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Geometric level raising for p-adic automorphic forms

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Abstract

We present a level-raising result for families of p-adic automorphic forms for a definite quaternion algebra D over $\mathbb{Q}$. The main theorem is an analogue of a theorem for classical automorphic forms due to Diamond and Taylor. We show that certain families of forms old at a prime l intersect with families of l-new forms (at a non-classical point). One of the ingredients in the proof of Diamond and Taylor's theorem (which also played a role in earlier work of Taylor) is the definition of a suitable pairing on the space of automorphic forms. In our situation one cannot define such a pairing on the infinite dimensional space of p-adic automorphic forms, so instead we introduce a space defined with respect to a dual coefficient system and work with a pairing between the usual forms and the dual space. A key ingredient is an analogue of Ihara's lemma which shows an interesting asymmetry between the usual and the dual spaces.

1. Introduction

Classical level-raising results typically show that if the reduction mod p of a level N modular form f has certain properties (depending on a prime $l \neq p$), then there exists a modular form g of level Nl, new at l, with $g \equiv f$ mod p. An example of a level-raising result for classical modular forms is the following, due to Ribet [Rib84].

THEOREM 1. *Let $f \in S_2(\Gamma_0(N))$ be an eigenform, and let $\mathfrak{p}|p$ be a finite place of $\bar{\mathbb{Q}}$ such that $p \geqslant 5$ and f is not congruent to an Eisenstein series modulo $\mathfrak{p}$. If $l \nmid Np$ is a prime number such that the following condition is satisfied,*

$$a_l(f)^2 \equiv (1+l)^2 \pmod{\mathfrak{p}},$$

then there exists a l-new eigenform $\tilde{f} \in S_2(\Gamma_0(Nl))$ congruent to f modulo $\mathfrak{p}$.

In this paper we prove an analogous level-raising result for families of p-adic automorphic forms. In [Buz04] and [Buz07, Part III], Buzzard defines modules of overconvergent p-adic automorphic forms for definite quaternion algebras, and constructs from these a so-called 'eigencurve'. The eigencurve is a rigid analytic variety whose points correspond to certain systems of eigenvalues for Hecke algebras acting on these modules of automorphic forms. This space p-adically interpolates the systems of eigenvalues arising from classical automorphic forms. Emerton has constructed eigenvarieties in a cohomological framework [Eme06], but in the following we will work with Buzzard's more concrete construction. Since we first wrote

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the present text, we have also proved some cases of level raising for p-adic modular forms using the completed cohomology spaces investigated by Emerton (see [New10]).

The first construction of an eigencurve was carried out for modular forms (automorphic forms for GL_2) in Coleman and Mazur's seminal paper [CM98]. An important recent result is the construction of a p-adic Jacquet–Langlands map between an eigencurve for a definite quaternion algebra and the GL_2 eigencurve (interpolating the usual Jacquet–Langlands correspondence), as carried out in [Che05].

We follow the general approach of the first part of Diamond and Taylor's paper [DT94], and our Theorem 12 is an analogue of [DT94, Theorem 1], but several new features appear in our work. In particular, the level-raising results in [DT94, Tay89] for definite quaternion algebras are proved by utilising a pairing on finite-dimensional vector spaces of automorphic forms. In our setting, the spaces of automorphic forms are Banach modules over an affinoid algebra, so we introduce spaces of 'dual' automorphic forms and work with the pairing between the usual space of automorphic forms and the dual space. We then prove suitable forms of Ihara's lemma, our Theorem 10 (cf. [DT94, Lemma 2]), for the usual and dual spaces of automorphic forms. An interesting asymmetry between the two situations can be observed.

This investigation of level-raising results was motivated by a conjecture made by Paulin, prompted by results on local–global compatibility on the eigencurve in his thesis [Pau07]. Paulin's conjecture was made for the GL_2-eigencurve; we may apply our theorem to the image of the p-adic Jacquet–Langlands map there to prove many cases of his conjecture. Since we have applications to the eigencurve for $\mathrm{GL}_2/\mathbb{Q}$ in mind we work with definite quaternion algebras over $\mathbb{Q}$ in this paper, but some of the methods of § 2 should apply to definite quaternion algebras over any totally real number field, although we do use the fact that weight space is one-dimensional in our arguments. We end this introduction by stating the conjecture made by Paulin.

1.1 A geometric level-raising conjecture

We fix two distinct primes p and l, and an integer N coprime to pl. Let $\mathcal{E}$ be the cuspidal eigencurve of tame level $\Gamma_0(Nl)$, parametrising overconvergent cuspidal p-adic modular eigenforms (see [Buz07] for its construction). If ϕ is a point of $\mathcal{E}$, corresponding to an eigenform f_ϕ, Paulin defines an associated representation of $\mathrm{GL}_2(\mathbb{Q}_l)$, denoted $\pi_{f_\phi,l}$. We call an irreducible connected component $\mathcal{Z}$ of the eigencurve *generically special* if the $\mathrm{GL}_2(\mathbb{Q}_l)$-representations associated with the points of $\mathcal{Z}$ away from a discrete set are special. We define *generically unramified principal series* similarly. Denote by α and β the roots of the polynomial $X^2 - t_l X + l s_l$, where t_l and s_l are the T_l and S_l eigenvalues of f_ϕ. Paulin makes the following conjecture.

Conjecture. Suppose $\mathcal{Z}$ is a generically unramified principal series component. Suppose further that there is a point ϕ on $\mathcal{Z}$ where the ratio of α to β becomes $l^{\pm 1}$ and $\pi_{f_\phi,l}$ is special. Then there exists a generically special component $\mathcal{Z}'$ intersecting $\mathcal{Z}$ at ϕ.

Chenevier raised the same question (in a slightly different form) in relation to the characterisation of the Zariski closure of the l-new classical forms in the eigencurve. We address this issue in § 3.2. Finally, in a recent preprint [Pau10] Paulin has proved versions of his level-raising (and lowering) conjectures (even for ramified principal series). His techniques are completely different to ours, making use of deformation theory and requiring a recent important result of Emerton showing that the space X_{fs} constructed by Kisin in [Kis03] is equal to the GL_2-eigencurve (if one restricts to pieces of the two spaces where certain conditions are satisfied by the relevant modulo p Galois representations).

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2. Modules of p-adic overconvergent automorphic forms and Ihara's lemma

In this section we will prove the results we need about modules of p-adic overconvergent automorphic forms for quaternion algebras.

2.1 Banach modules

Let K be a finite extension of $\mathbb{Q}_p$. We call a normed K-algebra A a *Banach algebra* if it satisfies the following properties.

- The K-algebra A is Noetherian.
- The norm $|-|$ is non-Archimedean.
- The K-algebra A is complete with respect to $|-|$.
- For any x, y in A we have $|xy| \leqslant |x||y|$.

We will normally assume A is a reduced affinoid algebra with its supremum norm. A *Banach A-module* is an A-module M endowed with a norm $|-|$ such that the following hold.

- For any $a \in A$, $m \in M$ we have $|am| \leqslant |a||m|$.
- The A-module M is complete with respect to $|-|$.

Given a set I we define the Banach A-module $c_I(A)$ to be functions $f: I \to A$ such that $\lim_{i\to\infty} f(i) = 0$, with norm the supremum norm. By a finite Banach A-module we mean a Banach A-module which is finitely-generated as an abstract A-module.

Suppose M is a Banach module over a Banach algebra A. We say that M is *ONable* if it is isomorphic (as a Banach module) to some $c_I(A)$. Note that this terminology differs slightly from that of [Buz07], where *ONable* refers to modules *isometric* to some $c_I(A)$ and *potentially ONable* replaces our notion of ONable. The Banach A-module P is said to satisfy the universal property (Pr) if for every surjection $f: M \to N$ of Banach A-modules and continuous map $\alpha: P \to N$, α lifts to a continuous map $\beta: P \to M$ such that the below diagram commutes.

$$\begin{tikzcd} & M \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists\beta"] \arrow[r, "\alpha"] & N \end{tikzcd}$$

Note that the universal property (Pr) is not quite the same as the property of being projective in the category of Banach A-modules, since an epimorphism of Banach modules is not necessarily a (set-theoretic) surjection. A module P having property (Pr) is equivalent to P being a direct summand of an ONable module. (See the end of § 2 in [Buz07]).

2.2 Some notation and definitions

Let p be a fixed prime. Let D be a definite quaternion algebra over $\mathbb{Q}$ with discriminant δ prime to p. Fix a maximal order $\mathcal{O}_D$ of D and isomorphisms $\mathcal{O}_D \otimes \mathbb{Z}_q \cong M_2(\mathbb{Z}_q)$ for primes $q \nmid \delta$. Note that these induce isomorphisms $D \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for $q \nmid \delta$. We define $D_f = D \otimes_{\mathbb{Q}} \mathbb{A}_f$, where $\mathbb{A}_f$ denotes the finite adeles over $\mathbb{Q}$. Write Nm for the reduced norm map from D_f to $\mathbb{A}_f^\times$. Note that if $g \in D_f$ we can regard the p component of g, g_p, as an element of $M_2(\mathbb{Q}_p)$.

For an integer $\alpha \geqslant 1$, we let $\mathbb{M}_\alpha$ denote the monoid of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}_p)$ such that $p^\alpha | c$, $p \nmid d$ and $ad - bc \neq 0$. If U is an open compact subgroup of $D_f^\times$ and $\alpha \geqslant 1$ we say that U has *wild level* $\geqslant p^\alpha$ if the projection of U to $\mathrm{GL}_2(\mathbb{Q}_p)$ is contained in $\mathbb{M}_\alpha$.

