

Minimal genus and fibering of canonical surfaces via disk decomposition

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ABSTRACT

This paper contains some applications of the description of knot diagrams by genus, and Gabai's methods of disk decomposition. We show that there exists no genus one knot of canonical genus 2, and that canonical genus 2 fiber surfaces realize almost every Alexander polynomial only finitely many times (partially confirming a conjecture of Neuwirth).

1. Introduction

Introducing the *genus* $g(K)$ of a knot K , Seifert [42] gave a construction of compact oriented surfaces in 3-space bounding the knot (*Seifert surface*) by an algorithm starting with some diagram of the knot (see [1, §4.3] or [40]). The surface given by this algorithm is called *canonical*. The *canonical genus* $\tilde{g}(K)$ of K is the minimal genus over all surfaces so obtained, the *genus* $g(K)$ the minimum extended over possible other Seifert surfaces of K . Clearly, $g(K) \leq \tilde{g}(K)$. The case of equality means the existence of a diagram which is *genus-minimizing* (or of *minimal genus*), that is, whose canonical Seifert surface realizes $g(K)$ (and not only $\tilde{g}(K)$). This problem has been studied over a long period. First, the minimal genus property was shown for alternating diagrams, independently by Crowell [11] and Murasugi [30]. Their algebraic proof uses the Alexander polynomial Δ [2] and the inequality $\max \deg \Delta \leq g$ (which thus they prove to be exact for alternating knots). Later Gabai [14] developed a geometric method using foliations, called disk decomposition, which, too, establishes genus minimality for alternating diagrams. Murasugi's sum operation [31] ($*$ -product) was shown to behave naturally with respect to the Alexander polynomial by himself, and later by Gabai [15, 16] in the geometrical context.

The aim of this paper is to apply the description of canonical genus 2 surfaces in [44], the methods of Gabai to detect minimal genus and fiber property, and some computational effort, to prove two related theorems.

THEOREM 1.1. *No knot has genus $g = 1$ and canonical genus $\tilde{g} = 2$.*

This statement cannot be considered expected, since for any other pair of integers a, b with $1 \leq a \leq b$, it is not too hard to construct knots with $g = a$ and $\tilde{g} = b$. A reformulation (using Theorem 2.9) is that almost every genus 2 diagram is *genus-minimizing*, except if it displays a (p, q, r) -pretzel knot, p, q, r odd, or a genus 1 2-bridge knot. In particular, we have the following corollary.

COROLLARY 1.1. *The canonical genus of any satellite knot of wrapping number zero is at least 3.*

In case of Whitehead doubles, there is a conjecture saying that their canonical genus is equal to the crossing number of the companion knot. The conjecture was confirmed first by Tripp [52]

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for doubles of $(2, n)$ -torus knots, and later more generally by Nakamura [33] for 2-bridge knots, and for alternating pretzel knots by Brittenham-Jensen [5]. Not much else seems known about it, though. The above corollary implies a partial statement towards the general case.

The proof of Theorem 1.1 consists in adapting a certain version of disk decomposition to deal with infinite families of canonical genus 2 surfaces. The failure of this procedure leaves only knots either of canonical genus 1, or with canonical genus 2 surfaces which can be Murasugi desummed. In the latter case, the summands can be inspected by induction.

The other result of disk decomposition, proved by essentially the same method, relates to the description of Alexander polynomials of fibered knots.

