Bull. Austral. Math. Soc. 34K25
Vol. 38 (1988) [255–261]

OSCILLATORY PROPERTIES OF THE SOLUTIONS OF LINEAR EQUATIONS OF NEUTRAL TYPE

D.D. Bainov, A.D. Myshkis and A.I. Zahariev

In this paper the oscillatory and asymptotic properties of solutions of the equation

$$\frac{d}{dt}\left(x(t)+\delta\int_0^\tau x(t-s)dr(s)\right)+\int_0^\sigma x(t-s)d\bar{r}(s)=0,$$

are investigated where $\delta = \pm 1$, $\tau > 0$, $\sigma > 0$, the functions $r(s)$ and $\bar{r}(s)$ are non-decreasing and $\bar{r}(\sigma) > \bar{r}(0) = r(0) = 0$.

The results of the investigation of the oscillatory and asymptotic properties of functional differential equations of neutral type, besides being of independent theoretical interest, have numerous important practical applications. Some examples of real objects simulated by equations of neutral type are given in [**3**, **7**, **8**]. We shall note that in comparison with the equations with a deviating argument considerably fewer results have been obtained about equations of neutral type. For the linear case oscillatory properties of the solutions in the case of one constant delay are given in [**4** - **6**] and for distributed delay in [**2**]. Oscillatory and asymptotic results on non-linear equations of neutral type have been obtained in [**1**, **12**, **14**, **15**]. We note that the oscillatory and asymptotic properties of linear equations and systems of delaying type with distributed delay have been studied in detail in [**11**].

In the present paper the oscillatory and asymptotic properties of the equation

$$\frac{d}{dt}\left(x(t)+\delta\int_0^\tau x(t-s)dr(s)\right)+\int_0^\sigma x(t-s)d\bar{r}(s)=0, \tag{1}$$

are investigated where $\delta = \pm 1$, $\tau > 0$, $\sigma > 0$, the function r and $\bar{r}$ are non-decreasing and $\bar{r}(\sigma) > \bar{r}(0) = r(0) = 0$. The results obtained in the paper generalise some of the assertions proved in [**5**].

Received 24 November 1987

DEFINITION 1: A function $x\colon [t_x, \infty) \to \mathbb{R}$ is called a *regular solution* of equation (1) if

$$x \in C[t_x, \infty),$$
$$\left(t \mapsto x(t) + \delta \int_0^\tau x(t-s)dr(s)\right) \in C^1[t_x+\alpha, \infty),$$
$$\alpha := \max(\tau, \sigma),$$

x satisfies equation (1) for almost all $t \geqslant t_x + \alpha$ and

$$\sup_{s\in[t,\infty)} |x(s)| > 0 \quad \text{for} \quad t > t_x + \alpha.$$

DEFINITION 2: We say that a property $Q(t)$ is *ultimately satisfied* if it is satisfied for all sufficiently large values of t.

DEFINITION 3: We shall call a regular solution $x(t)$ of equation (1) non-oscillating if it is either ulimately non-negative or ultimately non-positive. Otherwise we shall call the solution $x(t)$ oscillating. We shall denote the set of all ultimatley non-negative solutions of equation (1) by Ω^+.

Let x be a solution of equation (1) and define the sequence of functions

$$\begin{aligned} &x_k(t) = x_{k-1}(t) + \delta\int_0^\tau x_{k-1}(t-s)dr(s),\ t\in[t_x+k\tau, \infty), \\ &x_0(t) = x(t),\ t\in[t_x,\infty),\ k=1,2,\ldots. \end{aligned} \tag{2}$$

It is immediately verified by induction with respect to k that each function x_k is a solution of equation (1) and from the equality

$$x_k'(t) = -\int_0^\sigma x_{k-1}(t-s)d\bar{r}(s),\ t\in[t_x+(k-1)\tau, \infty),\ k=1,2,\ldots \tag{3}$$

it follows that $x \in C^k[t_x + k\alpha, \infty)$.