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We will be interested in two key examples of open compact subgroups of $D_f^\times$. For M any integer prime to δ, we define $U_0(M)$ (respectively $U_1(M)$) to be the subgroup of $D_f^\times$ given by the product $\prod_q U_q$, where $U_q = (\mathcal{O}_D \otimes \mathbb{Z}_q)^\times$ for primes $q|\delta$, and U_q are the matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$ (respectively $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$) mod $q^{\mathrm{val}_q(M)}$ for all other q. We can see that if p^α divides M, then $U_1(M)$ has wild level $\geqslant p^\alpha$.

Suppose we have $\alpha \geqslant 1$, U a compact open subgroup of $D_f^\times$ of wild level $\geqslant p^\alpha$ and A a module over a commutative ring R, with an R-linear right action of $\mathbb{M}_\alpha$. We define an R-module $\mathcal{L}(U, A)$ by

$$\mathcal{L}(U, A) = \{f : D_f^\times \to A : f(dgu) = f(g)u_p \ \forall d \in D^\times, g \in D_f^\times, u \in U\}$$

where $D^\times$ is embedded diagonally in $D_f^\times$. If we fix a set $\{d_i : 1 \leqslant i \leqslant r\}$ of double coset representatives for the finite double quotient $D^\times \backslash D_f^\times / U$, and write Γ_i for the finite group $d_i^{-1} D^\times d_i \cap U$, we have an isomorphism (see [Buz04, § 4])

$$\mathcal{L}(U, A) \to \bigoplus_{i=1}^{r} A^{\Gamma_i},$$

given by sending f to $(f(d_1), f(d_2), \ldots, f(d_r))$. If $U \subset U_1(N)$ for $N \geqslant 4$, then the groups Γ_i are trivial (this is proved in [DT94]).

For $f : D_f^\times \to A$, $x \in D_f^\times$ with $x_p \in \mathbb{M}_\alpha$, we define $f|x : D_f^\times \to A$ by $(f|x)(g) = f(gx^{-1})x_p$. Note that we can now also write

$$\mathcal{L}(U, A) = \{f : D^\times \backslash D_f^\times \to A : f|u = f \ \forall u \in U\}.$$

We can define double coset operators on the spaces $\mathcal{L}(U, A)$. If U, V are two compact open subgroups of $D_f^\times$ of wild level $\geqslant p^\alpha$, and A is as above, then for $\eta \in D_f^\times$ with $\eta_p \in \mathbb{M}_\alpha$ we may define an R-module map $[U\eta V] : \mathcal{L}(U, A) \to \mathcal{L}(V, A)$ as follows: we decompose $U\eta V$ into a finite union of right cosets $\coprod_i Ux_i$ and define

$$f|[U\eta V] = \sum_i f|x_i.$$

2.3 Overconvergent automorphic forms

Let $\mathcal{W}$ be the rigid analytic space $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{G}_m)$, defined over $\mathbb{Q}_p$. The reader may consult [Buz04, Lemma 2] for details of this space's construction and properties. For example, $\mathcal{W}$ is a union of finitely many open discs. The space $\mathcal{W}$ is the *weight space* for our automorphic forms. The $\mathbb{C}_p$-points w of $\mathcal{W}$ corresponding to characters $\kappa_w : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ with $\kappa_w(x) = x^k \varepsilon_p(x)$ for some positive integer k and finite order character ε_p are referred to as *classical* weights. Let X be a reduced connected K-affinoid subspace of $\mathcal{W}$, where $K/\mathbb{Q}_p$ is finite, and denote the ring of analytic functions on X by $\mathcal{O}(X)$. Such a space X corresponds to a character $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$ induced by the inclusion $X \subset \mathcal{W}$. If we have a real number $r = p^{-n}$ for some n, then we define $\mathbb{B}_{r,K}$ to be the rigid analytic subspace of affine 1-space over K with $\mathbb{C}_p$-points

$$\mathbb{B}_{r,K}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : \exists y \in \mathbb{Z}_p \text{ such that } |z - y| \leqslant r\}.$$

Similarly (for $r < 1$) we define $\mathbb{B}_{r,K}^\times$ to be the rigid analytic subspace of affine 1-space over K with $\mathbb{C}_p$-points

$$\mathbb{B}_{r,K}^\times(\mathbb{C}_p) = \{z \in \mathbb{C}_p : \exists y \in \mathbb{Z}_p^\times \text{ such that } |z - y| \leqslant r\}.$$

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A point $x \in X(\mathbb{C}_p)$ corresponds to a continuous character $\kappa_x : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. Such maps are analytic when restricted to the set $\{z \in \mathbb{Z}_p : |1-z| \leqslant r\}$ for small enough r. If κ_x and r have this property we call x an r-analytic point. A point is r-analytic if and only if its corresponding character extends to a morphism of rigid analytic varieties

$$\kappa_x : \mathbb{B}_{r,K}^\times \to \mathbb{G}_m.$$

Let X be a K-affinoid subspace of $\mathcal{W}$ as before, with associated character $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$. We say that κ is r-analytic if every point in $X(\mathbb{C}_p)$ is r-analytic. Fix a real number $0 < r < 1$ and let $\mathcal{A}_{X,r}$ be the $\mathcal{O}(X)$-Banach algebra $\mathcal{O}(\mathbb{B}_{r,K} \times_K X)$, endowed with the supremum norm. If κ is $rp^{-\alpha}$-analytic we can define a right action of $\mathbb{M}_\alpha$ on $\mathcal{A}_{X,r}$ by, for $f \in \mathcal{A}_{X,r}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{M}_\alpha$,

$$(f \cdot \gamma)(x, z) = \frac{\kappa_x(cz+d)}{(cz+d)^2} f\left(x, \frac{az+b}{cz+d}\right)$$

where $x \in X(\mathbb{C}_p)$ (with κ_x the associated character) and $z \in \mathbb{B}_{r,K}(\mathbb{C}_p)$.

Definition 1. Let X be a K-affinoid subspace of $\mathcal{W}$ as above, with $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$ the induced character. If we have a real number $r = p^{-n}$, some integer $\alpha \geqslant 1$ such that κ is $rp^{-\alpha}$-analytic, and U a compact open subgroup of $D_f^\times$ of wild level $\geqslant p^\alpha$, then define the space of r-overconvergent automorphic forms of weight X and level U to be the $\mathcal{O}(X)$-module

$$\mathbf{S}_X^D(U; r) := \mathcal{L}(U, \mathcal{A}_{X,r}).$$

If we endow $\mathbf{S}_X^D(U; r)$ with the norm $|f| = \max_{g \in D_f^\times} |f(g)|$, then the isomorphism

$$\mathbf{S}_X^D(U; r) \cong \bigoplus_{i=1}^{r} \mathcal{A}_{X,r}^{\Gamma_i} \tag{1}$$

induced by fixing double coset representatives d_i is norm preserving. Since the Γ_i are finite groups, and $\mathcal{A}_{X,r}$ is an ONable Banach $\mathcal{O}(X)$-module (it is the base change to $\mathcal{O}(X)$ of $\mathcal{O}(\mathbb{B}_{r,K})$, and all Banach spaces over a discretely valued field are ONable), we see that $\mathbf{S}_X^D(U; r)$ is a Banach $\mathcal{O}(X)$-module, and satisfies property (Pr).

Note that if $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}(X)$, corresponding to a point $x \in X(K')$ for K'/K finite, then taking the fibre of the module $\mathbf{S}_X^D(U; r)$ at $\mathfrak{m}$ gives the space of overconvergent forms $\mathbf{S}_x^D(U; r)$ corresponding to the point x of $\mathcal{W}(K')$ (note that a point of $\mathcal{W}(K')$ is a reduced connected K'-affinoid subspace).

These spaces of overconvergent automorphic forms were first defined in [Buz04], using ideas from the unpublished preprint [Ste94].

2.4 Dual modules

Suppose A is a Banach algebra. Given a Banach A-module $\mathbf{M}$ we define the *dual* $\mathbf{M}^*$ to be the Banach A-module of continuous A-module morphisms from $\mathbf{M}$ to A, with the usual operator norm. We denote the $\mathcal{O}(X)$-module $\mathcal{A}_{X,r}^*$ by $\mathcal{D}_{X,r}$.

If the map κ corresponding to X is $rp^{-\alpha}$-analytic, then $\mathbb{M}_\alpha$ acts continuously on $\mathcal{A}_{X,r}$, so $\mathcal{D}_{X,r}$ has an $\mathcal{O}(X)$-linear right action of the monoid $\mathbb{M}_\alpha^{-1}$ given by $(f \cdot m^{-1})(x) := f(x \cdot m)$, for $f \in \mathcal{D}_{X,r}$, $x \in \mathcal{A}_{X,r}$ and $m \in \mathbb{M}_\alpha$. If U is as in Definition 1 then its projection to $\mathrm{GL}_2(\mathbb{Q}_p)$ is contained in $\mathbb{M}_\alpha \cap \mathbb{M}_\alpha^{-1}$, so it acts on $\mathcal{D}_{X,r}$. This allows us to make the following definition.

Definition 2. For X, κ, r, α and U as above, we define the space of *dual* r-overconvergent automorphic forms of weight X and level U to be the $\mathcal{O}(X)$-module

$$\mathbf{V}_X^D(U;r) := \mathcal{L}(U, \mathcal{D}_{X,r}).$$

As in § 2.3, we have a norm-preserving isomorphism

$$\mathbf{V}_X^D(U;r) \cong \bigoplus_{i=1}^{r} \mathcal{D}_{X,r}^{\Gamma_i}. \tag{2}$$

Thus $\mathbf{V}_X^D(U;r)$ is a Banach $\mathcal{O}(X)$-module. We note that it will not usually satisfy property (Pr), since (unless X is a point) we expect that $\mathcal{D}_{X,r}$ will not be ONable.

