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A NOTE ON IMMERSING MANIFOLDS IN EUCLIDEAN SPACES

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Let M be a closed, connected smooth and 3-connected mod 2 (that is $H_i(M;\mathbb{Z}_2) = 0,\ 0 < i \le 3$) manifold of dimension $n = 7 + 8k$. Using a combination of cohomology operations on certain cohomology classes of M and on the Thom class of the stable normal bundle of M we show that under certain conditions M immerses in $\mathbb{R}^{2n-8}$. This extends previously known results for such a general manifold when the number of 1's in the dyadic expansion of n is less than 8.

1. Introduction

Let M be a smooth, closed, connected and 3-connected mod 2 manifold, whose dimension n is congruent to 7 mod 8. By Massey-Peterson [5] $\bar{w}_{n-i}(M) = 0$ for $i = 1,2,..,7$ where $\bar{w}_j(M)$ is the j-th mod 2 dual Stiefel-Whitney class of M. Then it is easily seen that M immerses in $\mathbb{R}^{2n-5}$. Following Ng [8, Theorem 1.2], if $Sq^1 H^{n-5}(M) \subset Sq^2 H^{n-6}(M)$, then M immerses in $\mathbb{R}^{2n-6}$.

We shall show that with certain additional hypotheses we can immerse M in $\mathbb{R}^{2n-7}$ or $\mathbb{R}^{2n-8}$.

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Throughout this paper all cohomology will be ordinary cohomology with mod 2 coefficients unless otherwise specified. Let $\dim M = n$.

2.

Let ν be a stable normal bundle of M. Then ν is classified by a map $g: M \to B\mathrm{spin}_N$ for some sufficiently large $N \geq n+1$ where $B\mathrm{spin}_j$ is the classifying space for spin j-plane bundles.

Consider the obvious inclusion $B\mathrm{spin}_{n-k} \to B\mathrm{spin}_N$ for $k = 7$ or 8. Then if $n \geq 15$ M immerses in $\mathbb{R}^{2n-k}$ if and only if g lifts to $B\mathrm{spin}_{n-k}$ if and only if the geometric dimension of $\nu \leq n-k$. Consider the n-modified Postnikov tower for $B\mathrm{spin}_{n-k} \to B\mathrm{spin}_N$ for $k = 7$ or 8. This is given in Table 1 or Table 2 of [7] where for $k = 8$ we drop the k_4^2 when $n \equiv 7 \bmod 16$. We list the results (only the k-invariants in the relevant dimensions) in Table 1 and Table 2. It is understood that if the need arises the tower is pulled back to $B\hat{SO}_N\langle 8\rangle$ the classifying space for spin N-plane bundles ξ satisfying $w_4(\xi) = 0$.

Let ϕ_4 and $\tilde{\phi}_5$ be the stable secondary cohomology operations associated with the following relations in the mod 2 Steenrod algebra A

$$\phi_4: \quad Sq^2(Sq^2Sq^1) = 0 \quad \text{and}$$

$$\tilde{\phi}_5: \quad (Sq^2Sq^1)(Sq^2Sq^1) + Sq^3Sq^3 = 0 \quad \text{respectively.}$$

It is easily seen that ϕ_4 and $\tilde{\phi}_5$ can be chosen to be spin trivial in the sense of [12]. That is to say for the Thom class U of the universal spin j-plane bundle over $B\mathrm{spin}_j$ for $j > 4$, $0 \in \phi_4(U)$ and $0 \in \tilde{\phi}_5(U)$.

As in [12] we derive the following relation:

$$\tilde{\psi}_5: \quad Sq^2\phi_4 + Sq^1\tilde{\phi}_5 = 0 .$$

Hence there is defined a stable tertiary operation $\tilde{\psi}_5$ associated with the above relation. Trivially $\tilde{\psi}_5$ is spin-trivial.

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It is appropriate at this point to say that all the theorems in Ng [7] hold with the operation ψ_5 replaced by $\tilde{\psi}_5$. This is readily deduced from the generating class theorem of Thomas [11] and the following proposition, which is inspired by Proposition 4.2 of [12].

PROPOSITION 2.1. (Thomas). *Let* w_{n-9} *be the* $(n-9)$*-th mod 2 universal Stiefel-Whitney class considered as in* $H^{n-9}(Bspin_{n-7})$.

