Two infinite integrals

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§ 1. *Introduction.*

1·1. The two functions $F(\lambda, \theta)$ and $G(\lambda, \theta)$, defined by the infinite integrals (1) and (2) respectively, below, occur in Kottler's theoretical discussion[1] of the diffraction of a monochromatic plane wave by a perfectly black half plane. Some properties of these functions have been investigated by several recent writers.

Copson and Ferrar[2] obtained, by a somewhat laborious method, the Fourier expansion of $F(\lambda, \theta)$ with respect to θ. The coefficients of this expansion turned out to be "cut" Bessel functions of the third kind with the argument λ. By the aid of this expansion Copson and Ferrar discussed the behaviour of $F(\lambda, \theta)$ for small values of λ.

Watson[3] obtained the expansions of both $F(\lambda, \theta)/\cos\frac{1}{2}\theta$ and $G(\lambda, \theta)/\cos\frac{1}{2}\theta$ into power series in $\sin\frac{1}{2}\theta$, convergent when $|\sin\frac{1}{2}\theta| < 1$, and also discussed the behaviour of $F(\lambda, \theta)$ and $G(\lambda, \theta)$ when θ approaches one of the values $\pm\pi$ lying on the circumference of the circle of convergence of his series.

In a recent note[4] I myself obtained a *definite* integral representation of $F(\lambda, \theta)$, (3), and emphasised that this integral representation seems to be a better expedient for discussing the behaviour of $F(\lambda, \theta)$ when λ is small.

1·2. In the following lines I propose to add some remarks to the theory of the functions $F(\lambda, \theta)$ and $G(\lambda, \theta)$.

The proof of the Fourier expansion of Copson and Ferrar is "less easy than one might expect," and Watson pointed out the series to be very slowly convergent. Both difficulties I believe to be caused by the fact that the function $F(\lambda, \theta)$ is *not* a *periodic* function in θ. The representation by a definite integral shows, however, that $F(\lambda, \theta)$

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[1] F. Kottler, *Ann. der Physik* 71 (1923), 457-508 (496, 499).

[2] E. T. Copson and W. L. Ferrar, *Proc. Edin. Math. Soc.* (2), 5 (1938), 159-68.

[3] G. N. Watson, *Proc. Edin. Math. Soc.* (2), 5 (1938), 173-81.

[4] A. Erdélyi, *Proc. Edin. Math. Soc.* (2), 6 (1939), 11.

is the sum of an elementary function, $\theta e^{-i\lambda\cos\theta}/2\pi$, and a function periodic in θ. It is therefore very probable that expanding only the second, periodic, term, a more rapidly convergent Fourier series can be found. In § 2 this series is quite easily obtained, using only the generating function of Bessel coefficients (i.e., of Bessel functions of first kind whose order is an integer), integrating term by term a uniformly convergent series, and using Bateman's generalisation of Kapteyn's integral. The convergence of the Fourier series obtained thus is comparable with the convergence of the exponential series, whereas the convergence of Copson and Ferrar's Fourier expansion is comparable with the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-)^n}{n} \sin 2n\theta .$$

In § 3 another expansion of $F(\lambda, \theta)$ is obtained, in terms of Lommel's function $s_{\mu,\nu}(\lambda)$. This expansion, being a power series in $\cos\theta$, is useful for numerical computation of $F(\lambda, \theta)$ for values of θ near to $\frac{1}{2}\pi$. Both series mentioned hitherto exhibit the behaviour of $F(\lambda, \theta)$ for small values of λ.

In § 4 I propose to show that the coefficients of Watson's expansions of $F(\lambda, \theta)$ and $G(\lambda, \theta)$ are expressible in terms of Whittaker's confluent hypergeometric function $W_{k,m}(z)$. Moreover, $G(\lambda, \theta)$ itself is expressible in terms of the Error function.

§ 2. *The Fourier expansion.*

2·1. The functions in question are those defined by the infinite integrals

$$F(\lambda, \theta) \equiv \frac{1}{2\pi} \int_0^\infty e^{i\lambda\cosh t} \frac{\sin\theta}{\cosh t + \cos\theta} dt \tag{1}$$

and

$$G(\lambda, \theta) \equiv \frac{1}{\pi} \int_0^\infty e^{i\lambda\cosh t} \frac{\cos\frac{1}{2}\theta \cosh\frac{1}{2}t}{\cosh t + \cos\theta} dt. \tag{2}$$

Both integrals are absolutely convergent if the restrictions $\mathfrak{J}(\lambda) > 0$, $|\mathfrak{R}(\theta)| < \pi$ are supposed to be fulfilled by the complex variables λ and θ.