Neuwirth, who introduced fibered knots, conjectured that for a given Alexander polynomial only finitely many fibered knots occur. This is clearly true for polynomials of degree[†] 1, since it was well known that only the trefoil and figure-8-knot are fibered genus 1 knots. Morton found the first counterexamples in [27], and later (after some other authors' contributions, for example [38]) managed to 'demolish the remains of this conjecture' [28] by constructing an infinite number of fibered knots for any Alexander polynomial of degree at least 2. His construction suggests that the description of general fiber surfaces is already too complicated even in genus 2. It seems reasonable to look then for some restrictions under which something more specific could be said about these surfaces. One such natural condition is to consider canonical surfaces. It was recently proved, independently by Nakamura [34] and myself [49], that each monic Alexander polynomial is realized by at least one canonical fiber surface. Moreover, for example by Stallings twists, one can construct infinitely many canonical fiber surfaces for many concrete Alexander polynomials. Nakamura's question thus becomes natural: do infinitely many canonical fiber surfaces (or equivalently fibered knots with such surfaces) exist for any monic Alexander polynomial of degree at least 2? We answer this question negatively for genus 2.

THEOREM 1.2. *All but finitely many monic degree 2 (admissible) polynomials are realized as Alexander polynomials by only finitely many (genus 2) knots with canonical fiber surfaces.*

This seems the first somewhat broader statement toward confirming (at least) partially Neuwirth's conjecture.

2. Preliminaries

2.1. Knots and knot diagrams

Knots/links and their diagrams must be oriented (even if orientation is not always displayed).

A crossing p in a knot diagram D is called *reducible* (or nugatory) if it looks like on the left of (2.1).

$$\begin{array}{c} \text{B} \quad \text{Q} \\ \diagdown \quad \diagup \\ \text{p} \\ \diagup \quad \diagdown \end{array} \longrightarrow \begin{array}{c} \text{P} \quad \text{Q} \\ \text{---} \quad \text{---} \end{array} \quad (2.1)$$

A diagram D is called *reducible* if it has a reducible crossing; otherwise, it is called *reduced*. The reducing of the reducible crossing p is the move depicted in (2.1). Each diagram D can be (made) reduced by a finite number of these moves. We assume in the following all diagrams reduced, unless otherwise stated.

The diagram on the right of

$$\begin{array}{c} \text{A} \\ \text{---} \end{array} \# \begin{array}{c} \text{B} \\ \text{---} \end{array} = \begin{array}{c} \text{A} \quad \text{B} \\ \text{---} \quad \text{---} \end{array}$$

[†]Here the maximal degree, and not the span (twice the maximal degree) is meant; see § 2.3.

is called the *connected sum* $A\#B$ of the diagrams A and B . We say that A and B are (*connected sum*) *factors* of D . If a diagram D can be represented as the connected sum of diagrams A and B , such that both A and B have at least one crossing, then D is called *composite*; otherwise, it is called *prime*. A knot or link K is *prime* if, whenever $D = A\#B$ is a composite diagram of K , one of A and B represents an unknotted arc (but not both; the unknot is not prime per convention). Every diagram (and link) has a decomposition into *prime factors*, unique up to permutation.



THEOREM 2.1 [24]. *If D is a prime alternating non-trivial diagram of a link K , then K is prime.*



A diagram D is *split* or *disconnected* if its plane curve is a disconnected set in \mathbb{R}^2 , that is, there is a simple closed curve disjoint from D , whose interior and exterior contain parts of D . A diagram which is not disconnected is *connected* or *non-split*. A link is a *split link* if it has a split diagram, and otherwise a *non-split link*.

The (*Seifert*) *genus* $g(K)$, respectively *Euler characteristic* $\chi(K)$ of a knot or link K , is said to be the minimal genus, respectively maximal Euler characteristic, of a Seifert surface of K . For a diagram D of K , we define $g(D)$ to be the genus of the Seifert surface obtained by Seifert’s algorithm on D , and $\chi(D)$ its Euler characteristic. For a diagram with $c(D)$ crossings, $n(D) = n(K)$ components, and $s(D)$ Seifert circles, we have $\chi(D) = s(D) - c(D)$ and $2g(D) = 2 - n(D) - \chi(D)$.