(a) $(0, 0) \in (\phi_4, \tilde{\phi}_5)(w_{n-9}) \subset H^{n-5}(Bspin_{n-7}) \oplus H^{n-4}(Bspin_{n-7})$.

(b) $0 \in \tilde{\psi}_5(w_{n-9}) \subset H^{n-4}(Bspin_{n-7})$.

Proof. Part (a): Let $j\colon Bspin_{n-9} \to Bspin_{n-7}$ be the inclusion. Then $j^*\colon H^*(Bspin_{n-7}) \to H^*(Bspin_{n-9})$ is an epimorphism. In dimension $\leq n-5$ j^* is a monomorphism while in dim $n-4$ Ker j^* is generated by $\{w_4.w_{n-8}\}$. Since $(\phi_4,\tilde{\phi}_5)$ is spin-trivial, $(0,0) \in (\phi_4,\tilde{\phi}_5)(w_{n-9}) \subset H^{n-5}(Bspin_{n-9}) \oplus H^{n-4}(Bspin_{n-9})$. Therefore there are classes $v \in H^{n-5}(Bspin_{n-7})$ and $u \in H^{n-4}(Bspin_{n-7})$ such that $(v,u) \in (\phi_4,\tilde{\phi}_5)(w_{n-9}) \subset H^{n-5}(Bspin_{n-7}) \oplus H^{n-4}(Bspin_{n-7})$ and $j^*(v,u) = (0,0)$. Thus $v = 0$ and $u = \alpha w_4.w_{n-8}$ for some $\alpha \in \mathbb{Z}_2$. But $Sq^5 w_{n-9} = w_4.w_{n-8}$ and so by redefining $\tilde{\phi}_5$ as $\tilde{\phi}_5 + \alpha Sq^5$ if need be we may assume that $u = 0$. Hence there is a choice of operation $(\phi_4,\tilde{\phi}_5)$ such that $(0,0) \in (\phi_4,\tilde{\phi}_5)(w_{n-9}) \subset H^{n-5}(Bspin_{n-7}) \oplus H^{n-4}(Bspin_{n-7})$. This proves part (a).

Part (b): First we claim that $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, Bspin_{n-9}) = j^*\mathrm{Indet}^{n-4}(\tilde{\psi}_5, Bspin_{n-7})$. Since $Sq^1(w_4.w_{n-9}) = w_4.w_{n-8}$ it follows that $0 \in \tilde{\psi}_5(w_{n-9}) \subset H^{n-4}(Bspin_{n-7})$. Now we shall establish the claim.

$\mathrm{Indet}^{n-4}(\tilde{\psi}_5, Bspin_{n-9})$ is the range of a cohomology operation

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defined on cohomology vectors $(x,y) \in H^{n-7}(B\mathrm{spin}_{n-9}) \times H^{n-7}(B\mathrm{spin}_{n-9})$ such that $\mathrm{Sq}^2 x = 0$ and $\mathrm{Sq}^2\mathrm{Sq}^1 x + \mathrm{Sq}^3 y = 0$. Since j^* is an eqimorphism there are classes x' and y' in $H^{n-7}(B\mathrm{spin}_{n-7})$ such that $j^*(x',y') = (x,y)$. Since j^* is a monomorphism in dim $n-5$, $j^*(\mathrm{Sq}^2 x') = \mathrm{Sq}^2 x = 0$ implies that $\mathrm{Sq}^2 x' = 0$. Since $j^*(\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^3 y') = 0$, $\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^3 y' = \alpha w_4.w_{n-8}$ for some $\alpha \in \mathbb{Z}_2$. We shall show that $w_4.w_{n-8} \notin \mathrm{Sq}^2\mathrm{Sq}^1\, {}^{n-7}(B\mathrm{spin}_{n-7}) + \mathrm{Sq}^3 H^{n-7}(B\mathrm{spin}_{n-7})$ and so $\alpha = 0$. Thus $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, B\mathrm{spin}_{n-9}) = j^*\mathrm{Indet}^{n-4}(\tilde{\psi}_5, B\mathrm{spin}_{n-7})$.