The first of these functions is also representable in the form

$$F(\lambda, \theta) = e^{-i\lambda\cos\theta} \left\{\frac{\theta}{2\pi} - \frac{1}{4}\sin\theta \int_0^\lambda H_0^{(1)}(t)\, e^{it\cos\theta}\, dt\right\}, \tag{3}$$

in which only a *definite* integral occurs. This representation I derived from Kottler's differential equation for $F(\lambda, \theta)$, but it can be as well derived immediately from (1), putting in (1)

$$e^{i\lambda \cosh t} \cdot \frac{1}{\cosh t + \cos \theta} = i e^{-i\lambda \cos\theta} \int_0^\lambda e^{is(\cosh t + \cos\theta)}\, ds + e^{-i\lambda\cos\theta} \cdot \frac{1}{\cosh t + \cos\theta},$$

and inverting the order of integration in the first of the terms obtained thus.

2·2. Copson and Ferrar obtained, in the paper quoted above, the Fourier expansion

$$F(\lambda, \theta) = \tfrac{1}{2} \sum_{n=1}^{\infty} i^{-n-1} h_n^{(1)}(\lambda) \sin n\theta. \tag{4}$$

In this expansion $h_n^{(1)}(\lambda)$ is a "cut Bessel function of the third kind," obtained by omitting all terms containing negative powers of λ from the expansion of $H_n^{(1)}(\lambda)$ in ascending powers of λ. This expansion is valid when $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$; when $\frac{1}{2}\pi < \pm\theta < \pi$, the term $\pm\frac{1}{2}e^{-i\lambda\cos\theta}$ has to be added to the expansion on the right.

Watson pointed out that this series, though very elegant, converges very slowly, since for fixed λ and large n we have

$$h_n^{(1)}(\lambda) = \frac{2i^{-n-1}}{n\pi} \cos \tfrac{1}{2}n\pi + O(n^{-2}).$$

A more rapidly convergent Fourier series can be found for the second term of the finite integral representation (3), that is to say for

$$F(\lambda, \theta) - \frac{\theta}{2\pi} e^{-i\lambda\cos\theta} = -\frac{1}{4} \sin\theta \int_0^\lambda H_0^{(1)}(t)\, e^{-i(\lambda - t)\cos\theta}\, dt. \tag{5}$$

$F(\lambda, \theta)$ is defined by this equation obviously for *any* finite values of λ and θ, real or complex, and the restrictions mentioned in 2.1 can be removed.

2·3. To expand (5) into a Fourier series, we differentiate Jacobi's expansion[1]

$$e^{-i(\lambda - t)\cos\theta} = J_0(\lambda - t) + 2 \sum_{n=1}^{\infty} i^{-n} J_n(\lambda - t) \cos n\theta$$

[1] G. N. Watson, *Theory of Bessel Functions* (Cambridge, 1922), § 2·22 (3), (4). This book will be referred to in the sequel as *Bessel Functions*.

with respect to θ. Term by term differentiation being permissible by reason of the uniform convergence of Jacobi's expansion in every finite domain of the variables $\lambda - t$ and θ, we obtain

$$\sin\theta \, . \, e^{-i(\lambda-t)\cos\theta} = -\frac{2}{\lambda - t}\sum_{n=1}^{\infty} i^{-n-1}\, n J_n(\lambda - t)\sin n\theta.$$

This series being absolutely and uniformly convergent for $\theta \leqq t \leqq \lambda$ (or on the path of integration, if λ happens to be complex), term by term integration is permissible and yields

$$\begin{aligned} F(\lambda, \theta) &= \frac{\theta}{2\pi} e^{-i\lambda\cos\theta} - \frac{1}{4}\sin\theta \int_0^{\lambda} H_0^{(1)}(t)\, e^{-i(\lambda-t)\cos\theta}\, dt \\ &= \frac{\theta}{2\pi} e^{-i\lambda\cos\theta} + \frac{1}{2}\sum_{n=1}^{\infty} i^{-n-1}\, n \sin n\theta \int_0^{\lambda} H_0^{(1)}(t) J_n(\lambda - t)\frac{dt}{\lambda - t}. \qquad (6) \end{aligned}$$