THEOREM 2.2 (See [11, 14, 30]). *If D is an alternating diagram of K , then $g(K) = g(D)$ and $\chi(D) = \chi(K)$.*

The *canonical genus* $\tilde{g}(K)$, respectively *canonical Euler characteristic* $\tilde{\chi}(K)$, is defined as the minimal genus, respectively maximal Euler characteristic, of all diagrams of K . In general we can have $g(K) < \tilde{g}(K)$, that is, no diagrams of K of minimal genus (see [26]).

The *writhe* or *sign* of a crossing is ± 1 . A crossing as  has writhe 1 and is called *positive*. A crossing as  has writhe -1 and is called *negative*. The writhe $w(D)$ of a link diagram D is the sum of the writhes of all its crossings. A diagram is *positive/negative* if all its crossings are such; a link with this type of diagram is also called *positive/negative*. (See, for example, [10, 36, 53, 54].)

DEFINITION 2.1. A *clasp* is a tangle that consists of two crossings, that is, a pair of crossings that bound a bigonal (2-corner) region in the plane complement of a diagram. The clasp is called *positive*, *negative* or *trivial*, if both crossings are positive/negative, respectively of different sign. Depending on the orientation of the involved strands (for an oriented diagram) we distinguish between a *reverse clasp*  and a *parallel clasp* . Thus a clasp is reverse if it contains a full Seifert circle, and parallel otherwise. The replacement of a crossing with a parallel clasp is called a *clasping*:

$$\begin{matrix} \diagup \diagdown \\ \diagdown \diagup \end{matrix} \rightarrow \begin{matrix} \diagup \diagdown \\ \diagdown \diagup \end{matrix} \text{ , } \quad \begin{matrix} \diagup \diagdown \\ \diagdown \diagup \end{matrix} \rightarrow \begin{matrix} \diagup \diagdown \\ \diagdown \diagup \end{matrix} \text{ .} \tag{2.2}$$

It is always compatible with component orientation, but changes the number of components.

DEFINITION 2.2. A *flype* is a move on a diagram as shown in Figure 1. We say that a crossing *admits* a flype if it can be represented as the distinguished crossing p in the diagrams in the figure, and both tangles have at least one crossing.

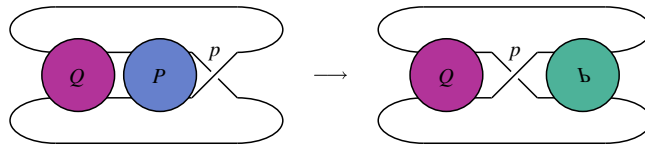
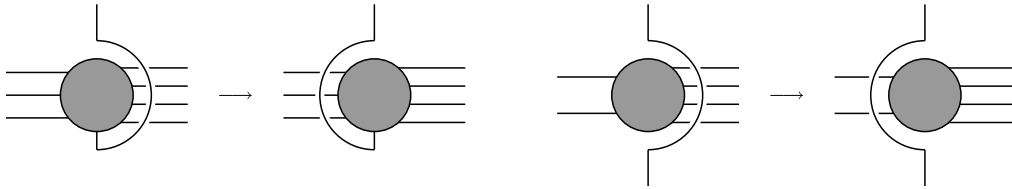
FIGURE 1. A flype near the crossing p .

FIGURE 2. Wave moves. The number of strands on the left and right of the shaded circle may vary. It is only important that the parities are equal, respectively different, and that the left-outgoing strands are fewer than the right-outgoing ones.

By the fundamental work of Menasco-Thistlethwaite, we have a proof of the Tait flyping conjecture.

THEOREM 2.3 [25]. *For two alternating diagrams of the same prime alternating link, there is a sequence of flypes taking the one diagram into the other.*

A wave move, or bridge rerouting, is a replacement of a too long bridge by a shorter one; see Figure 2 (or [50] for example).

A Seifert circle is called *separating* if both its complementary regions in the plane contain other Seifert circles. A diagram is *special* if no Seifert circle is separating. For such a diagram D , Seifert circles coincide with the boundary of a set of regions of D (connected components of the planar complement of D). We will call the remaining regions of D *hole regions*.