If U, V are two compact open subgroups of $D_f^\times$ of wild level $\geqslant p^\alpha$, then for $\eta \in D_f^\times$ with $\eta_p \in \mathbb{M}_\alpha^{-1}$ we get double coset operators $[U\eta V] : \mathbf{V}_X^D(U;r) \to \mathbf{V}_X^D(V;r)$.

2.5 Hecke operators

For an integer m, we define the *Hecke algebra away from* m, $\mathbb{T}^{(m)}$, to be the free commutative $\mathcal{O}(X)$-algebra generated by symbols T_π, S_π for π prime not dividing m. If δp divides m then we can define the usual action of $\mathbb{T}^{(m)}$ by double coset operators on $\mathbf{S}_X^D(U;r)$ as follows: for $\pi \nmid m$ define $\varpi_\pi \in \mathbb{A}_f$ to be the finite adele which is π at π and 1 at the other places. Abusing notation slightly, we also write ϖ_π for the element of $D_f^\times$ which is $\left(\begin{smallmatrix}\pi & 0\\ 0 & \pi\end{smallmatrix}\right)$ at π and the identity elsewhere. Similarly set $\eta_\pi = \left(\begin{smallmatrix}\varpi_\pi & 0\\ 0 & 1\end{smallmatrix}\right)$ to be the element of $D_f^\times$ which is $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$ at π and the identity elsewhere. On $\mathbf{S}_X^D(U;r)$ we let T_π act by $[U\eta_\pi U]$ and S_π by $[U\varpi_\pi U]$. Similarly on $\mathbf{V}_X^D(U;r)$ we define T_π to act by $[U\eta_\pi^{-1}U]$ and S_π by $[U\varpi_\pi^{-1}U]$. As usual we also have a compact operator acting on $\mathbf{S}_X^D(U;r)$, namely $U_p := [U\eta_p U]$.

2.6 A pairing

In this section X, κ, r, α and U will be as in Definition 1. We will denote by V another compact open subgroup of wild level $\geqslant p^\alpha$. We fix double coset representatives $\{d_i : 1 \leqslant i \leqslant r\}$ for the double quotient $D^\times \backslash D_f^\times / U$ and let γ_i denote the order of the finite group $d_i^{-1} D^\times d_i \cap U$. We can define an $\mathcal{O}(X)$-bilinear pairing between the spaces $\mathbf{S}_X^D(U;r)$ and $\mathbf{V}_X^D(U;r)$ by

$$\langle f, \lambda \rangle := \sum_{i=1}^{r} \gamma_i^{-1} \langle f(d_i), \lambda(d_i) \rangle,$$

where $f \in \mathbf{S}_X^D(U;r)$, $\lambda \in \mathbf{V}_X^D(U;r)$ and on the right-hand side of the above definition $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathcal{A}_{X,r}$ and $\mathcal{D}_{X,r}$ given by evaluation.

This pairing is independent of the choice of the double coset representatives d_i, since for every $d \in D^\times$, $g \in D_f^\times$, $u \in U$, $f \in \mathbf{S}_X^D(U;r)$ and $\lambda \in \mathbf{V}_X^D(U;r)$ we have

$$\langle f(dgu), \lambda(dgu) \rangle = \langle f(g)u_p, \lambda(g)u_p \rangle = \langle f(g)u_p u_p^{-1}, \lambda(g) \rangle = \langle f(g), \lambda(g) \rangle.$$

Combining this observation with the isomorphisms (1) and (2) we see that our pairing identifies $\mathbf{V}_X^D(U;r)$ with $\mathbf{S}_X^D(U;r)^*$.

The following proposition summarises a standard computation [DT94, Tay89] (although these assume the level group is small enough that the finite groups Γ_i are trivial), telling us how our pairing interacts with double coset operators. In particular, it implies that $\langle T_\pi f, \lambda \rangle = \langle f, T_\pi \lambda \rangle$ for $\pi \nmid \delta p$ when T_π acts in the usual way.

PROPOSITION 3. *Let* $f \in \mathbf{S}_X^D(U;r)$ *and let* $\lambda \in \mathbf{V}_X^D(V;r)$. *Let* $g \in D_f^\times$ *with* $g_p \in \mathbb{M}_\alpha$. *Then*

$$\langle f|[UgV], \lambda\rangle = \langle f, \lambda|[Vg^{-1}U]\rangle.$$

Proof. For $d \in D_f^\times$ set $\gamma(d) = \#(d^{-1}D^\times d \cap V)$. We have

$$f|[UgV] = \sum_{v \in (g^{-1}U\eta)\cap V\backslash V} f|(gv),$$

hence

$$\begin{aligned}
\langle f|[UgV], \lambda\rangle &= \sum_{d \in D^\times\backslash D_f^\times/V} \gamma(d)^{-1}\langle f|[UgV](d), \lambda(d)\rangle \\
&= \sum_{d \in D^\times\backslash D_f^\times/V} \sum_{v \in (g^{-1}Ug)\cap V\backslash V} \gamma(d)^{-1}\langle f|(gv)(d), \lambda(d)\rangle \\
&= \sum_{d \in D^\times\backslash D_f^\times/V} \sum_{v \in (g^{-1}Ug)\cap V\backslash V} \gamma(d)^{-1}\langle f(dv^{-1}g^{-1}) \cdot g_p v_p, \lambda(d)\rangle \\
&= \sum_{x \in D^\times\backslash D_f^\times/(g^{-1}Ug)\cap V} \langle f(xg^{-1}), \lambda(x) \cdot g_p^{-1}\rangle \\
&= \sum_{y \in D^\times\backslash D_f^\times/U\cap(gVg^{-1})} \langle f(y), \lambda(yg) \cdot g_p^{-1}\rangle \\
&= \langle f, \lambda|[Vg^{-1}U]\rangle
\end{aligned}$$

where we pass from the third line to the fourth line by counting double cosets and the final line follows by similar calculations to the first five lines. □

By the results of [Che04, § 5] and [Buz07, § 3] we know that, for a fixed $d \geqslant 0$, if X is a sufficiently small affinoid whose norm is multiplicative (with a precise bound given by [Che04, Théorème 5.3.1]) then, since U_p acts as a compact operator on $\mathbf{S}_X^D(U;r)$, we have a U_p stable decomposition

$$\mathbf{S}_X^D(U;r) = \mathbf{S}_X^D(U;r)^{\leqslant d} \oplus \mathbf{N},$$

where $\mathbf{S}_X^D(U;r)^{\leqslant d}$ is the space of forms of slope $\leqslant d$. We need X to be small enough that the Newton polygon of the characteristic power series for U_p acting on $\mathbf{S}_X^D(U;r)$ has the same slope $\leqslant d$ part when specialised to any point of X.

From now on we fix d and assume that X is such that this slope decomposition exists. The key example of such an X is an open ball of small radius.

The space $\mathbf{S}_X^D(U;r)^{\leqslant d}$ is a finite Banach $\mathcal{O}(X)$-module with property (Pr), i.e. a projective finitely generated $\mathcal{O}(X)$-module. In fact this decomposition must be stable under the action of $\mathbb{T}^{(\delta p)}$, since the T_π and S_π operators for $\pi \neq p$ commute with U_p. We define $\mathbf{V}_X^D(U;r)^{\leqslant d}$ to be the maps from $\mathbf{S}_X^D(U;r)$ to $\mathcal{O}(X)$ which are 0 on $\mathbf{N}$. This space is also stable under the action of $\mathbb{T}^{(\delta p)}$ and is naturally isomorphic to the dual of $\mathbf{S}_X^D(U;r)^{\leqslant d}$. The following lemma implies that our pairing is perfect when restricted to $\mathbf{S}_X^D(U;r)^{\leqslant d} \times \mathbf{V}_X^D(U;r)^{\leqslant d}$.

LEMMA 4. *Let* $\mathbf{M}$ *be a finite Banach* $\mathcal{O}(X)$*-module with property* (Pr). *Then the usual natural map* $\mathbf{M} \to (\mathbf{M}^*)^*$ *is an isomorphism. In other words, the* $\mathcal{O}(X)$*-module* $\mathbf{M}$ *is* reflexive.

Proof. Since $\mathbf{M}$ is finite we have a surjection of Banach $\mathcal{O}(X)$-modules $\mathcal{O}(X)^{\oplus n} \to \mathbf{M}$ for some n. Applying the universal property (Pr) to this surjection shows that we have a Banach

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$\mathcal{O}(X)$-isomorphism $\mathbf{M} \oplus \mathbf{N} \cong \mathcal{O}(X)^{\oplus n}$ for some module $\mathbf{N}$, so $\mathbf{M}$ is a projective $\mathcal{O}(X)$-module. Proposition 2.1 of [Buz07] states that the category of finite Banach $\mathcal{O}(X)$-modules, with continuous $\mathcal{O}(X)$-linear maps as morphisms, is equivalent to the category of finite $\mathcal{O}(X)$-modules, so we can just compute duals module-theoretically. We have exact sequences,

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathcal{O}(X)^{\oplus n} \longrightarrow \mathbf{N} \longrightarrow 0,$$
$$0 \longrightarrow \mathbf{N} \longrightarrow \mathcal{O}(X)^{\oplus n} \longrightarrow \mathbf{M} \longrightarrow 0,$$

and since $\mathbf{M}$, $\mathbf{N}$, $\mathbf{M}^*$ and $\mathbf{N}^*$ are all projective as $\mathcal{O}(X)$-modules we can take the dual of these exact sequences twice to get commutative diagrams with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{M} \arrow[r] \arrow[d] & \mathcal{O}(X)^{\oplus n} \arrow[r] \arrow[d] & \mathbf{N} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{M}^{**} \arrow[r] & \mathcal{O}(X)^{\oplus n} \arrow[r] & \mathbf{N}^{**} \arrow[r] & 0
\end{tikzcd}$$
$$\begin{tikzcd}
0 \arrow[r] & \mathbf{N} \arrow[r] \arrow[d] & \mathcal{O}(X)^{\oplus n} \arrow[r] \arrow[d] & \mathbf{M} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{N}^{**} \arrow[r] & \mathcal{O}(X)^{\oplus n} \arrow[r] & \mathbf{M}^{**} \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are the natural maps from a module to its double dual. Since the central maps are isomorphisms, we conclude that the outer maps are too. □