2·4. To evaluate the integrals in (6) we deal at first with the more general integral

$$\int_0^{\lambda} H_\nu^{(1)}(t)\, J_n(\lambda - t)\frac{dt}{\lambda - t}, \quad (n = 1, 2, \ldots.). \qquad (7)$$

In order to make this integral convergent we must suppose the real part of ν to be between -1 and $+1$. Now[1]

$$H_\nu^{(1)}(t) = \frac{J_{-\nu}(t) - e^{-\nu\pi i} J_\nu(t)}{i \sin \nu\pi},$$

and hence

$$\int_0^{\lambda} H_\nu^{(1)}(t) J_n(\lambda - t)\frac{dt}{\lambda - t} = \frac{1}{i\sin\nu\pi}\left\{\int_0^{\lambda} J_{-\nu}(t) J_n(\lambda - t)\frac{dt}{\lambda - t} - e^{-\nu\pi i}\int_0^{\lambda} J_\nu(t) J_n(\lambda - t)\frac{dt}{\lambda - t}\right\}.$$

According to Bateman's extension of Kapteyn's integral[2] we have

$$\int_0^{\lambda} J_\nu(t)\, J_n(\lambda - t)\frac{dt}{\lambda - t} = \frac{1}{n} J_{n+\nu}(\lambda),$$

and thus

$$n\int_0^{\lambda} H_\nu^{(1)}(t)\, J_n(\lambda - t)\frac{dt}{\lambda - t} = \frac{J_{n-\nu}(\lambda) - e^{-\nu\pi i} J_{n+\nu}(\lambda)}{i\sin\nu\pi}.$$

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[1] *Bessel Functions*, § 3·61 (5).

[2] H. Bateman, *Proc. London Math. Soc.* (2), 3 (1905), 111-23 (120). See also *Bessel Functions*, § 12·2 (3).

Approaching the limit $\nu \to 0$ on both sides of this equation—the carrying out of the limiting process under the sign of integration is easily justifiable—we obtain by the rule of L'Hospital

$$n \int_0^\lambda H_0^{(1)}(t)\, J_n(\lambda - t) \frac{dt}{\lambda - t} = -\frac{2}{\pi} i^{n-1} \left[\frac{\partial}{\partial \nu}\{e^{-\frac{1}{2}\nu\pi i} J_\nu(\lambda)\}\right]_{\nu=n}$$
$$= J_n(\lambda) - \frac{2}{\pi i}\left[\frac{\partial}{\partial \nu} J_\nu(\lambda)\right]_{\nu=n}. \tag{8}$$

2·5. Putting (8) in (6), we obtain, instead of Copson and Ferrar's expansion (4), the Fourier series

$$F(\lambda, \theta) = \frac{\theta}{2\pi} e^{-i\lambda\cos\theta} + \frac{1}{\pi}\sum_{n=1}^{\infty} \sin n\theta \left[\frac{\partial}{\partial\nu}\{e^{-\frac{1}{2}\nu\pi i} J_\nu(\lambda)\}\right]_{\nu=n}$$
$$= \frac{\theta}{2\pi} e^{-i\lambda\cos\theta} + \frac{1}{2}\sum_{n=1}^{\infty} i^{-n-1} \sin n\theta \left\{J_n(\lambda) - \frac{2}{\pi i}\left[\frac{\partial J_\nu(\lambda)}{\partial\nu}\right]_{\nu=n}\right\}. \tag{9}$$

From our derivation it is clear that this expansion is uniformly convergent in any finite domain of the (real or complex) variables λ and θ. Moreover, this expansion shows that $F(\lambda, \theta)$ can be written in the form

$$F(\lambda, \theta) = F_1(\lambda, \theta) + \log\lambda \,.\, F_2(\lambda, \theta),$$

with two functions, F_1 and F_2, which are integral functions of both of the complex variables λ and θ.

