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LOCAL-GLOBAL CRITERIA FOR OUTER PRODUCT RINGS

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Let R be a commutative ring with multiplicative identity. We say that R is an outer product ring (OP-ring) if each vector v in the exterior power $\bigwedge^n R^{n+1}$ is decomposable; i.e. $v = v_1 \wedge v_2 \wedge \ldots \wedge v_n$ with $v_i \in R^{n+1}$ (alternatively, each $n + 1$ tuple of elements in R is the tuple of $n \times n$ minors of some $n \times (n + 1)$ matrix with entries in R).

Lissner initiated the study of which commutative rings are OP-rings, showing that any Dedekind domain is an OP-ring [**13**]. Towber classified the local Noetherian OP-rings as those with maximal ideal generated by two elements, and Hinohara's reformulation of Towber's result extended the context to semi-local rings [**18**, **11**]. Assuming the stronger condition that all Plücker vectors are decomposable and designating such rings as Towber rings, Lissner and Geramita gave necessary conditions for a Noetherian ring to be a Towber ring [**14**]. The authors completed the latter investigation, showing that a reduced Noetherian ring is a Towber ring if and only if its localizations at prime ideals are Towber rings (regular local rings of dimension at most 2) and it has the property that its orientable modules are free ($\bigwedge^n P \simeq R$ with n a positive integer implies $P \simeq R^n$) [**7**].

In the spirit of this last result we provide in this note similar local-global criteria characterizing a class of Noetherian OP-rings. In particular, reduced Noetherian OP-rings are those reduced Noetherian rings whose orientable modules are free and whose localizations at maximal ideals are OP-rings.

We conclude with a series of consequences concerning the number of generators of ideals in OP-rings and their polynomial extensions analogous to those which appeared in [**7**].

1. OP-rings. The rings R within are commutative, Noetherian and have a multiplicative identity. R^n denotes a free n dimensional R module and $I(u)$ denotes the ideal in R generated by the coordinates of $u \in R^n$ in some (hence any) basis for R^n. We begin with the statements of two lemmas:

LEMMA 1. *Direct sums, homomorphic images, and localizations of* OP-*rings are* OP-*rings.*

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LEMMA 2. *Non-maximal prime ideals in semi-local* OP-*rings are principal.*

The proof of Lemma 1 is straightforward. Lemma 2 follows from Towber's proof in the local case (in [**18**]) and the Forster–Swan Theorem [**4**, Corollary 5].

LEMMA 3. *For each $u \in R^n$ there is a decomposition $u = rv$, $r \in R$ and $v \in R^n$, with the property that no principal minimal prime ideal of R contains $I(v)$.*

Proof. If P is a minimal prime containing $I(u)$ then P is the radical of a primary component $Q = Q_1$ in an irredundant primary representation $I(u) = Q_1 \cap \ldots \cap Q_t$. If P is principal generated by p then, as is an easy exercise, $Q = p^iR$ for some positive integer i. Set $u = p^iv$. We claim we may adjust v so that $I(v) \not\subseteq P$, so assume $I(v) \subseteq P$. Then $I(u) = p^iI(v) \subseteq p^{i+1}R$. Since the radical of Q_i is not contained in P for all $i \neq 1$, there exists $x \in \bigcap_{i\neq 1} Q_i - P$ (take $x = 1$ if $t = 1$). Since $p^ix \in I(u) \subseteq p^{i+1}R$, $(x + py)p^i = 0$, $y \in R$. Therefore $\text{ann}_R p^iR \not\subseteq pR$. Now select $w \in (\text{ann}_R p^iR)R^n$ so that $I(v + w) \not\subseteq pR$. Then

$$u = p^i(v + w) \quad \text{and} \quad I(v + w) \not\subseteq pR.$$

The lemma now follows by induction on the number of minimal prime ideals of R which contain $I(u)$.