We distinguish the following subcases of choices of δ and $r(\tau)$ for equation (1):

(i) $\delta = +1$;

(ii) $\delta = -1,\ r(\tau) < 1$;

(iii) $\delta = -1,\ r(\tau) = 1$;

(iv) $\delta = -1,\ r(\tau) > 1$.

LEMMA 1. *Let $x \in \Omega^+$.*

Then in cases (i)–(iii) the function x_k are ultimately non-increasing, $x_k \in \Omega^+$ and $x_k(\infty) = 0$, $k = 1,2,\ldots$. In case (iv) the function $(-1)^k x_k$ are ultimately non-decreasing and $(-1)^k x_k(\infty) = \infty$, $k = 1,2,\ldots$. For $k \geqslant 2$ in all cases the functions x_k are ultimately strictly monotone.

PROOF: From equation (3) it follows that all functions x_k, $k = 1, 2, \dots$ are regular and ultimately monotone solutions of equation (1) and, moreover, that x_1 is a non-increasing function. We shall prove that $x_k(\infty) \in \{-\infty, 0, \infty\}$, $k = 1, 2, \dots$.

If we suppose that this is not true, then substituting $x_k(t)$ into the equation and integrating, we obtain the relation

$$x_k(t) + \delta \int_0^\tau x_k(t-s)dr(s) = -x_k(\infty)\bar{r}(\sigma)t + 0(t), \; (t \to \infty),$$

which implies that the solution x_k is unbounded which contradicts our assumption.

From equation (2) it follows that if $x_1(\infty) = 0$, then $x_k(\infty) = 0$, $k \geqslant 1$ as well and in view of equation (3) we conclude that the functions x_k, $k \geqslant 1$ are ultimately non-increasing and $x_k \in \Omega^+$ for $k \geqslant 1$.

If we have case (i), then from equation (2) it follows that $x_1 \in \Omega^+$, hence $x_1(\infty) = 0$.

Suppose that in cases (ii) or (iii) the relation $x_1(\infty) = -\infty$ holds.

Then $\sup x(t) = \infty$ and for any sufficiently large number $\gamma \in \mathbb{R}$ there exists a point $t_\gamma > t_x$ such that $x(t_\gamma) = \gamma$, $x(t) < \gamma$, $t \in [t_x, t_\gamma)$ and, moreover, we have $\lim\limits_{\gamma \to \infty} t_\gamma = \infty$.

Hence equation (2) implies the inequaltiy

$$x_1(t_\gamma) = x(t_\gamma) - \int_0^\tau x(t_\gamma - s)dr(s) \geqslant \gamma - \int_0^\tau \gamma dr(s) = \gamma[1 - r(\tau)] \geqslant 0, \tag{4}$$

which contradicts the assumption that $x_1(\infty) = -\infty$, so $x_1(\infty) = 0$ in cases (ii) and (iii) as well.

Consider case (iv). If we suppose that $x_1(\infty) = 0$, then ultimately the inequality $x_1(t) > 0$ holds which implies that the function x_2 is ultimately decreasing. In view of $x_2(\infty) = 0$ we obtain that ultimately the inequality $x_2(t) > 0$ holds, hence the inequalities

$$x_1(t) > \int_0^\tau x_1(t-s)dr(s) \geqslant r(\tau)x_1(t) > x_1(t),$$

hold. This is impossible, so $x_1(\infty) = -\infty$. Analogously, if we repeat the arguments considering the function $-x_1(t)$ instead of $x(t)$, we obtain $x_2(\infty) = \infty$, etcetera. ∎

COROLLARY 1. *Suppose case (iv) holds.*
Then:

a) *if $x \in \Omega^+$, then $\sup x(t) = \infty$,*

b) *the characteristic equation*

$$z - z\int_0^\tau e^{-zs}dr(s) + \int_0^\sigma e^{-zs}d\bar{r}(s) = 0$$

has no real roots $z \leqslant 0$.