We can easily obtain information on the rapidity of convergence of (9), using[1]

$$\left[\frac{\partial J_\nu(\lambda)}{\partial\nu}\right]_{\nu=n} = \log(\tfrac{1}{2}\lambda)\, J_n(\lambda) - \sum_{m=0}^{\infty} \frac{(-)^m (\frac{1}{2}\lambda)^{n+2m}}{m!\,(n+m)!} \psi(n+m+1).$$

Hence for fixed λ and large values of n

$$J_n(\lambda) - \frac{2}{\pi i}\left[\frac{\partial J_\nu(\lambda)}{\partial\nu}\right]_{\nu=n} = \left\{1 - \frac{2}{\pi i}\log(\tfrac{1}{2}\lambda)\right\} J_n(\lambda) + \frac{2}{\pi i}\sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}\lambda)^{n+2m}}{m!\,(n+m)!}\psi(n+m+1)$$
$$= \left\{1 - \frac{2}{\pi i}\log(\tfrac{1}{2}\lambda) + \frac{2}{\pi i}\psi(n+1)\right\} \frac{(\frac{1}{2}\lambda)^n}{n!}\{1 + O(n^{-1})\}.$$

From this asymptotic form it is seen that the absolute convergence of (9) is comparable with the convergence of the exponential series with argument $\frac{1}{2}|\lambda|\exp\{|\mathfrak{J}(\theta)|\}$.

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[1] *Bessel Functions*, § 3·52.

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§ 3. *Expansion in a series of Lommel's functions.*

3·1. We start from[1]

$$\int^{\lambda} t^{\mu} H_0^{(1)}(t)\,dt = (\mu - 1)\,\lambda\, H_0^{(1)}(\lambda)\, S_{\mu-1,\,-1}(\lambda) + \lambda\, H_1^{(1)}(\lambda)\, S_{\mu,\,0}(\lambda), \qquad (10)$$

and put in this formula[2]

$$S_{\mu,\nu}(\lambda) = s_{\mu,\nu}(\lambda) + 2^{\mu-1}\,\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu + \tfrac{1}{2})\,\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2})$$
$$\times\,[\sin\tfrac{1}{2}(\mu-\nu)\pi\,.\,J_{\nu}(\lambda) - \cos\tfrac{1}{2}(\mu-\nu)\pi\,.\,Y_{\nu}(\lambda)]. \qquad (11)$$

Here

$$s_{\mu,\nu}(\lambda) = \frac{\lambda^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\;{}_1F_2(1;\,\tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{3}{2},\,\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{3}{2};\,-\tfrac{1}{4}\lambda^2) \quad (12)$$

denotes Lommel's function[3].

Putting (11) into (10) we obtain after some algebra

$$\int_0^{\lambda} t^{\mu} H_0^{(1)}(t)\,dt = (\mu - 1)\,\lambda\, H_0^{(1)}(\lambda)\, s_{\mu-1,\,-1}(\lambda) + \lambda\, H_1^{(1)}(\lambda)\, s_{\mu,\,0}(\lambda), \qquad (13)$$

this formula being valid for $\Re(\mu) > -1$.

3·2. Now, in virtue of (3),

$$F(\lambda,\theta) = e^{-i\lambda\cos\theta}\left\{\frac{\theta}{2\pi} - \tfrac{1}{4}\sin\theta\,.\int_0^{\lambda} H_0^{(1)}(t)\sum_{n=0}^{\infty}\frac{(it\cos\theta)^n}{n!}\,dt\right\}$$
$$= e^{-i\lambda\cos\theta}\left\{\frac{\theta}{2\pi} - \tfrac{1}{4}\sin\theta\sum_{n=0}^{\infty}\frac{i^n\cos^n\theta}{n!}\int_0^{\lambda} t^n H_0^{(1)}(t)\,dt\right\},$$

term by term integration being permissible by reason of the absolute and uniform convergence of $\Sigma\,(it\cos\theta)^n/n!$ in $0 \leqq t \leqq \lambda$, and of the absolute convergence of each of the integrals in the last equation.

Using (13) in the last equation we arrive at

$$F(\lambda,\theta) = e^{-i\lambda\cos\theta}\left\{\frac{\theta}{2\pi} - \tfrac{1}{4}\lambda\sin\theta\, H_0^{(1)}(\lambda)\sum_{n=0}^{\infty}\frac{n-1}{n!}\,i^n\cos^n\theta\, s_{n-1,\,-1}(\lambda)\right.$$
$$\left. - \tfrac{1}{4}\lambda\sin\theta\, H_1^{(1)}(\lambda)\sum_{n=0}^{\infty}\frac{i^n}{n!}\cos^n\theta\, s_{n,\,0}(\lambda)\right\}. \qquad (14)$$

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[1] *Bessel Functions*, § 10·74 (5). We have used further $H_{-1}^{(1)}(\lambda) = -H_1^{(1)}(\lambda)$.