2.6.1 *Direct limits and Fréchet spaces.* We should remark here that our use of the dual Banach modules $\mathbf{V}^D_X(U; r)$ is slightly unsatisfactory. For example, the modules do not satisfy property (Pr), and we must restrict to 'slope $\leqslant d$' subspaces to get a perfect pairing. One could alternatively work with modules of *all* overconvergent automorphic forms, rather than imposing r-overconvergence for a particular r. One defines

$$\mathbf{S}^D_X(U)^\dagger := \varinjlim_r \mathbf{S}^D_X(U; r),$$

where the (compact) transition maps in the direct system are induced by the inclusions $\mathbb{B}_{s,K} \subset \mathbb{B}_{r,K}$ for $s < r$. If X is a point (so $\mathcal{O}(X)$ is a field) then it is a standard result that the vector space $\mathbf{S}^D_X(U)^\dagger$ is reflexive (see [Sch02, Proposition 16.10]). Using this fact it is fairly straightforward to show that for any X, $\mathbf{S}^D_X(U)^\dagger$ is a reflexive $\mathcal{O}(X)$-module, with dual the Fréchet space

$$\mathbf{V}^D_X(U)^\dagger := \varprojlim_r \mathbf{V}^D_X(U; r).$$

2.7 Old and new

Fix an integer $N \geqslant 1$ (the tame level) coprime to p and fix an auxiliary prime $l \nmid Np\delta$. Let X be an affinoid subspace of weight space with associated character κ which is $rp^{-\alpha}$ analytic, for some integer $\alpha \geqslant 1$. Set $U = U_1(Np^\alpha)$, $V = U_1(Np^\alpha) \cap U_0(l)$. To simplify notation we set

$$L := \mathbf{S}^D_X(U; r)^{\leqslant d}, \quad L^* := \mathbf{V}^D_X(U; r)^{\leqslant d},$$
$$M := \mathbf{S}^D_X(V; r)^{\leqslant d}, \quad M^* := \mathbf{V}^D_X(V; r)^{\leqslant d}.$$

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We define a map $i: L \times L \to M$ by

$$i(f,g) := f|[U1V] + g|[U\eta_l V].$$

Since the map i is defined by double coset operators with trivial component at p it commutes with U_p and thus gives a well-defined map between these spaces of bounded slope forms. A simple calculation shows that these double coset operators act very simply. Regarding f and g as functions on $D_f^\times$ we have $f|[U1V] = f$, $g|[U\eta_l V] = g|\eta_l$. The image of i inside M will be referred to as the space of *oldforms*.

We also define a map $i^\dagger : M \to L \times L$ by

$$i^\dagger(f) := (f|[V1U], f|[V\eta_l^{-1}U]).$$

The kernel of $i^\dagger$ is the space of *newforms*. The maps i and $i^\dagger$ commute with Hecke operators T_q, S_q, where $q \nmid Npl\delta$.

The same double coset operators give maps,

$$j: L^* \times L^* \to M^*,$$
$$j^\dagger : M^* \to L^* \times L^*.$$

Using Proposition 3 we have

$$\langle i(f,g), \lambda\rangle = \langle (f,g), j^\dagger\lambda\rangle$$

for $f, g \in L$, $\lambda \in M^*$. Similarly

$$\langle f, j(\lambda,\mu)\rangle = \langle i^\dagger f, (\lambda,\mu)\rangle$$

for $f \in M$, $\lambda, \mu \in L^*$.

An easy calculation shows that $i^\dagger i$ acts on the product $L \times L = L^2$ by the matrix (acting on the right)

$$\begin{pmatrix} l+1 & [U\varpi_l^{-1}U][U\eta_l U] \\ [U\eta_l U] & l+1 \end{pmatrix} = \begin{pmatrix} l+1 & S_l^{-1}T_l \\ T_l & l+1 \end{pmatrix}.$$

We have exactly the same double coset operator formula for the action of $j^\dagger j$ on the product $L^* \times L^* = L^{*2}$. Since the Hecke operators S_l, T_l act by $[U\varpi_l^{-1}U]$, $[U\eta_l^{-1}U]$ respectively on L^* we deduce that, in terms of Hecke operators, $j^\dagger j$ acts on $L^* \times L^*$ by the matrix (again acting on the right)

$$\begin{pmatrix} l+1 & T_l \\ S_l^{-1}T_l & l+1 \end{pmatrix}.$$

If the affinoid X is sufficiently nice, then we can show that the map $i^\dagger i$ is injective. Before we prove this, we note that, in our setting, a family of p-adic automorphic eigenforms over an affinoid $X \subset \mathcal{W}$ is just a Hecke eigenform f in $\mathbf{S}_X^D(U; r)$.

Proposition 5. *If X is a one-dimensional irreducible connected smooth affinoid, then the map $i^\dagger i$ is injective.*

Proof. Let L_0 be the projective (since $\mathcal{O}(X)$ is a Dedekind domain) finite Banach $\mathcal{O}(X)$-module $\ker(i^\dagger i)$, and note that $L_0 \subset L^2$ is stable under the action of all the Hecke operators, since they all commute with $i^\dagger i$. Suppose L_0 is not zero. For (f,g) in L_0 we have $(l+1)f + S_l^{-1}T_l g = T_l f + (l+1)g = 0$. Eliminating g we get $T_l^2 f - (l+1)^2 S_l f = 0$, so projecting L_0 down to L

(taking either the first or the second factor) we see that the Hecke operator $T_l^2-(l+1)^2S_l$ acts as 0 on a non-zero projective submodule of L. This (applying the local eigenvariety construction as described in [Che04, § 6.2]) implies that there is a family of eigenforms over some one-dimensional sub-affinoid of X, all with the eigenvalue of $T_l^2-(l+1)^2S_l$ equal to 0. Now the Hecke algebra element $T_l^2-(l+1)^2S_l$ induces a rigid analytic function on the tame level N eigencurve for D (by taking the appropriate eigenvalue associated to a point), so this function must vanish on the whole irreducible component containing the one-dimensional family constructed above. However, every irreducible component contains a classical point, and these cannot be contained in the kernel of $T_l^2-(l+1)^2S_l$ since this would contradict the Hecke eigenvalue bounds given by the Ramanujan–Petersson conjecture. □

Note that the injectivity of $i^\dagger i$ implies the injectivity of i. The above shows that if X is as in the statement of Proposition 5, we have $\ker(i^\dagger)\cap\operatorname{im}(i)=0$ so our families in M are not both old and new at l. However, if X is just a point, then $i^\dagger i$ may have a kernel; this corresponds to p-adic automorphic forms which are both old and new at l.

2.8 Some modules

We denote the fraction field of $\mathcal{O}(X)$ by F. If A is an $\mathcal{O}(X)$-module we write A_F for the F-vector space $A\otimes_{\mathcal{O}(X)}F$.

We begin this section by noting that the injectivity of $i^\dagger i$ implies the injectivity of $j^\dagger j$.

Suppose $j^\dagger j(\lambda,\mu)=0$. Then $\langle(f,g),j^\dagger j(\lambda,\mu)\rangle=0$ for all $(f,g)\in L_F$, so (by Proposition 3) $\langle i^\dagger i(f,g),(\lambda,\mu)\rangle=0$ for all $(f,g)\in L_F$. Now since $i^\dagger i: L_F\to L_F$ is an injective endomorphism of a finite dimensional vector space, it is an isomorphism, so we see that $\lambda=\mu=0$. Hence $j^\dagger j$ (thus, *a fortiori*, j) is injective.

We now define two chains of modules which will prove useful:

$$\begin{array}{ll}
\Lambda_0:=L^2 & \Lambda_0^*:=L^{*2}\\
\Lambda_1:=i^\dagger M & \Lambda_1^*:=j^\dagger M^*\\
\Lambda_2:=i^\dagger(M\cap i(L_F^2)) & \Lambda_2^*:=j^\dagger(M^*\cap j((L_F^*)^2))\\
\Lambda_3:=i^\dagger iL^2 & \Lambda_3^*:=j^\dagger jL^{*2}.
\end{array}$$

We note that $\Lambda_0\supset\Lambda_1\supset\Lambda_2\supset\Lambda_3$, and that $\Lambda_2/\Lambda_3=i^\dagger(M\cap i(L_F^2)/iL^2)=i^\dagger((M/iL^2)^{\mathrm{tors}})$, with analogous statements for the starred modules.

We fix the usual action of $\mathbb{T}^{(N\delta pl)}$ on all these modules. We can now describe some pairings between them which will be equivariant under the $\mathbb{T}^{(N\delta pl)}$ action. They will not all be equivariant with respect to the action of T_l.

We have a (perfect) pairing $\langle\,,\rangle: L_F^2\times(L_F^*)^2\to F$ which, since j is injective, induces a pairing

$$\Lambda_0\times(M^*\cap j((L_F^*)^2))\to F/\mathcal{O}(X),$$

which in turn induces a pairing

$$P_1:\Lambda_0/\Lambda_1\times(M^*\cap j((L_F^*)^2)/j(L^{*2}))\to F/\mathcal{O}(X).$$

The fact that this pairing is perfect follows from the following lemma.