Consider the case $n = 15$, that is $n-7 = 8$. According to Quillen [9]

$$H^*(B\mathrm{spin}_{n-7}) = H^*(B\mathrm{spin}_8) = \mathbb{Z}_2[w_4,w_6,w_7,w_8] \otimes \mathbb{Z}_2[\eta_8] \text{ and}$$

$$H^*(B\mathrm{spin}_{n-9}) = H^*(B\mathrm{spin}_6) = \mathbb{Z}_2[w_4,w_6] \otimes \mathbb{Z}_2[\eta_8],$$

where η_8 corresponds to the vanishing of w_9. Therefore $H^8(B\mathrm{spin}_6) \simeq \langle w_4^2, \eta_8\rangle$. Note that $\mathrm{Sq}^1\eta_8 = 0 \in H^*(B\mathrm{spin}_{n-7})$. Clearly $\mathrm{Sq}^2\mathrm{Sq}^1 w_4^2 = \mathrm{Sq}^3 w_4^2 = 0$. Now if $\mathrm{Sq}^3\eta_8 = \alpha w_4 w_{n-8} = \alpha w_4.w_7 \in H^{11}(B\mathrm{spin}_{n-7})$, then $0 = \mathrm{Sq}^3\mathrm{Sq}^3\eta_8 = \mathrm{Sq}^3(\alpha w_4.w_7) = \alpha w_7^2$ and so $\alpha = 0$. Thus for $n = 15$, $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, B\mathrm{spin}_{n-9}) = j^*\mathrm{Indet}^{n-4}(\tilde{\psi}_5, B\mathrm{spin}_{n-7})$.

Now assume $n > 15$. According to [9], $H^*(B\mathrm{spin}_{n-7})$ is a polynomial algebra in dimension $\leq n-4$ generated by the universal mod 2 Stiefel-Whitney classes $A = \{w_i \mid 4 \leq i \leq n-7$ and i is not of the form $2^p + 1,\ p \geq 0\}$ except possibly for a non-trivial relation $v_{2^k+1} = 0$ in dimension $n-6$ for n of the form $7+2^k$ corresponding to the vanishing of $v_{2^k+1} = \mathrm{Sq}^{2^{k-1}}\mathrm{Sq}^{2^{k-2}}\ldots\mathrm{Sq}^2\mathrm{Sq}^1 w_2$ in $H^*(B\mathrm{spin}_{n-7})$. Let F be the polynomial algebra over $\mathbb{Z}_2$ generated by A. For a monomial

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$y = x_1^{e_1} x_2^{e_2} \dots x_k^{e_k}$ in F, $k \geq 1$, $e_i \geq 1$, $x_i \in A$, define the length $\ell(y)$ of y to the sum $e_1 + e_2 + \dots + e_k$. Define for a sum of monomials $y_1 + y_2 + \dots + y_j$, where the y_i's are distinct, the length to be $\ell(y_1 + y_2 + \dots + y_j) = \max\{\ell(y_k), 1 \leq k \leq j\}$. As convention we define $\ell(0) = \infty$. Consider F as an $\mathbb{A}$ algebra via the Wu formula, the relations $v_{2^k+1} = 0$, $1 < 2^k + 1 \leq n-7$ and $w_i = 0$, $i > n-7$. Then $F \to H^*(B\mathrm{spin}_{n-7})$ is an $\mathbb{A}$-isomorphism in dimension $\leq n-7$. Thus we can consider x' and y' as in F. Now $\ell(w_4 . w_{n-8}) = 2$. Clearly if $\ell(x') \geq 3$ then $\ell(\mathrm{Sq}^2 \mathrm{Sq}^1 x') \geq 3$. Similarly if $\ell(y') \geq 3$ then $\ell(\mathrm{Sq}^3 y') \geq 3$. So if $\ell(x') \geq 3$ or if $\ell(y') \geq 3$ then $\ell(\mathrm{Sq}^2 \mathrm{Sq}^1 x' + \mathrm{Sq}^3 y') \geq 3$. So we may assume that $\ell(x') = \ell(y') = 2$ since $\mathrm{Sq}^2 \mathrm{Sq}^1 w_{n-7} = \mathrm{Sq}^3 w_{n-7} = 0 \in H^*(B\mathrm{spin}_{n-7})$. Let G be the subalgebra of F generated by monomials of length 2. Then by using the Wu formula we see that elements in $(\mathrm{Sq}^2 \mathrm{Sq}^1 G)_{n-4}$ are of the form