Towber's characterization of local OP-rings as those local rings with maximal ideal having two generators and its consequences (Proposition 6 and its Corollary in [**7**]) provide the initial characterizations of OP-rings. Additionally, the OP-property for a ring R implies that R-oriented modules are free ($\bigwedge^n P \simeq R$ with n a positive integer implies $P \simeq R^n$) as is shown in Theorem 7 [**7**]. We isolate therefore the following conditions on a ring R:

(#) R-oriented modules are free and the localizations of R at maximal ideals are OP-rings.

Thus conditions (#) are satisfied by any OP-ring. We shall show in Theorem 1 below that conditions (#) suffice to imply that a reduced ring is an OP-ring. Note, for use in the proof of that theorem, that Hinohara's result [**11**] is that any semi-local ring satisfying (#) is an OP-ring.

If $E \subseteq R^n$ and μ represents this imbedding, let

$$\bigwedge^n \mu : \bigwedge^n E \to \bigwedge^n R^n$$

denote the induced mapping of the exterior powers. Let $N(E)$ denote the ideal generated by the coefficients of $(\bigwedge^n \mu)(v_1 \wedge \ldots \wedge v_n)$ with respect to a fixed generator of $\bigwedge^n R^n$ for all $v_i \in E$. For ideals I, J in R let $[J : I] = \{r \in R : rI \subseteq J\}$ denote the residual quotient of J by I.

LEMMA 4. *Let K be an ideal in R and E a submodule of R^n such that $N(E) \subseteq K$. Assume that for each maximal ideal $m \supseteq [N(E) : K]$ the localization R_m is regular of dimension at most 2 and the ideal KR_m is principal. Then there exists a submodule F of R^n containing E and such that $N(F) = K$.*

Proof. Let F denote a submodule of R^n maximal with respect to the conditions: $E \subseteq F$ and $N(F) \subseteq K$. It will suffice to show that $N(F)R_m = KR_m$ for each maximal ideal $m \supseteq [N(E) : K]$. Now R_m is a regular local ring of dimension at most 2 and $KR_m = sR_m$ for some $s \in K$. We may assume $s \neq 0$, in which case there is a submodule G of ${R_m}^n$ such that $F_m \subseteq G$ and $N(G) = sR_m$ [**7**, Theorem 1(i) and Proposition 4]. We claim that $F_m = G$, thereby completing the proof. Let $v \in G$ and select $t \notin m$ so that $tv \in R^n$ and $tv \wedge v_2 \wedge \ldots \wedge v_n \in K(\bigwedge^n R^n)$ for all choices v_i from a set of generators of F. Then $N(F + tvR) \subseteq K$, and $tv \in F$ follows. Hence $v \in F_m$.

THEOREM 1. *If R is reduced then conditions (#) are necessary and sufficient for R to be an* OP-*ring.*

Proof. The remarks following Lemma 3 give the necessity of conditions (#) in OP-rings. Now assume these conditions are present in R and let $0 \neq u \in \bigwedge^n R^{n+1}$. We shall attempt to find a non-zerodivisor $s \in R$ and $u_1, \ldots, u_n \in R^{n+1}$ giving the decomposition $su = u_1 \wedge \ldots \wedge u_n$, and such that the columns of the matrix U with rows $u_1, \ldots, u_n$ generate a submodule of a module $F \subseteq R^n$ having the property that $N(F) = sR$. For then $\bigwedge^n(F) \simeq sR \simeq R$ [**7**, Proposition 2], and our hypothesis implies that F is free. Thus if A denotes a matrix having as rows a basis for F then $(\det A)R = sR$ and $U = AV$ with V an $n \times (n + 1)$ matrix over R. If $v_1, \ldots, v_n$ denote the rows of V then $u = v_1 \wedge \ldots \wedge v_n$.