Lemma 6. *The pairing on $L_F^2\times(L_F^*)^2$ induces isomorphisms*

$$\operatorname{Hom}_{\mathcal{O}(X)}(\Lambda_1,\mathcal{O}(X))\cong M^*\cap j((L_F^*)^2)$$

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and

$$\mathrm{Hom}_{\mathcal{O}(X)}(\Lambda_0, \mathcal{O}(X)) \cong j(L^{*2}).$$

Proof. For the first isomorphism, the module $\mathrm{Hom}_{\mathcal{O}(X)}(\Lambda_1, \mathcal{O}(X))$ correspond to $l \in (L_F^*)^2$ such that $\langle i^\dagger m, l\rangle \in \mathcal{O}(X)$ for all $m \in M$. We have $\langle i^\dagger m, l\rangle = \langle m, jl\rangle$ so $\langle i^\dagger m, l\rangle \in \mathcal{O}(X)$ for all $m \in M$ if and only if $\langle m, jl\rangle \in \mathcal{O}(X)$ for all $m \in M$, i.e. if and only if $jl \in M^*$.

The second isomorphism is obvious, since j is injective. □

In exactly the same way, we have a perfect pairing

$$P_2 : (M \cap i(L_F^2))/i(L^2) \times \Lambda_0^*/\Lambda_1^* \to F/\mathcal{O}(X).$$

The final pairing we will need is induced by the pairing between M and M^*. It is straightforward to check that this gives a perfect pairing:

$$P_3 : \ker(i^\dagger) \times M^*/(M^* \cap j((L_F^*)^2)) \to \mathcal{O}(X).$$

2.9 An analogue of Ihara's lemma

In classical level-raising results (such as [DT94, Rib84, Tay89]) analogues of 'Ihara's lemma' [Iha75, Lemma 3.2] are used to show that prime ideals of a Hecke algebra containing the annihilators of certain modules of automorphic forms are in some sense 'uninteresting', or even to show that these modules are trivial. In this section we prove the appropriate analogue of Ihara's lemma in our setting.

From this section onwards we will assume that X is a one-dimensional irreducible connected smooth affinoid in weight space $\mathcal{W}$, so we can apply Proposition 5. We want to obtain information about the $\mathbb{T}^{(N\delta pl)}$ action on the quotients $\Lambda_2/\Lambda_3 \cong i^\dagger(M/iL^2)^{\mathrm{tors}}$, $\Lambda_2^*/\Lambda_3^* \cong j^\dagger(M^*/jL^{*2})^{\mathrm{tors}}$, Λ_0/Λ_1 and Λ_0^*/Λ_1^*. The pairings P_1 and P_2 allow us to use an analogue of Ihara's lemma (the following two propositions and theorem) to obtain crucial information about all four quotients. Recall that the radius of overconvergence r equals p^{-n} for some positive integer n. Fix a positive integer c such that $Nm(U_1(Np^{\alpha+n}))$ contains all elements of $\hat{\mathbb{Z}}^\times$ congruent to 1 modulo c. We first need a lemma allowing us to control certain forms with weight a point in weight space.

Lemma 7. *Let $x \in \mathcal{W}(K')$ for some K' a finite extension of $\mathbb{Q}_p$.*

(i) *Let $y \in \mathbf{S}_x^D(U; r)$ be non-zero. Suppose y factors through Nm, that is $y(g) = y(h)$ for all $g, h \in D_f^\times$ with $Nm(g) = Nm(h)$. Then κ_x is a classical weight $z \mapsto z^2\varepsilon_p(z)$, and for all but finitely many primes $q \equiv 1 \bmod c$, where c is the fixed integer chosen above, $(T_q - q - 1)y = 0$.*

(ii) *Let $y \in \mathbf{V}_x^D(U; r)$. If y factors through Nm, that is $y(g) = y(h)$ for all $g, h \in D_f^\times$ with $Nm(g) = Nm(h)$, then y is zero.*

Proof. We first prove part (i). Suppose y is as in the statement of that part. For $u_p \in \mathrm{SL}_2(\mathbb{Q}_p) \cap U$ we have $y(g) = y(gu_p) = y(g) \cdot u_p$ for all $g \in D_f^\times$. Noting that $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q}_p) \cap U$ for all $a \in \mathbb{Z}_p$, we see that $y(g)(z + a) = y(g)(z)$ for all $a \in \mathbb{Z}_p$, $z \in \mathbb{B}_{r,K'}$ so $y(g)(z)$ is constant in z, since non-constant rigid analytic functions have discrete zero sets. Recall that $U = U_1(Np^\alpha)$, so $u_0 := \left(\begin{smallmatrix} 1 & 0 \\ p^\alpha & 1 \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Q}_p) \cap U$, and for $z \in \mathbb{B}_{r,K'}$ we have

$$y(g)(z) = (y(g)u_0)(z) = \frac{\kappa_x(p^\alpha z + 1)}{(p^\alpha z + 1)^2} y(g),$$

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so κ_x must correspond to the classical weight given by $z \mapsto z^2\epsilon_p(z)$ for some character ϵ_p trivial on $1 + p^{\alpha+n}\mathbb{Z}_p$, where $r = p^{-n}$. This now implies that for each $g \in D_f^\times$ we have $y(g)\gamma = y(g)$ for all γ in the projection of $U_1(Np^{\alpha+n})$ to $\mathbb{M}_{\alpha+n}$, since these matrices all have bottom right-hand entry congruent to 1 mod $p^{\alpha+n}$.

We now follow [DT94] to complete the proof of the first part of the lemma. There is a $d_0 \in D^\times$ with $Nm(d_0) = q$, so $Nm(d_0^{-1}\eta_q) \in \mathbb{A}_f^\times$ is actually in $\hat{\mathbb{Z}}^\times$ and is congruent to 1 modulo c. Thus (by the way we picked c) there is $u_0 \in U_1(Np^{\alpha+n})$ such that $Nm(u_0) = Nm(d_0^{-1}\eta_q)$. Now we have

$$\begin{aligned} T_q(y)(g) &= \sum_{u \in (\eta_q^{-1}U\eta_q)\cap U\backslash U} y(gu^{-1}\eta_q^{-1}) \cdot u_p \\ &= \sum_{u \in (\eta_q^{-1}U\eta_q)\cap U\backslash U} y(g\eta_q^{-1}u^{-1}) \cdot u_p \\ &= \sum_{u \in (\eta_q^{-1}U\eta_q)\cap U\backslash U} y(g\eta_q^{-1}) = (q+1)y(g\eta_q^{-1}) \\ &= (q+1)y(g\eta_q^{-1}d_0d_0^{-1}) = (q+1)y(gu_0^{-1}d_0^{-1}) = (q+1)y(gu_0^{-1}) = (q+1)y(g), \end{aligned}$$

where to pass from the first line to the second we use the fact that y factors through Nm to commute y's arguments, from the second to the third we use that y is modular of level U and in the final line we first substitute u_0 for $d_0^{-1}\eta_q$ (since they have the same reduced norm), then commute y's arguments and use the left invariance of y under $D^\times$ followed by the fact that $u_0^{-1} \in U_1(Np^{\alpha+n})$ implies that $y(gu_0^{-1}) = y(g)u_{0,p}^{-1} = y(g)$.

We now give a proof of the second part of the lemma. First we perform a formal calculation. Fix an isomorphism

$$\mathcal{A}_{x,r} \cong \prod_{\alpha=1}^{n} K'\langle T\rangle$$

where $K'\langle T\rangle$ is the ring of power series with coefficients in K' tending to zero (T a formal variable), and n is a positive integer depending on r. Such an isomorphism exists since $\mathbb{B}_{r,K'}$ is just a disjoint union of finitely many affinoid discs. We then have an identification of $\mathcal{D}_{x,r}$ with $\prod_{\alpha=1}^{n} K'\langle[T]\rangle$, where $K'\langle[T]\rangle$ denotes the ring of power series with bounded coefficients in K', and the pairing between $\mathcal{A}_{x,r}$ and $\mathcal{D}_{x,r}$ is given on each component by $\langle \sum a_iT^i, \sum b_jT^j\rangle = \sum a_ib_i$. Now we can compute the action of $\gamma = \left(\begin{smallmatrix}1 & -1\\ 0 & 1\end{smallmatrix}\right)$ on $\mathcal{D}_{x,r}$. Let $f = (f_\alpha)_{\alpha=1,\dots,n}$ be an element of $\mathcal{D}_{x,r}$, with $f_\alpha = \sum b_{j,\alpha}T^j$. For each $\alpha = 1, \dots, n$ and $i \geqslant 0$ fix $e_{i,\alpha}$ to be the element of $\mathcal{A}_{x,r}$ which is T^i at the α component, and zero elsewhere. We have

$$\begin{aligned} \left\langle e_{i,\alpha}, f \cdot \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right\rangle &= \left\langle T^i, \left(\sum b_{j,\alpha}T^j\right) \cdot \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right\rangle \\ &= \left\langle T^i \cdot \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \sum b_{j,\alpha}T^j\right\rangle \\ &= \left\langle (T+1)^i, \sum b_{j,\alpha}T^j\right\rangle \\ &= \left\langle \sum_{k=0}^{i} \binom{i}{k} T^k, \sum b_{j,\alpha}T^j\right\rangle \\ &= \sum_{j=0}^{i} \binom{i}{j} b_{j,\alpha}. \end{aligned}$$

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We can now see that if we have $f = f \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ we get $\sum_{j=0}^{i} \binom{i}{j} b_{j,\alpha} = b_{i,\alpha}$ for all i and α, which implies that $f = 0$.