It is an easy observation that, for connected diagrams, two of the properties alternating, positive/negative and special imply the third. A diagram with these properties is called *special alternating*. A knot or link is special alternating if it has a special alternating diagram. Such knots were introduced and studied by Murasugi [29] and have a series of special features. Contrarily, all knots have a special (not necessarily alternating) diagram. Hirasawa [20] shows how to modify any knot diagram D into a special diagram D' so that $g(D) = g(D')$ (actually, the canonical surfaces of D and D' are isotopic).

The concept of (diagrammatic) Murasugi sum will also be of central importance here; we appeal to some nomenclature set up in [9, § 1] and [39].

DEFINITION 2.3. The *Seifert picture* (union of Seifert circles) of a link diagram D separates the plane into regions. A non-empty part of D lying in some such region is called a *block*. We call a *Murasugi atom* a prime factor of a block of D .

In the terms of Definition 2.3, a Seifert circle is separating if it bounds blocks on both sides, and a diagram is special if it has only one block. In particular, any of the blocks of D is special.

DEFINITION 2.4 (see [32] for example). The operation that reconstructs a diagram from its blocks by gluing them back along the separating Seifert circles is called (diagrammatic) **-product* or *Murasugi sum*.

This operation is (here) defined for a pair of diagrams A, B and a pair of distinguished non-separating Seifert circles a, b . We draw A so that all of it lies in the interior of a and B so that all of B is in the exterior of b . Then we identify a and b , such that at a place where a crossing is attached to b (respectively a) we delete an interval of a (respectively b). There may

be a sequence of consecutive deleted intervals along a or b ; we identify them. Then the number of such deleted intervals is an even integer; it can be called the *degree* of the Murasugi sum. A Murasugi sum of degree 2 is just a connected sum. A Murasugi sum of degree 4 is called a *plumbing*. If one of A or B is a Hopf band, it is a *Hopf plumbing*. The converse operations to (Hopf) plumbing and Murasugi sum are called (*Hopf*) *deplumbing* and *Murasugi desum*. (See, for example, [19] for some geometric pictures.)

The operation on the canonical surfaces of these diagrams is likewise called Murasugi sum. (This operation can be defined for Seifert surfaces in a more general form.)

DEFINITION 2.5. A knot or link L is *fibred*, if $S^3 \setminus L$ is a surface bundle over a circle.

THEOREM 2.4 (Neuwirth–Stallings). *If L is fibred, then a fiber surface is a minimal genus surface, and a minimal genus surface is unique.*

DEFINITION 2.6. A knot or link L will be called *canonically fibred*, if L is fibred, and the fiber is a canonical surface for some diagram D of L . We then also call D canonically fibred.

The proofs of Theorems 1.1 and 1.2 will rely fundamentally on the following theorem of Gabai.

THEOREM 2.5 (Gabai [16–18]). *If a surface L decomposes as Murasugi sum $A * B$, then L is of minimal genus if and only if A and B are so, and L is a fiber surface if and only if A and B are so.*

2.2. Knot and diagram notation

We will use, unless explicitly noted otherwise, KnotScape’s [21] notation for eleven or more crossing knots throughout the rest of the paper. It is organized so that non-alternating knots are appended to alternating ones instead of using ‘a’ and ‘n’ superscripts. For ten or fewer crossings we use the numbering of [40], with the Perko duplication removed (and index of the last four knots shifted down by 1).

It is convenient to define the *Dowker–Thistlethwaite (DT) notation* [12], in the form used in [21]. Several diagrams will be given, instead graphically, by this form.