[2] *Bessel Functions*, § 10·71 (3).

[3] *Bessel Functions*, § 10·7 (10).

Both infinite series in (14) converge absolutely and uniformly in every finite domain of the complex variables λ and θ. (14) is useful for calculating $F(\lambda, \theta)$ for values of θ near $\frac{1}{2}\pi$.

In order to judge the rapidity of the convergence of the two infinite series in (14), we remark, that, according to (12), for fixed λ and large values of n

$$s_{n-1,\,-1}(\lambda) = \frac{\lambda^n}{n^2 - 1}\{1 + O(n^{-1})\}$$

and

$$s_{n,\,0}(\lambda) = \frac{\lambda^{n+1}}{(n+1)^2}\{1 + O(n^{-1})\}.$$

Hence the series of the moduli of the terms of both infinite series in (14) converge like

$$\sum_{n=0}^{\infty} \frac{|\lambda \cos\theta|^n}{(n+1)!} \text{ and } |\lambda| \sum_{n=0}^{\infty} \frac{|\lambda\cos\theta|^n}{(n+1)(n+1)!}$$

respectively, that is to say, like the power series of $\exp|\lambda\cos\theta|$.

3·3. From (14) we obtain especially

$$F(\lambda, \tfrac{1}{2}\pi) = \tfrac{1}{4} + \tfrac{1}{4}\lambda H_0^{(1)}(\lambda)\, s_{-1,\,-1}(\lambda) - \tfrac{1}{4}\lambda H_1^{(1)}(\lambda)\, s_{0,\,0}(\lambda). \tag{15}$$

It is important to remark that in this formula only tabulated functions occur.

Indeed, if the two parameters of Lommel's function happen to be equal, Lommel's function is expressible in terms of Struve's function $\mathbf{H}_\nu$. For (12) yields

$$s_{\nu,\nu}(\lambda) = \frac{\lambda^{\nu+1}}{2\nu+1}\, {}_1F_2(1;\tfrac{3}{2}, \nu+\tfrac{3}{2}; -\tfrac{1}{4}\lambda^2);$$

but[1]

$$\mathbf{H}_\nu(\lambda) = \frac{(\frac{1}{2}\lambda)^{\nu+1}}{\Gamma(\frac{3}{2})\Gamma(\nu+\frac{3}{2})}\, {}_1F_2(1;\tfrac{3}{2}, \nu+\tfrac{3}{2}; -\tfrac{1}{4}\lambda^2),$$

and therefore

$$s_{\nu,\nu}(\lambda) = 2^\nu\,\Gamma(\tfrac{3}{2})\,\Gamma(\nu+\tfrac{1}{2})\,\mathbf{H}_\nu(\lambda). \tag{16}$$

In particular

$$s_{0,\,0}(\lambda) = \tfrac{1}{2}\pi\,\mathbf{H}_0(\lambda) \tag{17}$$

and

$$s_{-1,\,-1}(\lambda) = -\tfrac{1}{2}\pi\,\mathbf{H}_{-1}(\lambda) = \tfrac{1}{2}\pi\,\mathbf{H}_1(\lambda) - 1. \tag{18}$$

[1] *Bessel Functions*, § 10·4 (2).

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In (18) the recurrence formula of Struve functions[1]

$$\mathbf{H}_{\nu-1}(\lambda) + \mathbf{H}_{\nu+1}(\lambda) = \frac{2\nu}{\lambda}\mathbf{H}_\nu(\lambda) + \frac{(\frac{1}{2}\lambda_\nu)}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{3}{2})}$$

with $\nu = 0$ was used.

Thus we can write instead of (15)

$$F(\lambda, \tfrac{1}{2}\pi) = \tfrac{1}{4} - \tfrac{1}{4}\lambda H_0^{(1)}(\lambda) + \tfrac{1}{8}\pi\lambda\{H_0^{(1)}(\lambda)\mathbf{H}_1(\lambda) - H_1^{(1)}(\lambda)\mathbf{H}_0(\lambda)\}. \quad (19)$$

Tables of Struve's functions occurring here are found in Watson's *Bessel Functions*[2] and in Jahnke-Emde's *Tables of functions*[3].