PROOF: Assertion a) follows from the fact that $x_1(\infty) = -\infty$ and assertion b) follows directly from assertion a). ∎

Remark 1. In (iv), if $x \in \Omega^+$, then the limit $x(\infty)$ may not exist. Thus, for instance, the equation

$$\{x(t) - [10e^{\pi}x\Big(t - \frac{\pi}{2}\Big) + 4e^{2\pi}x(t-\pi)]\}' + \frac{1}{2}x(t) + 25e^{\pi}x\Big(t - \frac{\pi}{2}\Big) + \frac{1}{2}e^{2\pi}x(t-\pi) = 0,$$

has a particular solution $x(t) = e^{2t}(1 + \cos t)$.

THEOREM 1. *Suppose that either case (i) or case (ii) holds. If the function $x \in \Omega^+$, then $x(\infty) = 0$.*

PROOF: In case (i), if $x \in \Omega^+$, then ultimately the inequality $x(t) < x_1(t)$ holds and then Theorem 1 follows immediately from Lemma 1.

Let us consider case (ii) and let $x \in \Omega^+$. Then $\sup x(t) < \infty$ since otherwise from relation (4) passing to the limit for $\gamma \to \infty$ we would obtain $\limsup_{t\to\infty} x_1(t) = \infty$ which contradicts Lemma 1. Let us write $\limsup_{t\to\infty} x(t) = x^0$, and choose a sequence $\{t_n\}$, $\lim\limits_{n\to\infty} t_n = \infty$ such that $\lim\limits_{n\to\infty} x(t_n) = x^0$. Then we obtain the inequality

$$\liminf_{n\to\infty} x_1(t_n) \geqslant \lim_{n\to\infty} x(t_n) - \limsup_{n\to\infty} \int_0^t x(t_n - s)dr(s) = x^0[1 - r(\tau)] > 0,$$

which contradicts Lemma 1. Hence the result. ∎

LEMMA 2. *([**9, 10, 13**]). Let the constants $p, \mu \in \mathsf{R}_+$ and $p\mu > e^{-1}$.*

Then each of the inequalities

$$x' - px(t+\mu) \geqslant 0,\ x' + px(t-\mu) \leqslant 0$$

has no ultimately positive solutions.

THEOREM 2. *Suppose case (i) holds and there exists a constant $\sigma_1 \in [0, \sigma]$ satisfying the inequality*

$$(\sigma_1 - \tau)[\bar{r}(\sigma) - \bar{r}(\sigma_1^-)]] \geqslant \frac{1}{e}[1 + r(\tau)]. \tag{5}$$

(if $\sigma_1 = 0$. then we put $\bar{r}(\sigma_1^-) = 0$).

Then each regular solution of equation (1) oscillates.

PROOF: Let $x \in \Omega^+$. Then from Lemma 1 and equation (3) it follows that ultimately the inequalities $x_2(t) > 0$, $x_2'(t) < 0$, $x_2''(t) \geqslant 0$ hold. Hence we obtain

$$\begin{aligned} 0 &= x_2'(t) + \int_0^\tau x_2'(t-s)dr(s) + \int_0^\sigma x_2(t-s)d\bar{r}(s) \\ &\geqslant x_2'(t-\tau) + \int_0^\tau x_2'(t-\tau)dr(s) + \int_{\sigma_1^-}^\sigma x_2(t-\sigma_1)d\bar{r}(s) \\ &= [1 + r(\tau)]x_2'(t-\tau) + [\bar{r}(\sigma) - \bar{r}(\sigma_1^-)]x_2(t-\sigma_1) \end{aligned} \tag{6}$$

https://doi.org/10.1017/S0004972700027520 Published online by Cambridge University Press

which implies the inequality

$$x_2'(t) + \frac{\bar{r}(\sigma) - \bar{r}(\sigma_1^-)}{1 + r(\tau)} x_2(t - (\sigma_1 - \tau)) \leqslant 0.$$

Using inequality (5) and applying Lemma 2 to the above inequality we conclude that it has no ultimately non-negative solutions which contradicts the assumption that $x \in \Omega^+$. ∎