A knot diagram D of n crossings is represented as a sequence of $n + 2$ integers of the form (n, k, a_1, \dots, a_n) . The first integer specifies the crossing number, n . The second integer is a knot identifier; it has a merely technical purpose, and in the notation we reproduce its value will be specified subsequently. The a_i are obtained as follows. Each crossing of D has two crossing points, an overcrossing and an undercrossing. We number the crossing points by 1 to $2n$ cyclically along the circle. Each crossing then has a crossing point of an odd number $2k - 1$ (for some $1 \leq k \leq n$) and one of an even number l . We set $a_k = l$ if l is the overcrossing, and $a_k = -l$ otherwise. We obtain thus a sequence of n even integers a_i , which determines a *prime diagram* D up to mirroring. We fix the mirroring ambiguity so that $a_1 > 0$ if and only if its crossing (the crossing with crossing point 1) is positive (as defined in § 2.1). Beside saving space, the DT notation thus has a further advantage of assigning an order to the crossings, which will be very convenient for us.

The *mirror image* of a diagram D is written $!D$, and $!K$ is the mirror image of K . Clearly $g(!D) = g(D)$ (and therefore $\tilde{g}(K) = \tilde{g}(!K)$), and $g(!K) = g(K)$.

We will repeatedly mention (specific) rational, pretzel, and Montesinos tangles, diagrams and links, and thus we clarify here (our convention of) terminology. In the following we use the approach of Conway [7].

A *tangle diagram* is a diagram consisting of strands crossing each other, and having four ends.

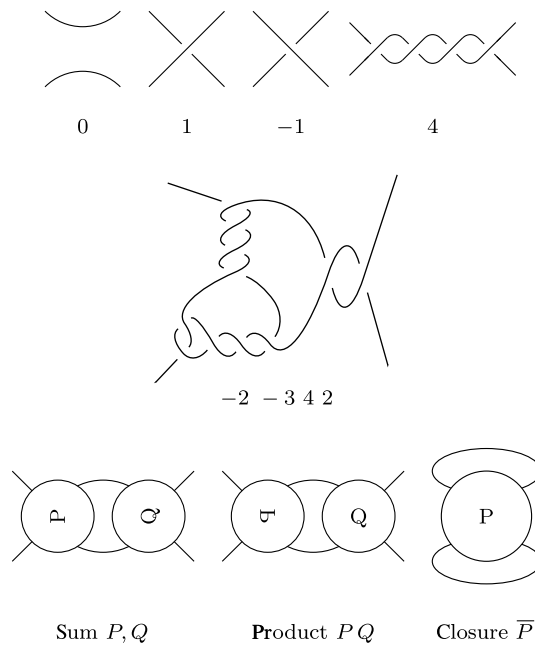


FIGURE 3. Conway's tangles and operations.

DEFINITION 2.7. A *rational* tangle diagram is the one that can be obtained from the primitive Conway tangle diagrams by iterated left-associative product in the way displayed in Figure 3. A simple but typical example is shown in the middle of the figure. The convention in composing tangles, which will extend to Montesinos diagrams below, is that the diagram is alternating if and only if all occurring numbers have the same sign. A *rational (2-bridge) knot* or link is one with a diagram which is the closure of a rational tangle diagram.

Montesinos links (see, for example, [23]) are generalizations of pretzel and rational links and special types of arborescent links. They are obtained by summing (simultaneously, not successively) n rational tangles P_i . If $n < 3$, an easy observation shows that the Montesinos link is in fact a rational link. A *pretzel link* (a_1, \dots, a_n) for integers a_i is a Montesinos link with all P_i being primitive Conway tangles a_i . A *Montesinos diagram* (respectively rational or pretzel diagram) is the diagram of a closed Montesinos (or rational, or pretzel) tangle. Geometric properties of Montesinos links are discussed in detail in [6].

2.3. The Alexander-Conway polynomial

Let $X \in \mathbb{Z}[t, t^{-1}]$. The *minimal* or *maximal degree* $\min \deg X$ or $\max \deg X$ is the minimal, respectively maximal, exponent of t with non-zero coefficient in X . Let $\text{span}_t X = \max \deg_t X - \min \deg_t X$. The coefficient in degree d of t in X is denoted $[X]_{t^d}$ or $[X]_d$. The *leading coefficient* \max of X of X is its coefficient in degree $\max \deg X$.

