ON DYNAMICAL SYSTEMS WITH ONE DEGREE OF FREEDOM

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1. Introduction. Consider the (vector, n-component) system of differential equations

$$x' = f(x) \quad (' = d/dt), \tag{1}$$

where $f(x)$ is of class C^1. Let Ω denote a set of points, x, consisting of unrestricted solution paths $x(t)$, so that the $x(t)$ exist and lie in Ω for $-\infty < t < \infty$. Let $t = t_0$ be arbitrary but fixed; then the solution $x = x(t)$ will be called stable (with respect to Ω) if for every $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ such that $|x(t) - y(t)| < \epsilon$ whenever $y(t)$ is in Ω and $|x(t_0) - y(t_0)| < \delta$. For a discussion of this type of stability (called A-stability in **(7)**), see Liapounoff **(4**, pp. 210–211; **8**, pp. 98–99**)**, wherein are given references to Minding and Dirichlet.

It was shown by Hartman and Wintner **(3)** that a solution $x(t)$ of (1) which is dense on a compact set Ω is almost periodic (in the sense of Bohr) if and only if it is stable in a certain sense. The type of stability considered there (called B-stability in **(7)**), however, is more restrictive than the A-stability, and, in the sequel, the term "stability" will refer only to that (A-stability) defined at the beginning of this section. It was shown in **(7)** that if (1) is of the incompressible type, so that

$$\operatorname{div} f \equiv \Sigma \partial f_k / \partial x_k = 0, \tag{2}$$

and if the space Ω is suitably restricted, then all stable solutions of (1) do have certain properties possessed by almost periodic solutions. Whether all such solutions, for n arbitrary, are actually almost periodic will remain undecided. (If (2) is not assumed, then stability surely does not imply almost periodicity even if the dimension number n of (1) is unity; see **(7)**.)

The present paper will be devoted to a consideration of (1), subject to the (measure-preserving) condition (2), in the special case when $n = 2$. It is known that the system (1) is then equivalent to a conservative Hamiltonian system **(8**, p. 88**)**, and hence, in view of the existence of the energy integral, is completely integrable. In what follows then, only Hamiltonian systems of one degree of freedom, that is, systems of the type

$$p' = -\partial H/\partial q, \; q' = \partial H/\partial p \quad (H = H(p, q), \; ' = d/dt), \tag{3}$$

will be considered. The identification with (1) is $x_1 = p, x_2 = q, f_1 = -\partial H/\partial q$,

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$f_2 = \partial H/\partial p$; the assumption that $f(x)$ of (1) be of class C^1 is now that the Hamiltonian $H(p, q)$ be of class C^2. The following theorem will be proved:

I. *Let* $H(p, q)$ *be of class* C^2 *on the* p, q *space, and let* Ω *denote an unrestricted invariant set of finite positive measure. Then through every point* $x = (p, q)$ *of* Ω, *except possibly for those belonging to a set* Z *of measure* 0, *there exists either an equilibrium solution* ($x \equiv$ const.) *or a periodic solution of* (3).

It will turn out that the set Z is the set excluded from the assertion of the Poincaré recurrence theorem on Ω (see §3 below). It was shown in **(7)**, however, that, even for general systems (1) satisfying (2) with dimension number n arbitrary, if the set Ω satisfies

$$\text{meas}\, \Omega\Sigma > 0, \tag{4}$$

where x is an arbitrary point of Ω and Σ is any open sphere (disk, in the present case) with center at x, then no point of a stable path can belong to Z. As a consequence, there follows the theorem

II. *Under the same assumptions as in* (I), *along with the additional condition* (4), *every stable solution path* $x(t) = (p(t), q(t))$ *is periodic* (*possibly constant*) *on* $-\infty < t < \infty$.

Needless to say, the condition (4) is fulfilled if, for instance, the set Ω is open or is the closure of an open set.

It should be noted that the existence of a closed (Jordan) curve in the p, q space of the form $H =$ const. does not necessarily imply that this is the path of a periodic solution of (3). One need only consider the physical example of a simple pendulum oscillating with an energy just sufficient to raise the pendulum (asymptotically) to its greatest possible height, corresponding to a position of unstable equilibrium.