Begin by letting $P_1, \ldots, P_a$ denote the minimal prime ideals in R. Lemma 1 allows us to assume that R cannot be expressed as a direct product of two rings in a nontrivial way, hence

$$R \neq P_i + \textstyle\bigcap_{j \neq i} P_j \quad \text{for each} \quad 1 \leqq i \leqq a.$$

Let S denote the complement in R of the union of the associated prime ideals of R, the associated prime ideals of $I(u)$, and if $a > 1$ the associated prime ideals of the ideals $P_i + \bigcap_{j \neq i} P_j$, $\quad i = 1, \ldots, a$.

Since R_S is a semi-local OP-ring, Lemmas 2, 3 imply the existence of $r \in R_S$, $v = v_1 \wedge \ldots \wedge v_n \in \bigwedge^n(R^{n+1})$ such that $u = rv$ and $I(v)$ is not contained in P_iR_S whenever $P_i \supseteq I(u)$. (Note P_iR_S is not maximal since $P_i \supseteq I(u)$ and R reduced imply $a > 1$. Hence P_i is properly contained in an associated prime ideal of $P_i + \bigcap_{j \neq i} P_j$.) It follows that there exist $s \in S$, $t \in R$ and $u_1, \ldots, u_n \in R^{n+1}$ such that

$$su = tu_1 \wedge \ldots \wedge u_n$$

and no P_i contains both $I(u)$ and $I(u_1 \wedge \ldots \wedge u_n)$.

Now let E denote the submodule of R^n generated by sR^n, the columns of the matrix having rows $tu_1, u_2, \ldots, u_n$, and all vectors in R^n with first entry 0 and remaining entries from $\operatorname{ann}_R tR$. This choice of E satisfies

$$N(E) \subseteq sR \quad \text{and} \quad [N(E) : sR] \supseteq s^{n-1}R + I(u) + (\operatorname{ann}_R tR)^{n-1}.$$

It remains only to show that the maximal ideals containing $[N(E) : sR]$ have height 2, for then Lemma 4 implies the existence of the submodule in the opening remarks of this proof.

Assuming the contrary, we let $m \supseteq [N(E) : sR]$ be a maximal ideal of height $\leqq 1$. Since $s \in m \cap S$ is a non-zerodivisor, m has height 1. Also, $m \supseteq I(u)$ and fails to be an associated prime of $I(u)$. Therefore $m \supseteq P_i \supseteq I(u)$ for some i, $1 \leqq i \leqq a$, and $a > 1$ since $I(u) \neq 0$ and R is reduced. Since $I(u)$ contains $tI(u_1 \wedge \ldots \wedge u_n)$ and $I(u_1 \wedge \ldots \wedge u_n)$ is not contained in P_i, $t \in P_i$. Since $m \supseteq \operatorname{ann}_R tR$ and $t \in P_i$ implies $\operatorname{ann}_R tR \supseteq \bigcap_{i \neq j} P_j$,

$$m \supseteq P_i + \bigcap_{j \neq i} P_j.$$

Since m has height 1, m is an associated prime ideal of $P_i + \bigcap_{j \neq i} P_j$, a contradiction to the fact that m meets S.

The proof of Theorem 1 is complicated by the fact that the minimal prime ideals of R need not be principal. Assuming minimal prime ideals were principal, we would proceed via Lemma 3 to the case where $I(u)$ is not contained in any minimal prime ideal. The technicalities involving the minimal primes in the proof of Theorem 1 can then be discarded.

COROLLARY 1. *If R satisfies conditions (#) and minimal prime ideals in R are principal then R is an* OP*-ring.*

The context of Corollary 1 includes the possible OP-rings R for which the polynomial ring $R[x]$ is an OP-ring [**9**].

Actually, it suffices for the proof of Theorem 1 to exhibit the module E so that the residual quotient $[N(E) : sR]$ is contained in only maximal ideals having regular localizations. This is easily accomplished should there be only finitely many maximal ideals m for which R_m is not regular; simply exclude these from the multiplicative system S.