The (one variable) *Alexander polynomial* [2] Δ can be defined by being 1 on the unknot and the (skein) relation

$$\Delta\left(\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array}\right) - \Delta\left(\begin{array}{c} \nwarrow \\ \times \\ \nearrow \end{array}\right) = (t^{1/2} - t^{-1/2}) \Delta\left(\begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array}\right). \tag{2.3}$$

(The Alexander polynomial is often defined only up to units in $\mathbb{Z}[t, t^{-1}]$; the present normalization is so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$. For links there are also multi-variable versions, but we use only the one-variable polynomial throughout the paper.)

We will sometimes use, instead of Δ , the *Conway polynomial* [7] $\nabla(z)$ with $\nabla(t^{1/2} - t^{-1/2}) = \Delta(t)$. It satisfies the skein relation (with D_* the diagrams in (2.3))

$$\nabla(D_+) - \nabla(D_-) = z\nabla(D_0). \tag{2.4}$$

Note that $\max \deg \nabla = 2 \max \deg \Delta$. It is well known that for a link L of n components, $\nabla_L \in z^{n-1}\mathbb{Z}[z^2]$, and that each such polynomial occurs for some L , except for knots ($n = 1$), where we pose the additional condition $[\nabla]_0 = 1$. We abbreviate by ∇_i the coefficient $[\nabla]_i$ of z^i in ∇ , and for $i = 2, 4$ we sometimes write v_i for ∇_i .

Since we have $2 \max \deg \Delta \leq 1 - \chi$ for every link, it is clear that in any diagram $2 \max \deg \Delta(D) \leq 1 - \chi(D)$. Murasugi proved the multiplicativity of the maximal (possible) coefficients under Murasugi sum.

THEOREM 2.6 (Murasugi [31]; see also [32]). *If a diagram D decomposes as Murasugi sum $A * B$, then*

$$[\Delta(D)]_{(1-\chi(D))/2} = [\Delta(A)]_{(1-\chi(A))/2} \cdot [\Delta(B)]_{(1-\chi(B))/2}.$$

The notion of a homogeneous diagram/link was introduced by Cromwell [9], in an attempt to extend certain results on positive and alternating links. We say that D is *homogeneous* if all blocks D_i of D are positive or negative (that is, special alternating). A link is homogeneous if it has a homogeneous diagram. Since in positive/negative diagrams all blocks are positive/negative, such diagrams are homogeneous. Alternating diagrams are also homogeneous, this time so that blocks which are bordering along a Seifert circle have opposite sign. Theorem 2.6 (algebraically) and Theorem 2.5 (geometrically) extend the minimal genus property of alternating diagrams to homogeneous diagrams and links. (For positive diagrams this property follows from yet another source, the work of Bennequin [3] on contact structures; see [41].)

The following property of the Alexander polynomial of fibered links is well known. We call a polynomial Δ in $\mathbb{Z}[t^{\pm 1}]$ *monic* if $\max \text{cf } \Delta = \pm 1$.

THEOREM 2.7 (See, for example, [40]). *If L is fibered, then $2 \max \deg \Delta_L = 1 - \chi(L)$ and Δ_L is monic.*

2.4. Genus generators

Now let us recall, from [44, 45], some basic facts concerning knot generators of given genus. An explanation is also given in [8, § 5.3]. (There are several equivalent forms of these definitions, and we choose here one that leans on the previously explained DT code.) We will also fix some notation and conventions used below.

DEFINITION 2.8. Let D be a knot diagram, and p and q be two of its crossings. If the crossing points of p, q are passed in cyclic order $ppqq$, we call the crossings *linked*, and write $p \cap q$. Otherwise (that is, if the cyclic order is $ppqq$), p, q are *unlinked*, and we denote this by $p \not\cap q$.