For a general discussion of systems (1) when $n = 2$, see the series of papers of Poincaré **(5)**, especially the one of 1885. It should be noted that the notion of stability considered there **(5**, pp. 167–172**)** is not that of the present paper, but what is sometimes termed stability in the sense of Poisson. In this connection, compare the recurrence theorem of Poincaré **(6**, pp. 67 ff**)** cited at the beginning of §3 below. For further references to the case $n = 2$, see Birkhoff **(1)**, in particular pp. 123–124, and Brown **(2)**.

Another corollary of I is

III. *Let* $H = H(p, q)$ *be of class* C^2 *and suppose that the point* (p_0, q_0) *is an isolated equilibrium point of the system* (3). *If* (p_0, q_0) *is a stable point* (*that is, if the solution* $p \equiv p_0$, $q \equiv q_0$ *is stable*), *then it is either a* (*local*) *maximum or a minimum point of the funtion* $H(p, q)$.

The question as to whether there is a theorem corresponding to III for the case of a conservative dynamical system with n degrees of freedom was pointed out by Wintner **(8**, p. 101**)** and will remain undecided. It is known

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that the conclusion of III can become false if the restriction that (p_0, q_0) be an isolated equilibrium point is dropped (**8**, pp. 100–101).

The proof of III as a consequence of Theorem I will be given in §2; the proof of I will be given in §3.

2. Proof of III. Grant Theorem I. Since $x_0 = (p_0, q_0)$ is a stable equilibrium point, there exists a sequence of invariant (and, if desired, open) sets containing, and closing down upon, the point x_0 (Poincaré-Birkhoff criterion). Choose one of these sets and call it Ω; it is clear that the assumptions of I are now fulfilled.

Since x_0 is an isolated equilibrium point, it follows from I that through almost all points sufficiently close to x_0 there exist (non-constant) periodic solution of (3), corresponding to closed Jordan curves in the p, q space. Consider a sequence of such curves $C_1, C_2, \ldots$, which, in view of the stability assumption on x_0, tend to the point x_0. Since the energy integrals $H =$ const. of (2) are the equations of the solutions in the p, q plane, each of the curves C_n is a level curve of H. Hence the function H attains either a (local) maximum or a minimum value at at least one point, say x_n, *inside* each (Jordan) curve C_n. Clearly the x_n are equilibrium points and satisfy $x_n \to x_0$ as $n \to \infty$. Since x_0 is an isolated equilibrium point, then $x_n = x_0$ for n sufficiently large (and hence for some one value of n). Consequently, x_0 is either a maximum or a minimum point of H and the proof of III is complete.

3. Proof of I. The assumptions on Ω are those guaranteeing the validity of the Poincaré recurrence theorem (cf., in this connection, **6**, pp. 67 ff.; **8**, p. 91; **7**). Let Z denote the set of measure zero excluded from the assertion of this theorem, so that, if x_0 is not in Z, the solution path $x(t)$ through x_0 has the following property: if t^* is arbitrary, there exists a sequence of dates t_n, where $n = \pm 1, \pm 2, \ldots$, such that $t_n \to \infty$ or $-\infty$ according as $n \to \infty$ or $-\infty$ and $x(t_n) \to x(t^*)$ as $|n| \to \infty$. It will be shown that if $x_0 = (p_0, q_0)$ is in $\Omega - Z$, then the (vector) function $x(t)$, where $x_0 = x(t_0)$, is periodic or constant (for $-\infty < t < \infty$).

Suppose, if possible, that $x(t) \not\equiv$ const. and not periodic. Since $x_0 = x(t_0)$ is in $\Omega - Z$, there exist values t_n and points $x_n = x(t_n)$ such that $x_n \neq x_0$ and $x_n \to x_0$ as $n \to \infty$. Since $x(t) \not\equiv$ const., so that x_0 is not an equilibrium point, the general existence theorem along with the attending continuity properties for solutions of ordinary differential systems (1), guarantees the existence of points X_n on the curve $x(t)$ which lie on the normal line to this curve at the point x_0, and satisfy $X_n (\neq x_0) \to x_0$. Since the solution path curve $x(t)$ constitutes a branch of the locus $H(p, q) = c$, for some constant c, it is clear that the directional derivatives of H, taken along the tangent and the normal to the path $x(t)$ at the point x_0, are zero. Consequently, the vector grad H is zero at this point; and so x_0 is an equilibrium point, in contradiction with the supposition at the beginning of this paragraph. This completes the proof of I.

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