathbf{KH} \\ \mathbf{K} \end{bmatrix}$$

The first matrix on the RHS is the controllability matrix $\mathbf{C_o}$ in (9-7). Since the Ackermann's formula is applied to only completely state controllable systems, the inverse of $\mathbf{C_o}$ exists, and we can write

$$\begin{bmatrix} \alpha_1\mathbf{K} + \alpha_2\mathbf{KH} + \mathbf{KH}^2 \\ \alpha_2\mathbf{K} + \mathbf{KH} \\ \mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix}^{-1} \alpha_c(\mathbf{A})$$

Finally, we multiply the equation with [0 0 1] to extract **K**:

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix}^{-1} \alpha_c(\mathbf{A}) \qquad (7)$$

This is clearly the Ackermann's formula in (9-22) when $n = 3$. Or we can consider (9-22) as the general extension of our simplified illustration here.