Now we return to the statement in the lemma and suppose $y \in \mathbf{V}_x^D(U; r)$ factors through Nm. Let u_1 be the element of $U \subset D_f^\times$ with p component equal to γ and all other components the identity. For all $g \in D_f^\times$, $y(gu_1) = y(g)\gamma$ since $u_1 \in U$ and $y(gu_1) = y(g)$ since $Nm(u_1) = 1$. Hence $y(g) = y(g)\gamma$ and the above calculation shows that $y(g) = 0$ for all g. □

The following two propositions apply the preceding lemma to give a form of Ihara's lemma for modules of overconvergent automorphic forms and dual overconvergent forms.

Proposition 8. *For all but finitely many primes $q \equiv 1 \bmod c$, $T_q - q - 1$ annihilates the module $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ of $\mathcal{O}(X)$.*

Proof. Fix $q \equiv 1 \bmod c$ with $q \nmid Np\delta l$ and set $H_q := T_q - q - 1$. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}(X)$ and set $K' = \mathcal{O}(X)/\mathfrak{m}$. The maximal ideal $\mathfrak{m}$ corresponds to a point x of $X(K')$, with corresponding weight $\kappa_x : \mathbb{Z}_p^\times \to K'^\times$ the specialisation of κ at $\mathfrak{m}$.

We have a short exact sequence,

$$0 \longrightarrow L^2 \xrightarrow{i} M \longrightarrow M/iL^2 \longrightarrow 0\,.$$

Noting that L^2 and M are $\mathcal{O}(X)$-torsion free, hence flat, and taking derived functors of $-\otimes_{\mathcal{O}(X)} K'$ gives an exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, K') \arrow[r, "\delta"] & L^2 \otimes_{\mathcal{O}(X)} K' \arrow[d, bend left, "i"] \\ 0 & M/iL^2 \otimes_{\mathcal{O}(X)} K' \arrow[l] & M \otimes_{\mathcal{O}(X)} K' \arrow[l] \end{tikzcd}$$

We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & L^2 \arrow[r, "i"] \arrow[d, "H_q"] & M \arrow[r] \arrow[d, "H_q"] & M/iL^2 \arrow[r] \arrow[d, "H_q"] & 0 \\ 0 \arrow[r] & L^2 \arrow[r, "i"] & M \arrow[r] & M/iL^2 \arrow[r] & 0 \end{tikzcd}$$

so by the naturality of the long exact sequence for Tor the diagram

$$\begin{tikzcd} \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, K') \arrow[r, "\delta"] \arrow[d, "H_q"] & L^2 \otimes_{\mathcal{O}(X)} K' \arrow[d, "H_q"] \\ \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, K') \arrow[r, "\delta"] & L^2 \otimes_{\mathcal{O}(X)} K' \end{tikzcd}$$

commutes. To complete the proof it suffices to prove that H_q annihilates the kernel of

$$i : L^2 \otimes_{\mathcal{O}(X)} K' \to M \otimes_{\mathcal{O}(X)} K'.$$

We proceed by viewing these modules as spaces of automorphic forms with weight x (a single point in weight space). We define two finite-dimensional K'-vector spaces:

$$L_x := \mathbf{S}_x^D(U; r)^{\leqslant d},$$
$$M_x := \mathbf{S}_x^D(V; r)^{\leqslant d}.$$

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There are maps $i_x : L_x^2 \to M_x$ and $i_x^\dagger : M_x \to L_x$ as defined in § 2.7 (taking the weight X in that section to be the point x), but note that as now the weight is just a point in weight space, Proposition 5 does not apply. In particular the map i_x might not be injective.

Recall that L and M are finitely generated $\mathcal{O}(X)$ modules, whence $L \otimes_{\mathcal{O}(X)} K'$ and $M \otimes_{\mathcal{O}(X)} K'$ are finite-dimensional K'-vector spaces. Since the Newton polygon of the characteristic power series for U_p acting on $\mathbf{S}_X^D(U;r)$ has the same slope $\leqslant d$ part when specialised to any point of X, and the specialisation of the Banach $\mathcal{O}(X)$-modules $\mathbf{S}_X^D(U;r)$ and $\mathbf{S}_X^D(V;r)$ at $\mathfrak{m}$ gives $\mathbf{S}_x^D(U;r)$ and $\mathbf{S}_x^D(V;r)$ respectively, we have isomorphisms $L \otimes_{\mathcal{O}(X)} K' \to L_x$ and $M \otimes_{\mathcal{O}(X)} K' \to M_x$ which commute suitably with $i : L^2 \otimes_{\mathcal{O}(X)} K' \to M \otimes_{\mathcal{O}(X)} K'$ and $i_x : L_x^2 \to M_x$. These isomorphisms also commute with double coset operators, so to prove the proposition it suffices show that H_q annihilates the kernel of i_x.

Suppose $i_x(y_1, y_2) = 0$. Then $y_1 = -y_2|\eta_l$, so we have $y_2 \in \mathbf{S}_x^D(U;r)$, $y_2|\eta_l \in \mathbf{S}_x^D(U;r)$. Therefore y_2 and $y_2|\eta_l$ are both invariant under the action of the group U, so y_2 is invariant under the action of the group generated by U and $\eta_l U \eta_l^{-1}$ in $D_f^\times$. (Note that every element of this group has projection to its pth component lying in $\mathbb{M}_\alpha$.) Since by [Ser80, II.1.4, Corollary 1]

$$\mathrm{SL}_2(\mathbb{Q}_l) = \left\langle \mathrm{SL}_2(\mathbb{Z}_l), \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}_l) \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right\rangle,$$

and the l-factor of U is $\mathrm{GL}_2(\mathbb{Z}_l)$, we have that y_2 is invariant under $\mathrm{SL}_2(\mathbb{Q}_l)$, where we embed $\mathrm{SL}_2(\mathbb{Q}_l)$ into $D_f^\times$ in the obvious way.

Denote by $D^{Nm=1}$ the algebraic subgroup of $D^\times$ whose elements are of reduced norm 1. We have $D^{Nm=1}(\mathbb{Q}_l) \cong \mathrm{SL}_2(\mathbb{Q}_l)$, since D is split at l. The strong approximation theorem applied to $D^{Nm=1}$ implies that $D^{Nm=1}(\mathbb{Q}) \cdot \mathrm{SL}_2(\mathbb{Q}_l)$ is dense in $D_f^{Nm=1} := D^{Nm=1}(\mathbb{A}_f)$, where $D^{Nm=1}(\mathbb{Q})$ is embedded diagonally in $D_f^{Nm=1}$. For each $g \in D_f^\times$ we define

$$X^g := \{h \in D_f^{Nm=1} : y_2(gh) = y_2(g)\}.$$

Since y_2 is continuous, X^g is closed, and for $\delta \in D^{Nm=1}(\mathbb{Q})$, $\gamma \in \mathrm{SL}_2(\mathbb{Q}_l)$ we have $y_2(gg^{-1}\delta\gamma g) = y_2(\delta\gamma g) = y_2(\gamma g) = y_2(gg^{-1}\gamma g) = y_2(g)$, since $g^{-1}\gamma g \in \mathrm{SL}_2(\mathbb{Q}_l)$. Therefore X^g contains the dense set $g^{-1}D^{Nm=1}(\mathbb{Q})\mathrm{SL}_2(\mathbb{Q}_l)g$, so X^g is the whole of $D_f^{Nm=1}$. This shows that y_2 factors through Nm. Now the first part of Lemma 7 applies. □

PROPOSITION 9. *The module* $\mathrm{Tor}_1^{\mathcal{O}(X)}(M^*/jL^{*2}, \mathcal{O}(X)/\mathfrak{m})$ *is* 0 *for all maximal ideals* $\mathfrak{m}$ *of* $\mathcal{O}(X)$.

Proof. We again set $K' = \mathcal{O}(X)/\mathfrak{m}$, and let $x \in X(K')$ be the point corresponding to $\mathfrak{m}$. Proceeding as at the beginning of the proof of Proposition 8 we see that we must show that the map

$$j : L^{*2} \otimes_{\mathcal{O}(X)} K' \to M^* \otimes_{\mathcal{O}(X)} K'$$

is injective. We define

$$L_x^* := \mathbf{V}_x^D(U;r)^{\leqslant d},$$
$$M_x^* := \mathbf{V}_x^D(V;r)^{\leqslant d}.$$

As in the previous proposition, it is sufficient to show that the map $j_x : L_x^{*2} \to M_x^*$ is injective. Now we continue as in the proof of Proposition 8, and finally apply the second part of Lemma 7. □

The following consequence of the preceding two propositions will be the most convenient analogue of Ihara's lemma for our applications.

THEOREM 10.

(i) *There is a positive integer e such that for all but finitely many primes $q \equiv 1 \bmod c$, $(T_q - q - 1)^e$ annihilates $(M/iL^2)^{\mathrm{tors}}$. Therefore these Hecke operators annihilate the modules Λ_2/Λ_3 and, by consideration of the pairing P_2, Λ_0^*/Λ_1^*.*

(ii) *The module $(M^*/jL^{*2})^{\mathrm{tors}}$ is equal to 0. Therefore the modules Λ_2^*/Λ_3^* and, by consideration of the pairing P_1, Λ_0/Λ_1 are also equal to 0.*

Proof. We first prove the first part of the theorem. Fix a $q \equiv 1 \bmod c$ with $q \nmid Np\delta l$. The module $(M/iL^2)^{\mathrm{tors}}$ is finitely generated (and torsion) over the Dedekind domain $\mathcal{O}(X)$, so it is isomorphic as an $\mathcal{O}(X)$-module to $\bigoplus_i \mathcal{O}(X)/\mathfrak{m}_i^{e_i}$ for some finite set of maximal ideals $\mathfrak{m}_i$ in $\mathcal{O}(X)$. We set e to be the maximum of the e_i. Set $H_q := T_q - q - 1$ as before. We will show that H_q^e annihilates $(M/iL^2)^{\mathrm{tors}}$.