THEOREM 3. *Suppose case (ii) holds and one of the following two conditions hold:*

a) *there exists a constant $\sigma_1 \in [0, \sigma]$ satisfying the inequality*

$$[\bar{r}(\sigma) - \bar{r}(\sigma_1^-)] > \frac{1}{e}[1 - r(\tau)], \tag{7}$$

b) *there exist constants $\tau_1 \in [0, \tau]$ and $\sigma_1 \in [0, \sigma]$ satisfying the inequality*

$$[r(\tau) - r(\tau_1^-)][\bar{r}(\sigma) - \bar{r}(\sigma_1^-)](\sigma_1 + \tau_1) \geqslant 1. \tag{8}$$

Then each regular solution of equation (1) is oscillating.

PROOF: Let $x \in \Omega^+$. Then, arguing as in the proof of inequality (6), we conclude that for each constant $\sigma_1 \in [0, \sigma]$ ultimately the following inequality holds

$$0 \geqslant [1 - r(\tau)]x_2'(t) + [\bar{r}(\sigma) - \bar{r}(\sigma_1^-)]x_2(t - \sigma_1). \tag{9}$$

If inequality (7) holds and we apply Lemma 2 to inequality (9), then as in Theorem 1 we arrive at a contradiction to the assumption that $x \in \Omega^+$.

Provided that inequality (8) holds, substituting the solution x_2 into equation (1) and integrating from t to $t + \sigma_1 + \tau_1$ we obtain that ultimately the following inequality holds

$$\begin{aligned} x_2(t + \sigma_1 + \tau_1) - x_2(t) - \int_0^\tau x_2(t + \sigma_1 + \tau_1 - s)dr(s) + \int_0^\tau x_2(t - s)dr(s) \\ + (\tau_1 + \sigma_1).\beta.x_2(t + \tau_1) < 0, \beta := \bar{r}_2(\sigma) - \bar{r}_2(\sigma_1^-). \end{aligned} \tag{10}$$

Inequality (10), since the function x_2 is ultimately decreasing, implies the inequality

$$x_2(t + \tau_1 + \sigma_1) - x_2(t) + \beta(\tau_1 + \sigma_1)x(t + \tau_1) < 0. \tag{11}$$

Replace t by $t - s$ in inequality (11) and integrate with respect to $r(s)$ from $s = 0$ to $s = \tau$; then adding inequality (10) to the inequality obtained, we get

$$x_2(t + \tau_1 + \sigma_1) + \beta.(\tau_1 + \sigma_1)x_2(t + \tau_1) + \{(\tau_1 + \sigma_1)\beta[r(\tau) - r(\tau_1^-)]\} < 0,$$

which contradicts inequality (8). ∎

THEOREM 4. *In case (iii) each regular solution of equation (1) oscillates.*

PROOF: If $x \in \Omega^+$, then setting $\sigma_1 = 0$ we obtain inequality (9) which contradicts the assertion of Lemma 2. ∎

THEOREM 5. *Suppose case (iv) holds and that there exist constants* $\tau_1 \in [0, \tau)$, σ_1
$\in [0, \sigma]$ *satisfying the inequaltiy*

$$\bar{r}(\sigma_1^+)(\tau_1 - \sigma_1) > \frac{1}{e}[r(t) - 1],\ r(t) \equiv 0,\ t \in [0, \tau_1) \tag{12}$$

(if $\sigma_1 = \sigma$*, then we set* $\bar{r}(\sigma_1^+) = \bar{r}(\sigma)$*).*

Then each regular solution of equation (1) oscillates.