- (i) We call p and q *equivalent*, $q \simeq p$, if for all $r \neq p, q$ we have $r \cap p \iff r \cap q$.
- (ii) We call p and q *~-equivalent* and write $p \sim q$ if p and q are equivalent and $p \not\cap q$.
- (iii) Similarly p and q are called *~*-equivalent*, $p \approx q$, if p and q are equivalent and $p \cap q$.
- (iv) We call p and q *twist equivalent*, if there is a sequence of crossings $p = p_0, \dots, p_n = q$ such that p_i and p_{i+1} form a clasp (cf. Definition 2.1).

We sometimes also call a ~-equivalence class a *group* of crossings. A ~-equivalence class consisting of one crossing is called *trivial*, a class of more than one crossing *non-trivial*.

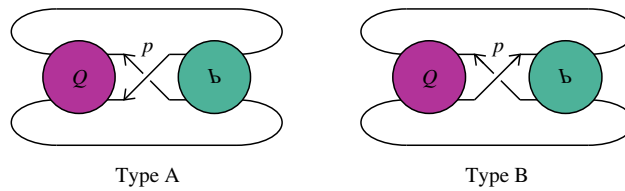


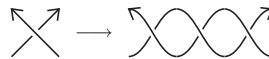
FIGURE 4. A flype of type A and B.

It is an exercise to check that \sim is an equivalence relation, and two crossings are \sim -equivalent, if after a sequence of flypes they can be made to form a reverse clasp. The same remark applies to \approx , with a parallel clasp instead of a reverse one. (In this way, one can also define these equivalences in link diagrams, where linkedness of crossings cannot be defined as done here.)

It is readily seen that linkedness is a feature of two \sim -equivalence (in fact, even equivalence) classes, and thus we also call \sim -equivalence classes (un)linked, depending on whether any crossing in the one is (un)linked with any crossing in the other. For instance, the 3 crossing trefoil diagram has three (trivial) mutually linked \sim -equivalence classes, which form a single equivalence class. The 4 crossing figure-8-knot diagram has two pairs of unlinked crossings, forming two linked \sim -equivalence (and equivalence) classes.

DEFINITION 2.9. Let $t(D)$ be the number of \sim -equivalence classes of the diagram D . When a DT notation of D is given, there is a natural ordering of its \sim -equivalence classes by the first integer in the sequence (giving a crossing in the class). A DT notation (a_i) also assigns a number $i = 1, \dots, c(D)$ to each crossing of D , and we write $\{i\}$ for the \sim -equivalence class of crossing i .

DEFINITION 2.10. A \bar{t}_2 move or twist at a crossing p in a diagram D is a move which creates a pair of \sim -equivalent crossings to p as shown in the following diagram.



DEFINITION 2.11. An alternating diagram D is called \bar{t}_2 irreducible or generating diagram, if all its \sim -equivalence classes have at most two crossings. The diagrams obtained from D by \bar{t}_2 moves and crossing changes form the sequence or series $\langle D \rangle$ of D . An alternating knot K is called a generator if some of its alternating diagrams are generating.

Observe that, since \sim -equivalence is invariant under flypes, Theorem 2.3 implies that some alternating diagram of K is generating if and only if all its alternating diagrams are so.

A flype in Figure 1 is called trivial if one of the tangles contains only crossings equivalent to the crossing admitting the flype.

We introduced (see [51]) a distinction of flypes according to the orientation near the crossing p at which the flype is performed. See Figure 4, and compare with the right of Figure 1. A flype is of type A if the strand orientation is so that strands on the left/right side of each tangle are directed equally with respect to the tangle (that is, both enter or both exit). Otherwise it is a flype of type B. So the property to admit a type A, respectively type B, flype is an invariant of the \approx , respectively \sim -equivalence, class.

An important observation is that each crossing admits at most one of the types A and B of flypes, and this remains so after applying any sequence on flypes on the diagram.