Indeed, suppose that $m \in M$ represents a non-zero torsion class in M/iL^2. Thus, there exists a non-zero $\alpha \in \mathcal{O}(X)$ such that $\alpha m \in iL^2$. We have $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/(\alpha)) = \{m \in M/iL^2 : \alpha m = 0\}$. So we are required to prove that H_q^e annihilates $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/(\alpha))$. Since

$$(M/iL^2)^{\mathrm{tors}} \cong \bigoplus_i \mathcal{O}(X)/\mathfrak{m}_i^{e_i},$$

we can assume that $(\alpha) \supset \prod_i \mathfrak{m}_i^{e_i}$, so it is enough to show that H_q^e annihilates

$$\bigoplus_i \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/\mathfrak{m}_i^{e_i}).$$

Taking derived functors of $-\otimes_{\mathcal{O}(X)} M/iL^2$ of the short exact sequence

$$0 \longrightarrow \mathfrak{m}_i/\mathfrak{m}_i^{e_i} \longrightarrow \mathcal{O}(X)/\mathfrak{m}_i^{e_i} \longrightarrow \mathcal{O}(X)/\mathfrak{m}_i \longrightarrow 0$$

and applying Proposition 8 and induction on e_i (note that $\mathfrak{m}_i/\mathfrak{m}_i^{e_i}$ is isomorphic to $\mathcal{O}(X)/\mathfrak{m}_i^{e_i-1}$), we see that for each i, $H_q^{e_i}$ annihilates $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/\mathfrak{m}_i^{e_i})$. Now $e_i \leqslant e$ for all i, so H_q^e annihilates all of $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/(\alpha))$. Since α was arbitrary, H_q^e annihilates $(M/iL^2)^{\mathrm{tors}}$.

The second part of the theorem follows easily from Proposition 9. □

2.10 Supporting Hecke ideals

We will call a maximal ideal $\mathfrak{M}$ in the Hecke algebra $\mathbb{T}^{(N\delta p)}$ *Eisenstein* (compare [Maz77]) if there is some positive integer c such that for all but finitely many primes $q \equiv 1 \bmod c$, $T_q - q - 1 \in \mathfrak{M}$. The motivation for this definition is the following well-known lemma.

LEMMA 11. *Suppose we have a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ and a positive integer c such that for all but finitely many primes $q \equiv 1 \bmod c$, the trace of Frobenius at q, $\mathrm{Tr}(\mathrm{Frob}_q) = q + 1$. Then ρ is reducible.*

Proof. The formula for the traces implies that ρ restricted to the cyclotomic field $\mathbb{Q}(\zeta_c)$ is isomorphic to $\mathbf{1} \oplus \chi$, where $\mathbf{1}$ is the trivial one-dimensional representation, and χ is the l-adic cyclotomic character. Denoting $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by G and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_c))$ by H and applying Frobenius

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reciprocity we conclude that

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\mathbf{1} \oplus \chi), \rho)$$

is non-zero. $\mathrm{Ind}_H^G(\mathbf{1} \oplus \chi)$ is just a direct sum of one-dimensional representations, so ρ is reducible. □

We recall that for an arbitrary ring R the *support* of an R-module A is the set of prime ideals $\mathfrak{p} \lhd R$ such that the localisation $A_{\mathfrak{p}}$ is non-zero. If A is finitely generated as an R-module then the support of A is equal to the set of prime ideals in R containing the annihilator of A. We write $\mathbb{T}_L$ for the image of $\mathbb{T}^{(N\delta p)}$ in $\mathrm{End}_{\mathcal{O}(X)}(L)$ and similarly $\mathbb{T}_M$ for the image of $\mathbb{T}^{(N\delta pl)}$ in $\mathrm{End}_{\mathcal{O}(X)}(M)$. Analogously we define $\mathbb{T}_{L^*}$ and $\mathbb{T}_{M^*}$ to be the image of the Hecke algebra in the endomorphism rings of the relevant dual modules. Note that there are natural embeddings $\mathbb{T}_M \hookrightarrow \mathbb{T}_L$ and $\mathbb{T}_{M^*} \hookrightarrow \mathbb{T}_{L^*}$. If I is an ideal of $\mathbb{T}^{(N\delta p)}$ we write I_L for the image of I in $\mathbb{T}_L$ and I'_M for the image of $I \cap \mathbb{T}^{(N\delta pl)}$ in $\mathbb{T}_M$.

We can now state and prove the main theorem of this section.

Theorem 12. *Suppose $\mathfrak{M}$ is a non-Eisenstein maximal ideal of $\mathbb{T}^{(N\delta p)}$ containing $T_l^2 - (l+1)^2 S_l$. Further suppose that $\mathfrak{M}_L$ is in the support of the $\mathbb{T}_L$-module L. Then $\mathfrak{M}'_M$ is in the support of the $\mathbb{T}_M$-module $\ker(i^\dagger) \subset M$.*

Proof. We write $\mathfrak{M}_{L^*}$ for the image of $\mathfrak{M}$ in $\mathbb{T}_{L^*}$ and $\mathfrak{M}'_{M^*}$ for the image of $\mathfrak{M} \cap \mathbb{T}^{(N\delta pl)}$ in $\mathbb{T}_{M^*}$. Since $\mathfrak{M}_L$ is in the support of L, and the perfect pairing between L and L^* is equivariant with respect to all of $\mathbb{T}^{(N\delta p)}$ (including T_l), we know that $\mathfrak{M}_{L^*}$ is in the support of L^*. Consider the module

$$Q := \Lambda_0^*/\Lambda_3^* = L^{*2} \Big/ L^{*2} \begin{pmatrix} l+1 & T_l \\ S_l^{-1} T_l & l+1 \end{pmatrix}.$$

Since $L^{*2}_{\mathfrak{M}_{L^*}} \neq 0$ and $\det\begin{pmatrix} l+1 & T_l \\ S_l^{-1}T_l & l+1 \end{pmatrix} \in \mathfrak{M}$ we know that $Q_{\mathfrak{M}_{L^*}} \neq 0$, i.e. $\mathfrak{M}_{L^*}$ is in the support of Q. We can view Q as a $\mathbb{T}_{M^*}$-module, with $\mathfrak{M}'_{M^*}$ in its support.

Theorem 10 implies that if $\mathfrak{M}'_{M^*}$ is in the support of Λ_0^*/Λ_1^* or Λ_2^*/Λ_3^* then it is Eisenstein, so it must be in the support of Λ_1^*/Λ_2^*. This quotient is a homomorphic image of $M^*/(M^* \cap j(L_F^{*2}))$, so $\mathfrak{M}'_{M^*}$ is in the support of $M^*/(M^* \cap j(L_F^{*2}))$. Finally we can apply pairing P_3 (which is equivariant with respect to $\mathbb{T}^{(N\delta pl)}$) to conclude that $\mathfrak{M}'_M$ is in the support of $\ker(i^\dagger)$. □

3. Applications

In this section we explain some applications of the preceding results, including the proof of some cases of the conjecture mentioned in the introduction.

3.1 Geometric level raising for *p*-adic modular forms

We firstly describe the application of Theorem 12 to the conjecture of our introduction. Chenevier [Che05] extended the classical Jacquet–Langlands correspondence to a rigid analytic embedding from the eigencurve for a definite quaternion algebra to some part of the GL_2 eigencurve. We may use this to translate the results of the previous section to the GL_2 eigencurve.

We state the case of [Che05, Theorem 3] that we will use. We have primes p, l and a coprime integer $N \geqslant 1$. Fix another prime q and a character ε of $\mathbb{Z}/Np\mathbb{Z}$. Let $\mathcal{E}$ be the tame level $\Gamma_1(N) \cap \Gamma_0(q)$ and character ε reduced cuspidal eigencurve. Let $D/\mathbb{Q}$ be a quaternion algebra

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ramified at the infinite place and at q, and let $\mathcal{E}^D$ be the corresponding reduced eigencurve of tame level $U_1(N)$ and character ε.

Theorem 13. *There is a closed rigid analytic immersion*

$$JL_p : \mathcal{E}^D \hookrightarrow \mathcal{E}$$

whose image is the Zariski closure of the classical points of $\mathcal{E}$ that are new at q. This map is defined over weight space and is Hecke equivariant.

We get the same result if we change the level of $\mathcal{E}$ to $\Gamma_1(N) \cap \Gamma_0(ql)$ (call this eigencurve $\mathcal{E}'$) and change the level of $\mathcal{E}^D$ to $U_1(N) \cap U_0(l)$ (call this eigencurve $\mathcal{E}^{D'}$), where we construct these eigencurves using the Hecke operators at l in addition to the usual Hecke operators away from the level. This allows us to relate $\mathcal{E}'$ and the two-covering $\mathcal{E}^{\text{old}}$ of $\mathcal{E}$ corresponding to taking roots of the lth Hecke polynomial.

Lemma 14. *There is a closed embedding $\mathcal{E}^{\text{old}} \hookrightarrow \mathcal{E}'$, with image the Zariski closure of the classical l-old points in $\mathcal{E}'$.*

Proof. Given X an affinoid subdomain of $\mathcal{W}$, let M_X and M'_X denote the Banach $\mathcal{O}(X)$-modules of families (weight varying over X) of overconvergent modular forms of tame levels $\Gamma_1(N) \cap \Gamma_0(q)$ and $\Gamma_1(N) \cap \Gamma_0(ql)$ respectively, as defined in [Buz07, § 7]. The two degeneracy maps from level $\Gamma_1(N) \cap \Gamma_0(ql)$ to level $\Gamma_1(N) \cap \Gamma_0(q)$ give a natural embedding $M_X^2 \to M'_X$. Denote the image of this map by M_X^{old}; it is stable under all Hecke operators (including at l). The lemma follows from the observation that applying the eigenvariety machine of [Buz07] to the Banach modules M_X^{old} (with X varying) gives the space $\mathcal{E}^{\text{old}}$. $\square$

Let $\mathcal{Z} \subset \mathcal{E}'$ be the Zariski closure of the classical points in $\mathcal{E}'$ corresponding to forms new at l. Proposition 4.7 of [Che05] shows that $\mathcal{Z}$ can be identified with the points of $\mathcal{E}'$ lying in a one-dimensional family of l-*new* points, where l-new means they come from overconvergent modular forms in the kernel of the map analogous to $i^\dagger$ in the GL_2 setting. The following theorem corresponds to the conjecture in the introduction for points of $\mathcal{E}'$ in the image of JL_p.