$$w_{8k-4j+1} . w_{4j+2} + w_{8k-4j+2} . w_{4j+1} ,$$

where $n-7 = 8k$ and $1 \leq j \leq (n-9)/4$. A similar analysis shows that the elements in $(\mathrm{Sq}^3 G)_{n-4}$ are of the form

$$w_{8k-4j+3} . w_{4j} + w_{8k-4j+2} . w_{4j+1} + w_{8k-4j+1} . w_{4j+2} + w_{8k-4j} . w_{4j+3} ,$$

where $1 \leq j < (n-7)/4$. Thus $(\mathrm{Sq}^2 \mathrm{Sq}^1 G + \mathrm{Sq}^3 G)_{n-4}$ is generated by

$$\{w_{8k-4j+1} . w_{4j+2} + w_{8k-4j+2} . w_{4j+1},\ w_{8k-4j+3} . w_{4j} + w_{8k-4j} . w_{4j+3},\ 1 \leq j < (n-7)/4\}.$$

Here w_{2^p+1} is thought of as in F via the relations $v_{2^j+1} = 0$. Hence we conclude that $w_4 . w_{n-8}$ could not be in $\mathrm{Sq}^2 \mathrm{Sq}^1 H^{n-7}(B\mathrm{spin}_{n-7}) + \mathrm{Sq}^3 H^{n-7}(B\mathrm{spin}_{n-7})$. This completes the proof of part (b).

Table 1.

The n-Postnikov tower for $\pi : B\mathrm{spin}_{n-7} \to B\mathrm{spin}_N$

k-invariant	Dimension	Defining Relation
k_1^1	$n-6$	$k_1^1 = \delta w_{n-7}$
k_2^1	$n-5$	$k_2^1 = w_{n-5}$
k_3^1	$n-3$	$k_3^1 = w_{n-3}$
k_1^2	$n-5$	$Sq^2 k_1^1 = 0$
k_2^2	$n-4$	$Sq^2 k_2^1 + Sq^3 k_1^1 = 0$
k_3^2	$n-3$	$(Sq^4 + w_4) k_1^1 = 0$
k_6^2	n	$(Sq^4 + w_4) k_3^1 = 0$
k_1^3	$n-4$	$Sq^2 k_1^2 = 0$
k_4^3	n	$(\chi Sq^4 + w_4) k_3^2 + Sq^2 Sq^4 k_1^2 = 0$

Table 2

The n-Postnikov tower for $\pi : B\mathrm{spin}_{n-8} \to B\mathrm{spin}_N$

k-invariant	Dimension	Defining Relation
k^1	$n-7$	$k^1 = w_{n-7}$
k_1^2	$n-5$	$Sq^2 Sq^1 k^1 = 0$
k_2^2	$n-3$	$(Sq^4 + w_4) Sq^1 k^1 = 0$
$k_4^2 (n \equiv 15(16))$	n	$(Sq^8 + w_8) k^1 = 0$
k_1^3	$n-4$	$Sq^2 k_1^2 = 0$
k_3^3	n	$Sq^2 Sq^4 k_1^2 + (\chi Sq^4 + w_4) k_2^2 = 0$

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Recall ζ_6 and ζ_8 (for $n \equiv 15 \bmod 16$) are the stable cohomology operations of Hughes-Thomas type associated with the relations in the mod 2 Steenrod algebra,

$$\zeta_6 : Sq^4 Sq^{n-3} + Sq^2(Sq^{n-3}Sq^2) + Sq^1(Sq^{n-3}Sq^3 + Sq^{n-1}Sq^1) = 0$$

and

$$\zeta_8 : Sq^8(Sq^{n-7}) + Sq^4(Sq^{n-7}Sq^4) + Sq^2(Sq^{n-3}Sq^2 + Sq^{n-7}Sq^2Sq^4)$$
$$+ Sq^1(Sq^{n-1}Sq^1 + Sq^{n-5}Sq^5 + Sq^{n-3}Sq^3 + Sq^{n-7}Sq^7) = 0 .$$

In [7] we have defined a stable tertiary operation Ω realizing the k-invariant k_4^3 of Table 1 or k_3^3 of Table 2. Let $\phi_{1,1}$ be the Adams basic operation associated with the relation $Sq^2Sq^2 + Sq^3Sq^1 = 0$. Then we have the following theorem.