COROLLARY 2. *If R satisfies conditions (#) and has only finitely many maximal ideals with non-regular localizations then R is an* OP*-ring.*

We suspect conditions (#) will suffice to show that such a Noetherian ring is an OP-ring, but a nonzero nilradical still poses a problem.

2. Complete intersections. In this section we extend to OP-rings several results presented in Section 4 of [**7**] concerning the minimal number of generators $v_R(I)$ of ideals I in a Towber ring R.

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LEMMA 5. *Let R be semi-local and $0 \neq u \in \bigwedge^p R^n$ a Plücker vector such that the maximal ideals which contain $I(u)$ have regular localizations of dimension at most 2. Then u is decomposable.*

Proof. From the exact sequence

$$0 \longrightarrow u^{\perp} \longrightarrow R^n \underset{\wedge u}{\longrightarrow} \bigwedge^{p+1} R^n \longrightarrow \operatorname{cok}(\wedge u) \longrightarrow 0$$

where $u^{\perp} = \{v \in R^n : v \wedge u = 0\}$, we conclude that $u_Q{}^{\perp}$ is a free module of rank p for each prime ideal $Q \supseteq I(u)$ [**17**, Theorem 1.1] and also if $Q \not\supseteq I(u)$ [**17**, Theorem 2.1]. Hence $u^{\perp}$ is a free module and u is decomposable [**17**, Theorem 2.2].

LEMMA 6. *Let $u \in \bigwedge^p R^n$ with su decomposable for some non-zerodivisor s in R. If R_m is regular of dimension at most 2 for each maximal ideal $m \supseteq sR + I(u)$ and R-oriented modules are free, then u is decomposable.*

Proof. The proof is outlined in the opening comments of the proof of Theorem 1.

THEOREM 2. *If R satisfies conditions (#) and I is an ideal in R such that height $I \geqq 1$ and $v_{R_m}(I) \leqq 2$ for each maximal ideal m in R then $v_R(I) \leqq 2$.*

Proof. By virtue of the Forster–Swan Theorem [**4**, Corollory 5] we may assume I can be generated by three elements. Let $I = aR + bR + cR$ and let $S \subseteq R^3$ be the module of solutions (x, y, z) of $ax + by + cz = 0$. Since I_P has two generators for each associated prime ideal P of I, two of a, b, c generate I_P. Thus there exists $u_P \in S_P$ such that $I(u_P)_P \not\subseteq P_P$. It now follows that there is a solution $u = (x, y, z) \in S$ such that $I(u)$ is not contained in any associated prime of I. Set $(u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}) = (a, b, c, z, -y, x)$. Then

$$u = \sum u_{ij} e_i \wedge e_j \in \bigwedge^2 R^4$$

is a Plücker vector with the property that $I(u)$ has height at least 2. Lemma 5 implies that u is decomposable over the ring of quotients of R, and Lemma 6 implies that u is decomposable over R. Let $u = v \wedge w$ with $v = (v_1, v_2, v_3, v_4)$, $w = (w_1, w_2, w_3, w_4) \in R^4$. Since $u_{1i} = v_1 w_i - w_1 v_i$,

$$I \subseteq v_1 R + w_1 R.$$

If I has primary component Q with radical P, then one of u_{ij}, $2 \leqq i < j \leqq 4$, is not in P. Since u_{ij} is the determinant of the matrix for the homogeneous system $v_1 w_i - w_1 v_i \equiv v_1 w_j - w_1 v_j \equiv 0$ modulo P,

$$(v_1, w_1) u_{ij} \equiv 0 \text{ modulo } P.$$

Since $v_1, w_1 \in Q$ follows, $I = v_1 R + w_1 R$.