3·4. A few words more may be said concerning the computation of $F(\lambda, \theta)$ for any values of θ from (14). It is only necessary to deal with the computation of $s_{\nu+n,\nu}(\lambda)$ $(\nu = 0, 1;\ n = 0, 1, 2, \ldots)$.

We have seen that $s_{\nu,\nu}(\lambda)$ can be computed by the aid of tables of Struve's functions. Having this, the recurrence formula[4] of Lommel's functions

$$s_{\mu+2,\nu}(\lambda) = \lambda^{\mu+1} - [(\mu+1)^2 - \nu^2]\, s_{\mu,\nu}(\lambda) \quad (20)$$

furnishes us successively with $s_{\nu+2,\nu}(\lambda)$, $s_{\nu+4,\nu}(\lambda)$, ...

Again,

$$s_{\nu+1,\nu}(\lambda) = \frac{\lambda^{\nu+2}}{4(\nu+1)}\,{}_1F_2(1;\, 2, \nu+2;\, -\tfrac{1}{4}\lambda^2) = \lambda^\nu - 2^\nu\Gamma(\nu+1)J_\nu(\lambda). \quad (21)$$

From this equation, in connection with (20), $s_{\nu+1,\nu}(\lambda)$, $s_{\nu+3,\nu}(\lambda)$, ... can be computed.

For small values of λ, or θ near to $\frac{1}{2}\pi$, only a few terms of the infinite series in (14) are to be taken in account.

§ 4. *Expansions in ascending powers of* $\sin\frac{1}{2}\theta$.

4·1. Watson obtained the expansions

$$F(\lambda;\theta) = \tfrac{1}{4}\cos\tfrac{1}{2}\theta \sum_{n=0}^{\infty} f_n(\lambda)\sin^{2n+1}\tfrac{1}{2}\theta$$

and

$$G(\lambda;\theta) = \tfrac{1}{4}\cos\tfrac{1}{2}\theta \sum_{n=0}^{\infty} g_n(\lambda)\sin^{2n}\tfrac{1}{2}\theta,$$

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[1] *Bessel Functions*, § 10·4 (5).

[2] Pp. 666-97.

[3] Third Edition (1938), 218-23.

[4] *Bessel Functions*, § 10·72 (1).

in which f_n and g_n are defined by the integrals

$$f_n(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{e^{i\lambda\cosh t}\,dt}{\cosh^{2n+2}\frac{1}{2}t}, \tag{22}$$

and

$$g_n(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{e^{i\lambda\cosh t}\,dt}{\cosh^{2n+1}\frac{1}{2}t}. \tag{23}$$

I propose to show that f_n and g_n can be expressed in terms of Whittaker's confluent hypergeometric function $W_{k,m}(z)$.

4·2. Both $f_n(\lambda)$ and $g_n(\lambda)$ can be expressed in terms of the function

$$\phi_\nu(z) \equiv \frac{2}{\pi}\int_0^\infty e^{-\frac{1}{2}z\cosh t}\cosh^{-4\nu}\tfrac{1}{2}t\,dt. \tag{24}$$

This integral is absolutely convergent if $|\arg z| < \frac{1}{2}\pi$.

The substitution $\cosh t = 2u + 1$ in (24) yields the integral representation

$$\phi_\nu(z) = \frac{2}{\pi}e^{-\frac{1}{2}z}\int_0^\infty e^{-uz}u^{-\frac{1}{2}}(u+1)^{-\frac{1}{2}-2\nu}\,du. \tag{25}$$

Comparing this with the definition of Whittaker's $W_{k,m}$-function[1]

$$W_{k,m}(z) = \frac{z^k e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty e^{-t}t^{-\frac{1}{2}-k+m}\left(1+\frac{t}{z}\right)^{-\frac{1}{2}+k+m}dt,$$

$[\Re(\frac{1}{2}-k+m) > 0]$, we immediately see that (26)

$$\phi_\nu(z) = \frac{2}{\sqrt{\pi}}z^{\nu-\frac{1}{4}}W_{-\nu,-\nu}(z). \tag{27}$$