THEOREM 2.2. *Let $N > n$ and η be an N-plane bundle over M with $w_4(\eta) = w_4(M)$. Suppose $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = Sq^2H^{n-6}(M)$ (hence $Sq^1H^{n-5}(M) \subset Sq^2H^{n-6}(M)$).*

(a) (Case $k=7$). Suppose $Sq^2H^{n-7}(M;\mathbb{Z}) = Sq^2H^{n-7}(M)$ and $\mathrm{Indet}^n(k_4^3, M) \neq 0$, where k_4^3 is defined by Table 1. Then the geometric dimension of $\eta \leq n-7$ if and only if

$$\delta w_{n-7}(\eta) = 0,\ w_{n-5}(\eta) = 0,\ 0 \in \phi_4(w_{n-9}(\eta)),\ 0 \in \phi_{1,1}(w_{n-7}(\eta)) ,$$
$$\zeta_6(U(\eta)) = 0 \ \text{and}\ 0 \in \tilde{\psi}_5(w_{n-9}(\eta)) .$$

(b) (Case $k=8$). Suppose $\mathrm{Indet}^n(k_3^3, M) \neq 0$ where k_3^3 is defined by Table 2.

(i) Suppose $n \equiv 7\ (16)$ with $n > 7$ and $Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M)$. Then geometric dimension of $\eta \leq n-8$ if and only if

$$w_{n-7}(\eta) = 0,\ 0 \in \phi_4(w_{n-9}(\eta)) \ \text{and}\ 0 \in \tilde{\psi}_5(w_{n-9}(\eta)) .$$

(ii) Suppose $n \equiv 15 \bmod 16$ with $n > 15$ and $w_4(\eta) = 0$. Suppose <u>either</u> $w_8(\eta) = w_8(M)$ and $Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M)$ or

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$Sq^2 H^5(M) = 0$. Then geometric dimension of $\eta \le n-8$ if and only if $w_{n-7}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \zeta_8(U(\eta))$ and $0 \in \tilde{\psi}_5(w_{n-9}(\eta))$.

Proof. Part (a) is a consequence of Proposition 2.1 and [7, Theorem 7.1] since all the k-invariants are stable. Part (b) follows from [7, Theorem 7.2] noting that we need only consider stable k-invariants.

For any bundle ξ over M classifed by a map g from M into $B\hat{S}O_j\langle 8\rangle$, $j \ge 4$, define $\nu_4(\xi)$ to be $g^*(\nu_4)$, where $\nu_4 \in H^4(B\hat{S}O_j\langle 8\rangle) \approx \mathbb{Z}_2$ is a generator. We can easily extend this definition to a stable bundle ξ satisfying $w_4(\xi) = w_2(\xi) = w_1(\xi) = 0$.

We have the following theorem when the top dimensional tertiary obstruction has trivial indeterminacy.

THEOREM 2.3. *Let $N > n$ and η be an N-plane bundle over M with $w_4(\eta) = w_4(M) = 0$. Suppose $Sq^3(\nu_4(-\eta) + \nu_4(-\tau)) = 0$ and $Indet^{n-4}(\tilde{\psi}_5, M) = Indet^{n-4}(k_1^3, M)$, where k_1^3 is defined by Table 1 if $k = 7$ and by Table 2 if $k = 8$. (Hence $Sq^1 H^{n-5}(M) \subset Sq^2 H^{n-6}(M)$.)*

(a) (Case $k=7$). Suppose $Sq^2 H^{n-7}(M;\mathbb{Z}) = Sq^2 H^{n-7}(M)$, and $Indet^n(k_4^3, M) = 0$, where k_4^3 is defined by Table 1. Then the geometric dimension of $\eta \le n-7$ if and only if $\delta w_{n-7}(\eta) = 0$, $w_{n-5}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \phi_{1,1}(w_{n-7}(\eta))$, $\zeta_6(U(\eta)) = 0$, $0 \in \tilde{\psi}_5(w_{n-9}(\eta))$ and $\Omega(U(\eta)) = 0$.

(b) (Case $k=8$). Suppose $Sq^2 H^{n-7}(M) = Sq^2 Sq^1 H^{n-8}(M)$ and $Indet^n(k_3^3, M) = 0$ where k_3^3 is defined by Table 2.

(i) Suppose $n \equiv 7 \mod 16$ with $n > 7$. Then the geometric dimension of $\eta \le n-8$ if and only if $w_{n-7}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \tilde{\psi}_5(w_{n-9}(\eta))$ and $\Omega(U(\eta)) = 0$.