If A is a finitely generated module over a ring R of finite Krull dimension we denote by

$$e_R(A) = \max \{v_{R_P}(A) + \dim R/P : \dim R/P < \dim R\}.$$

A conjecture of Evans and Eisenbud [**5**], recently shown to be true by A. Sathaye [**15**] and N. Mohan Kumar [**12**] states that any such module A may be generated by $e_R(A)$ elements in $R = S[x]$. A key portion of the proof of this conjecture was the following:

THEOREM. [**12**, Theorem 1] *Let N be the nilradical of R. If $v_{R/N}(I) \leqq e_{R/N}(I)$ for each ideal I in R/N of height $\geqq 1$ then $v_R(A) \leqq e_R(A)$ for each finitely generated R module A.*

The above theorem allows a sharpening of the usual Forster–Swan bounds for generating modules over rings satisfying conditions (#). Corollary 3 below extends Theorem 2 herein and Proposition 9a [**1**] for two dimensional rings.

COROLLARY 3. *If R satisfies conditions (#) and R has Krull dimension 2 then $v_R(A) \leqq e_R(A)$ for any finitely generated R module A.*

Proof. Lemma 1 and Theorem 2.26 [**16**] imply that R/N inherits the hypothesis on R. In view of the above theorem, we may assume R is reduced. We need only show that $v_R(I) \leqq e_R(I)$ for each ideal of R of height at least 1. If $e_R(I) \geqq 3$ then $v_R(I) \leqq e_R(I)$ follows from the Forster–Swan Theorem. If $e_R(I) = 2, v_R(I) \leqq e_R(I)$ follows from Theorem 2. There remains only to see that $e_R(I) \leqq 1$ cannot occur. Since R has dimension 2 there is a non-maximal prime ideal P having height 1. Since

$$e_R(I) \geqq v_{R_P}(IR_P) + \dim R/P = v_{R_P}(IR_P) + 1,$$

$e_R(I) \leqq 1$ implies $IR_P = 0$. But R reduced and height $I \geqq 1$ implies $\operatorname{ann}_R I = 0$. Therefore $e_R(I) > 1$.

We close this note with some results about the generation of ideals in polynomial rings. These results and their proofs represent mild modifications of similar results having previously appeared. We list them here in a context which includes OP-rings.

THEOREM 3. *Assume that R has Krull dimension at most 2, that oriented R-modules are free and that R_m is a regular local ring for each maximal ideal m of height 2. Then oriented $R[x]$ modules are free.*

Proof. The proof follows along the lines of that of Theorem 2.3.1 [**8**]. The needed modifications are straightforward.

COROLLARY 4. *Let R be as in Theorem 3 and set $T = R[x_1, \ldots, x_n]$. Let m be a maximal ideal of T and let $m \cap R = P$. If either height $P > 0$ or*

$n \geqq 2$ then

$$v_T(m) = v_{T_m}(mT_m).$$

In particular, if T_m is regular then $v_T(m) =$ height m, hence m is a complete intersection.

Proof. All cases are covered in [**3**] except the case $n = 1$, height $P = 1$ and T_m regular. But in this case Theorem 3 [**7**] applies in light of Theorem 3 above.

In the above result the restriction on the height of P may be dropped if P is assumed maximal. Otherwise, Proposition 3.1 [**6**] indicates this restriction is necessary but can be removed with certain additional assumptions (for example, if R is a domain and each projective $R[x]$ module is extended from R (Corollary 11 [**2**])).

We include the last case mentioned in the proof of Corollary 5 in the following special case of Theorem 3 [**7**].

COROLLARY 5. *Let R be as in Theorem 3. If I is an ideal in $R[x]$ such that $R[x]/I$ is semi-local and I locally has two generators, then I has two generators.*

For ideals of the type described in Corollary 5 the result is an improvement on the number of generators predicted by the Evans and Eisenbud conjecture.

COROLLARY 6. *Let R be as in Theorem 3. If I is an ideal in $R[x]$ such that I has homological dimension one and I/I^2 is free over $R[x]/I$, then I has two generators. Moreover, I is generated by an R-sequence.*

Proof. See the proof given for Proposition 5 [**10**].

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