4·3. Now the equations

$$f_n(\lambda) = \phi_{\frac{1}{2}+\frac{1}{2}n}(-2i\lambda) = \frac{2}{\sqrt{\pi}}(-2i\lambda)^{\frac{1}{2}n}W_{-\frac{1}{2}-\frac{1}{2}n,\,-\frac{1}{2}-\frac{1}{2}n}(-2i\lambda) \tag{28}$$

and

$$g_n(\lambda) = \phi_{\frac{1}{4}+\frac{1}{2}n}(-2i\lambda) = \frac{2}{\sqrt{\pi}}(-2i\lambda)^{\frac{1}{2}n-\frac{1}{4}}W_{-\frac{1}{4}-\frac{1}{2}n,\,-\frac{1}{4}-\frac{1}{2}n}(-2i\lambda) \tag{29}$$

at once follow. In both of these equations $|\arg(-2i\lambda)| < \pi$ is to be taken.

Thus, the coefficients in Watson's expansions can be expressed

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[1] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge, 1927), § 16·12. To compare (25) and (26) put $t = uz$.

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in terms of Whittaker's function. The connection of f_{-1} with Bessel functions, namely

$$f_{-1}(\lambda) = \tfrac{1}{2}\pi i\, H_0^{(1)}(\lambda),$$

and the recurrence formulae

$$(2n+1) f_n(\lambda) = (2n + 4i\lambda) f_{n-1}(\lambda) - 4i\lambda f_{n-2}(\lambda)$$

and

$$2n\, g_n(\lambda) = (2n - 1 + 4i\lambda)\, g_{n-1}(\lambda) - 4i\lambda\, g_{n-2}(\lambda),$$

obtained by Watson, at once follow from the theory of the $W_{k,m}$ function.

It is easily seen that the functions $g_n(\lambda)$ are expressible in terms of the Error function and its derivatives[1].

§ 5. *Expression of $G(\lambda, \theta)$ in terms of the Error function.*

5·1. I conclude by noticing that $G(\lambda, \theta)$ itself is expressible in terms of Gauss's Error function

$$\operatorname{Erfc}(x) \equiv \int_x^\infty e^{-t^2}\, dt$$

with complex argument.

To exhibit the connection between $G(\lambda, \theta)$ and the Error function we put in (2)

$$\cosh t = 1 + i\frac{v}{\lambda},$$

v being the new variable of integration. This substitution yields, rotating the path of integration through a right angle,

$$\begin{aligned} G(\lambda, \theta) &= \frac{(-2i\lambda)^{-\frac{1}{2}} e^{i\lambda}}{2\pi \cos\frac{1}{2}\theta} \int_0^\infty e^{-v} v^{-\frac{1}{2}} \left(1 + \frac{v}{-2i\lambda\cos^2\frac{1}{2}\theta}\right)^{-1} dv \\ &= \frac{(-2i\lambda)^{-\frac{1}{4}} e^{i\lambda \sin^2\frac{1}{2}\theta}}{2\sqrt{(\pi\cos\frac{1}{2}\theta)}}\, W_{-\frac{1}{4}, -\frac{1}{4}}(-2i\lambda\cos^2\tfrac{1}{2}\theta), \end{aligned}$$

according to (26).

Now[2],

$$\operatorname{Erfc}(x) = \tfrac{1}{2} x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}\, W_{-\frac{1}{4}, \pm\frac{1}{4}}(x^2),$$

and hence

$$G(\lambda, \theta) = \frac{1}{\sqrt{\pi}} e^{-i\lambda\cos\theta} \operatorname{Erfc}(\sqrt{(-2i\lambda)}\cos\tfrac{1}{2}\theta) = \frac{1}{\sqrt{\pi}} e^{-i\lambda\cos\theta} \int_{\sqrt{(-2i\lambda)}\,.\cos\frac{1}{2}\theta}^\infty e^{-t^2}\, dt. \quad (30)$$

[1] See also *Modern Analysis*, § 16·2.

[2] *Modern Analysis*, § 16·12.

It does not seem to be very easy to deduce (30) directly from (2). So far as I see either double (improper) integrals or fractional integration by parts must be used.

5·2. There seems to be no equally simple expression for $F(\lambda, \theta)$. I only succeeded in expressing this function in terms of a certain kind of confluent hypergeometric function of *two* variables. I omit the deduction of this expression, however, because it does not seem to be of any use in the computation of $F(\lambda, \theta)$.

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