(ii) Suppose $n \equiv 15 \mod 16$ with $n > 15$ and either $w_8(\eta) = w_8(M)$ or $Sq^2 H^5(M) = 0$. Then the geometric dimension of

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$$\eta \leq n-8 \text{ if and only if } w_{n-7}(\eta) = 0,\ 0 \in \phi_4(w_{n-9}(\eta)),$$
$$0 \in \zeta_8(U(\eta)),\ 0 \in \tilde{\psi}_5(w_{n-9}(\eta)) \text{ and } \Omega(U(\eta)) = 0.$$

Proof. Part (a) is a consequence of Proposition 2.1 and [7, Theorem 8.1] and Part (b) is a consequence of Proposition 2.1 and [7, Theorem 8.2].

3. Immersion Theorems

Let M' be a closed, connected and smooth spin manifold of dimension $n \equiv 7 \bmod 8$ with $n > 7$. Following Massey-Peterson [5] we deduce that $\bar{w}_{n-i}(M') = 0$ for $i = 0,1,2,\ldots,7$. In particular if the number of 1's in the dyadic expansion of n $\alpha(n)$ is greater than or equal to 6, then $\bar{w}_{n-9}(M') = 0$. If furthermore $w_4(M') = 0$ then $\bar{w}_{n-9}(M') = 0$ for $n \equiv 15 \bmod 16$ or $n \equiv 7 \bmod 16$ and $\alpha(n) \geq 6$.

Take a Spivak normal bundle ν for M. Then the top class of the Thom space $T(\nu)$ is spherical. Therefore $\zeta_6(U(\nu))$, $\zeta_8(U(\nu))$ and $\Omega(U(\eta))$ whenever they are defined are all zero modulo zero indeterminacy.

Therefore applying Theorem 2.2 together with the preceding paragraph we have the following theorem.

THEOREM 3.1. *Suppose* $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = Sq^2H^{n-6}(M)$.

(a) *Suppose* $\alpha(n) \geq 6$, $Sq^2H^{n-7}(M;\mathbb{Z}) = Sq^2H^{n-7}(M)$ *and* $\mathrm{Indet}^n(k_4^3(\nu), M) \neq 0$, *where* k_4^3 *is defined by Table 1. Then* M *immerses in* $\mathbb{R}^{2n-7}$.

(b) *Suppose* $\mathrm{Indet}^n(k_3^3(\nu), M) \neq 0$ *where* k_3^3 *is defined by Table 2 and* $Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M)$. *Then* M *immerses in* $\mathbb{R}^{2n-8}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$.

Similarly from Theorem 2.3 we have

THEOREM 3.2. *Let* $w_4(M) = 0$.

(a) *Suppose* $Sq^2H^{n-7}(M;\mathbb{Z}) = Sq^2H^{n-7}(M)$ *and* $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = \mathrm{Indet}^{n-4}(k_1^3(\nu), M)$, *where* k_1^3 *is defined by Table 1. Then* M *immerses in* $\mathbb{R}^{2n-7}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or* $n \equiv 15 \bmod 16$.

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(b) *Suppose* $Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M)$ *and* $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = \mathrm{Indet}^{n-4}(k_1^3(\nu), M)$, *where* k_1^3 *is defined by Table 2. Then* M *immerses in* $\mathbb{R}^{2n-8}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or* $n \equiv 15 \bmod 16$ *with* $n > 15$.

Combining Theorem 3.1 and Theorem 3.2 we have the following theorem.

THEOREM 3.3. *Suppose* $w_4(M) = 0$ *and* $\mathrm{indet}^{n-4}(\tilde{\psi}_5, M) = Sq^2H^{n-6}(M)$.

(a) *Suppose* $Sq^2H^{n-7}(M;\mathbb{Z}) = Sq^2H^{n-7}(M)$. *Then* M *immerses in* $\mathbb{R}^{2n-7}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or if* $n \equiv 15 \bmod 16$.

(b) *Suppose* $Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M)$. *Then* M *immerses in* $\mathbb{R}^{2n-8}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or if* $n \equiv 15 \bmod 16$ *and* $n > 15$.

If M is 4-connected mod 2 then $\mathrm{Indet}^n(k_4^3(\nu), M) = 0$. Thus by Theorem 3.2 we have the following immediate corollary.