Theorem 15. *Suppose we have a point $\phi \in \mathcal{E}$ lying in the image of JL_p, with $T_l^2(\phi) - (l+1)^2 S_l(\phi) = 0$. Let the roots of the lth Hecke polynomial corresponding to ϕ be α and $l\alpha$ where $\alpha \in \mathbb{C}_p$. Then the point over ϕ of $\mathcal{E}^{\text{old}}$ corresponding to α lies in $\mathcal{Z}$.*

Proof. We pick d, r and α such that the automorphic form corresponding to the preimage of ϕ under JL_p is r-overconvergent of slope $\leqslant d$ and level $U_1(Np^\alpha)$. Now fix a closed ball in $\mathcal{W}$, containing the weight of ϕ, which is small enough (note that 'small enough' depends on d, r and α) to apply the local eigenvariety construction described in [Che04, § 6.2]. Denote this ball by X. As usual the $rp^{-\alpha}$-analytic character $\mathbb{Z}_p^\times \to \mathcal{O}(X)$ induced by the embedding $X \hookrightarrow \mathcal{W}$ is denoted by κ.

The system of Hecke eigenvalues given by ϕ corresponds to a maximal ideal $\mathfrak{M}$ in the $\mathcal{O}(X)$ Hecke algebra $\mathbb{T}^{(Npq)}$. We know that $T_l^2 - (l+1)^2 S_l \in \mathfrak{M}$. If we set $L = \mathbf{S}_X^D(U_1(Np^\alpha); r)^{\leqslant d}$ as before, and use the notation of the previous section, then $\mathfrak{M}_L$ is in the support of L. At this stage Theorem 12 applies to the ideal $\mathfrak{M}$, which is not Eisenstein since the Galois representation attached to ϕ is irreducible (recall that we are working on the cuspidal part of the eigencurve). Therefore we know that $\mathfrak{M}'_M$ is in the support of $\ker(i^\dagger) \subset M$. We can then take a height one prime ideal $\mathfrak{p} \subset \mathfrak{M}'_M$ in the support of $\ker(i^\dagger) \subset M$, and then this corresponds (by [Che04, Proposition 6.2.4]) to a p-adic family of automorphic forms, new at l, passing through a point ϕ'

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with system of Hecke eigenvalues the same as those for ϕ away from l. Now applying the map JL_p we see that one of the points over ϕ must lie in $\mathcal{Z}$. A calculation using the fact that ϕ' comes from an eigenform in the kernel of $i^\dagger$ shows that the point corresponds to the root α. $\square$

To translate this theorem into the language of the introduction, note that $\mathcal{E}^{\mathrm{old}}$ corresponds to the generically unramified principal series components of $\mathcal{E}'$, whilst $\mathcal{Z}$ corresponds to the generically special or supercuspidal components. We may identify the point over ϕ lying in $\mathcal{Z}$ as the one whose attached $\mathrm{GL}_2(\mathbb{Q}_l)$-representation is special.

3.2 Eigenvarieties of newforms

We return to the situation of a definite quaternion algebra D over $\mathbb{Q}$ with arbitrary discriminant δ prime to p. Denote the levels $U_1(Np^\alpha)$ by U and $U_1(Np^\alpha) \cap U_0(l)$ by V, as before. We denote by $\mathcal{E}^D$ the tame level $U_1(N) \cap U_0(l)$ reduced eigencurve for D. Suppose ϕ is a point of $\mathcal{E}^D$, with weight x. We say that ϕ is *l-new* if it corresponds to a Hecke eigenform in the kernel of the map $i_x^\dagger : \mathbf{S}_x^D(V;r) \to \mathbf{S}_x^D(U;r)$, where this is defined as in § 2.7 by $i_x^\dagger(f) := (f|[V1U], f|[V\eta_l^{-1}U])$. Denote by $\mathcal{Z}$ the Zariski closure of the points in $\mathcal{E}^D$ arising from *classical* l-new forms. We have the following proposition, due to Chenevier.

Proposition 16.

- (i) *The set of x in $\mathcal{E}^D$ that are l-new is the set of points of a closed reduced analytic subspace $\mathcal{E}^D_{\mathrm{new}} \subset \mathcal{E}^D$.*
- (ii) *The space $\mathcal{Z}$ is a closed subspace of $\mathcal{E}^D_{\mathrm{new}}$, and its complement is the union of irreducible components of dimension 0 in $\mathcal{E}^D_{\mathrm{new}}$.*
- (iii) *A point of $\mathcal{E}^D_{\mathrm{new}}$ lies in $\mathcal{Z}$ if and only if it lies in a one-dimensional family of points in $\mathcal{E}^D_{\mathrm{new}}$.*

Proof. Exactly as for [Che05, Proposition 4.7]. $\square$

We now apply the results of § 2 to show that a point of $\mathcal{E}^D_{\mathrm{new}} \backslash \mathcal{Z}$ lies in a one-dimensional family of points in $\mathcal{E}^D_{\mathrm{new}}$, so by contradiction we can conclude that $\mathcal{E}^D_{\mathrm{new}}$ is equal to $\mathcal{Z}$.

Theorem 17. *The space $\mathcal{E}^D_{\mathrm{new}}$ is equal to $\mathcal{Z}$. In particular, $\mathcal{E}^D_{\mathrm{new}}$ is equidimensional of dimension one.*

Proof. Let ϕ be a point of $\mathcal{E}^D_{\mathrm{new}} \backslash \mathcal{Z}$ with weight x. The proof will proceed by showing that ϕ is also l-old, then raising the level at ϕ, as in the previous theorem, to show that it lies in a family of l-new points.

We pick d, r and α such that ϕ comes from a r-overconvergent automorphic form of slope d and level $U = U_1(Np^\alpha)$. Now fix a closed ball in $\mathcal{W}$, containing the weight of ϕ, which is small enough (note that 'small enough' depends on d, r and α) to apply the local eigenvariety construction described in [Che04, § 6.2]. Denote this ball by X. The point x in X corresponds to a maximal ideal $\mathfrak{m}$ of $\mathcal{O}(X)$. If we set $M = \mathbf{S}_X^D(V;r)^{\leqslant d}$ as before, then ϕ lies in a family corresponding to a Hecke eigenvector in M.

As in the previous subsection, we have a closed embedding $\mathcal{E}^D_{\mathrm{old}} \hookrightarrow \mathcal{E}^D$, where $\mathcal{E}^D_{\mathrm{old}}$ is a two-covering of the tame level U reduced eigencurve for D, and the image of this embedding is the Zariski closure of the classical l-old points in $\mathcal{E}^D$, which also equals the space of l-old points in $\mathcal{E}^D$ (as is clear from applying the proof of Lemma 14 to modules of overconvergent automorphic forms for D). Since the space $\mathcal{E}^D$ is equidimensional of dimension one, but ϕ does not lie in a family of points in $\mathcal{E}^D_{\mathrm{new}}$, ϕ must lie in a family of points in $\mathcal{E}^D_{\mathrm{old}}$. So ϕ is l-old and l-new.

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We now 'raise the level' at ϕ. As in the proof of Proposition 8 we specialise the $\mathcal{O}(X)$ modules $L = \mathbf{S}_X^D(U;r)^{\leqslant d}$ and M at the maximal ideal $\mathfrak{m}$ to give vector spaces L_x and M_x. Since ϕ is l-new and l-old, it arises from an eigenform g in $\mathrm{im}(i_x) \cap \ker(i_x^\dagger)$. Now a calculation using the explicit matrix for the map $i_x^\dagger i_x$ shows that g is of the form $i_x(\alpha f, -f)$, where f is an eigenform in L_x with $(T_l^2 - (l+1)^2 S_l)f = 0$, and the roots of the lth Hecke polynomial for f are α and $l\alpha$ (note that with our normalisations g is the l-stabilisation of f corresponding to the root α). Now applying the proof of Theorem 15 we see that ϕ lies in $\mathcal{Z}$, so we have a contradiction and therefore must have $\mathcal{E}_{\mathrm{new}}^D = \mathcal{Z}$. $\square$

3.2.1 *Modules of newforms and the eigenvariety machine.* Define the Banach $\mathcal{O}(X)$-module $M_{X,r}^{\mathrm{new}}$ for varying affinoid subdomains $X \subset \mathcal{W}$ (with corresponding character κ) to be the kernel of the map

$$i^\dagger : \mathbf{S}_X^D(U_1(Np^\alpha) \cap U_0(l); r) \to \mathbf{S}_X^D(U_1(Np^\alpha); r) \times \mathbf{S}_X^D(U_1(Np^\alpha); r).$$

One might wish to construct the rigid analytic space $\mathcal{E}_{\mathrm{new}}^D$ from these modules using Buzzard's eigenvariety machine [Buz07]. The key issue is to show that the modules $M_{X,r}^{\mathrm{new}}$ behave well under base change between affinoid subdomains. The author is not sure whether this should be true or not; the results in this paper can be viewed as showing that these modules behave well under base change from a sufficiently small affinoid to a point and it is not clear that one can conclude something about base change between open affinoids from this.

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