PROOF: Let $x \in \Omega^+$. Then from Lemma 1 and equality (3) it follows that $x_2'(\infty) = \infty$ and ultimately the inequality $x_2'(t) \geqslant 0$ holds, whence we obtain

$$\begin{aligned}
0 &= x_2'(t) - \int_{\tau_1^-}^{\tau} x_2'(t-s)\,dr(s) + \int_0^{\sigma} x_2(t-s)\,d\bar{r}(s) \\
&\geqslant x_2'(t-\tau_1) - \int_{\tau_1}^{\tau} x_2'(t-\tau_1)\,dr(s) + \int_0^{\sigma_1^+} x_2(t-\sigma_1)\,d\bar{r}(s) \\
&\quad - [r(\tau) - 1]x_2'(t-\tau_1) + \bar{r}(\sigma_1^+)x_2(t-\sigma_1),
\end{aligned}$$

hence ultimately the inequaltiy

$$x_2'(t) - \frac{\bar{r}(\sigma_1^+)}{r(\tau) - 1}x_2(t + \tau_1 - \sigma_1) \geqslant 0$$

holds, which, in view of inequaltiy (12), contradicts the assertion of Lemma 2. ∎

REFERENCES

[1] D.D. Bainov, A.I. Zahariev and A.D. Myshkis, 'Oscillatory properties of the solutions of a class of integro-differential equations of neutral type', *IX Intern. Conf. on Nonlin. Osc* **2**, pp. 39–40 (Naukova Dumka, Kiev, 1984). (in Russian) .

[2] D.D. Bainov, A.D. Myshkis and A.I. Zahariev, 'Necessary and sufficient conditions for oscillations of the solutions of linear autonomous functional differential equations of neutral type with distributed delay' (to appear).

[3] R. Braiton, 'Nonlinear oscillations in a distributed network', *Quart. Appl. Math.* **24** (1967), 289–301.

[4] M.K. Grammatikopoulos, E.A. Grove and G. Ladas, 'Oscillation and asymptotic behaviour of neutral differential equations with deviating arguments', *Applicable Anal.* (to appear).

[5] M.K. Grammatikopoulos, E.A. Grove and G. Ladas, 'Oscillations of first order neutral delay differential equations', *J. Math. Anal. Appl.* (to appear).

[6] M.K. Grammatikopoulos, E.A. Grove and G. Ladas, 'Oscillations and asymptotic behaviour of second order neutral differential equations with deviating arguments' (to appear).

[7] J. Hale, *Theory of functional differential equations* (Springer-Verlag, New York, 1977).

[8] V.B. Kolmanovskii and V.R. Nosov, *Stability and periodic régimes of controllable systems with aftereffect* (Moscow, 1981). (in Russian) .

[9] R.G. Koplatadze and T.A. Canturia, 'On oscillatory and monotonic solutions of first order differential-difference equations with retarded arguments', *Differentsial'nye Uravaneniya* **8** (1982), 1463–1465. (in Russian) .

[10] G. Ladas and I.P. Stavroulakis, 'On delay differential inequalities of first order', *Funkcial Ekvac.* **25** (1982), 105–113.

[11] A.D. Myshkis, *Linear differential equations with a delaying argument* (Moscow, 1972). (in Russian) .

[12] A.D. Myshkis, D.D. Bainov and A.I. Zahariev, 'Oscillatory and asymptotic properties of a class of operator-differential inequalities', *Proc. Roy. Soc. Edinburgh* **Sect A 96** (1984), 5–13.

[13] H. Onose, 'Oscillatory properties of the first order differential inequalities with deviating arguments', *Funkcial Ekvac.* **26** (1983), 189–195.

[14] A.I. Zahariev and D.D. Bainov, 'Oscillating properties of the solutions of a class of neutral type functional differential equations', *Bull. Austral. Math. Soc.* **22** (1980), 365–372.

[15] A.I. Zahariev and D.D. Bainov, 'Oscillating and asymptotic properties of a class of functional differential equations with maxima', *Czechoslovak Math. J.* **34** (1984), 247–251.

Dr D.D. Bainov
P.O. Box 45
1504 Sofia
Bulgaria

Dr A.D. Myshkis
Moscow Institute of Railway Transport Engineers

Dr. A.I. Zahariev
Plovdiv University
"Paissii Hilendarski"