COROLLARY 3.4. *Suppose* M *is* 4-*connected mod* 2.

(a) *Suppose* $Sq^2H^{n-7}(M;\mathbb{Z}) = Sq^2H^{n-7}(M)$. *Then* M *immerses in* $\mathbb{R}^{2n-7}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or if* $n \equiv 15 \bmod 16$.

(b) *Suppose* $Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M)$. *Then* M *immerses in* $\mathbb{R}^{2n-8}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or if* $n \equiv 15 \bmod 16$ *and* $n > 15$.

Assume now $w_4(M) = 0$. From the definition of $\tilde{\psi}_5$ we deduce that if either $\mathrm{Sq}^3H^{n-7}(M) = 0$ or $\mathrm{Sq}^2\mathrm{Sq}^1H^{n-7}(M) = 0$ or equivalently if either $\mathrm{Sq}^2\mathrm{Sq}^1H^4(M) = 0$ or if $\mathrm{Sq}^3H^4(M) = 0$, then $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = \phi_3 D^{n-7} + \zeta_3 \tilde{D}^{n-7}$, where ϕ_3 and ζ_3 are stable operations associated with the relations

$$\phi_3 \colon \mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^1(\mathrm{Sq}^2\mathrm{Sq}^1) = 0 \quad \text{and}$$

$$\zeta_3 \colon \mathrm{Sq}^1\mathrm{Sq}^3 = 0 \quad \text{respectively;}$$

$D^{n-7} = \{x \in H^{n-7}(M) \mid \mathrm{Sq}^2x = \mathrm{Sq}^2\mathrm{Sq}^1x = 0\}$ and $\tilde{D}^{n-7} = \{x \in H^{n-7}(M) \mid \mathrm{Sq}^3x = 0\}$.

We can choose ζ_3 to be $\phi_{0,0} \circ \mathrm{Sq}^2$ where $\phi_{0,0}$ is the operation associated with the relation $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. If $H_6(M;\mathbb{Z})$ has no 2-

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torsion then $Sq^3 \nu_4(-\tau) = 0$ and so $\mathrm{Indet}^n(k_4^3, M) = 0$ by S-duality. If further $Sq^1 H^{n-5}(M) \subset Sq^2 H^{n-6}(M)$ and $Sq^2 H^5(M) = 0$ then $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$, where k_1^3 is defined by Table 1. If in addition that $H_7(M;\mathbb{Z})$ has no free parts and its 2-torsion elements are all of order 2, then $\mathrm{Indet}^{n-4}(\tilde{\psi}_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$, where k_1^3 is defined by Table 2.

Thus we have from Theorem 3.2

THEOREM 3.5. *Suppose* $w_4(M) = 0$, $Sq^2 H^5(M) = 0$, $Sq^1 H^{n-5}(M) \subset Sq^2 H^{n-6}(M)$ *and* $H_6(M, \mathbb{Z})$ *has no 2-torsion elements. Then*

(a) M *immerses in* $\mathbb{R}^{2n-7}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or* $n \equiv 15 \bmod 16$.

(b) *Suppose* $H_7(M;\mathbb{Z})$ *has no free parts and its 2-torsion elements are at most of order 2. Then* M *immerses in* $\mathbb{R}^{2n-8}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or* $n \equiv 15 \bmod 16$ *and* $n > 15$.

Suppose now $Sq^1 H^4(M) = 0$ and $\phi_{0,0} H^4(M) = 0$. By Poincaré daulity one readily deduces that $\phi_{0,0} H^{n-5}(M) = 0$. As for Theorem 3.5 we deduce from Theorem 3.2 the following:

COROLLARY 3.6. *Suppose* $w_4(M) = 0$, $Sq^1 H^4(M) = 0$, $\phi_{0,0} H^4(M) = 0$ *and* $H_6(M;\mathbb{Z})$ *has no 2-torsion elements.*

(a) M *immerses in* $\mathbb{R}^{2n-7}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or* $n \equiv 15 \bmod 16$.

(b) *Suppose* $H_7(M;\mathbb{Z})$ *has no free parts and its 2-torison elements are at most of order 2. Then* M *immerses in* $\mathbb{R}^{2n-8}$ *if* $n \equiv 7 \bmod 16$ *and* $\alpha(n) \geq 6$ *or* $n \equiv 15 \bmod 16$ *and* $n > 15$.

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