THEOREM 2.8 [4, 45]. There exist only finitely many generators of given genus. All diagrams of that genus can be obtained from diagrams of these generators, under \bar{t}_2 twists, flypes, and crossing changes.

In [46] we obtained rather sharp estimates on the maximal number of crossings and \sim -equivalence classes of generators. This, together with the examples in [51], determined the maximal crossing number of a knot generator to be $10g - 7$ for genus $g \geq 2$, and the maximal number of \sim -equivalence classes to be $6g - 3$.

In practice (in particular as we will see below) it is important to obtain the list of generators for small genus. We recall the result for genus at most 2. (For genus 2, it is sufficient to find prime diagrams, and by Theorem 2.1, prime generators. Genus 1 was also observed independently; see [4, 41].)

THEOREM 2.9 [44, 45]. *There are two generators of genus 1, the trefoil and figure-8-knot. There are 24 prime generators of genus 2, as shown in Figure 5.*

A classification, by means of obtaining the list of prime generators, is possible for genus $g \leq 4$, and is explained in [44, 48]. The rapid growth of complexity not only raised an enormous computational challenge for the generator list of each next genus, but also required an entirely different (and considerably more efficient) method of compilation.

It follows from Theorem 2.3 that the series of different (alternating) diagrams of the same generating knot are equivalent up to mutations. However, in §3, the sensitivity of the geometric methods with respect to mutation requires us to work with multiple diagrams D of each generator K . In that case, the series $\langle K \rangle$ will stand for the union of $\langle D \rangle$ over diagrams D of K that need to be treated. (One can still use $t(K) = t(D)$, since $t(D)$ is equal for all diagrams D that can be taken.)

Let $D(x_1, \dots, x_l)$ be the diagram in $\langle D \rangle$ where group i has crossings of total writhe x_i . A feature of Vassiliev invariants, studied in [47], is that when v is a Vassiliev invariant of degree n , then the map

$$(x_1, \dots, x_l) \mapsto v(D(x_1, \dots, x_l))$$

is a polynomial in x_i of degree (taken with respect to all variables together) at most n . In case of the coefficients of ∇ , such as $v = v_2$ or v_4 , a consequence of the skein relation (2.4) of ∇ is that this polynomial is additionally square-free (no monomial contains a variable in a power higher than 1; see also [43]). In several calculations below it will be necessary to determine these polynomials (under some restrictions on x_i). This was usually done by evaluating v on several simple instance diagrams $D(x_1, \dots, x_l)$ and recovering the polynomial from these particular values.

2.5. Disk decomposition

Here we briefly review the work of Gabai in [14, 17], which will be used in §3. For a set S , the expression $|S|$ denotes in the sequel the cardinality of S . The symbol ' \subset ' stands for a (not necessarily proper) inclusion.

A *sutured manifold* in the sense of Gabai [14] can be understood as a pair (L, H) consisting of a closed 3-dimensional submanifold H of \mathbb{R}^3 with boundary $S = \partial H$ a connected surface, and a set of oriented disjoint loops $L \subset H$, called *suture*. We require that one can orient the connected components of $S \setminus L$ so that the induced orientations on L coincide from both sides of L (in particular a connected component of $S \setminus L$ never bounds to itself along a loop of L), and are given by the orientation of L .

Let F be a connected Seifert surface of a(n oriented) link $L = \partial F$. We embed F as $F \times \{0.5\}$ into the *bicolor* $H = F \times I$ (with $I = [0, 1]$). Then (L, H) becomes a sutured manifold. We call it a *canonical sutured manifold* $C(F)$ of F .

Now let X be a special diagram of a knot K and F its canonical Seifert surface. Then in $C(F) = (K, H)$, the bicolor H is a handlebody, with $2g + 1$ holes, corresponding to the hole regions of X (including possibly the unbounded one), where $g = g(X)$.

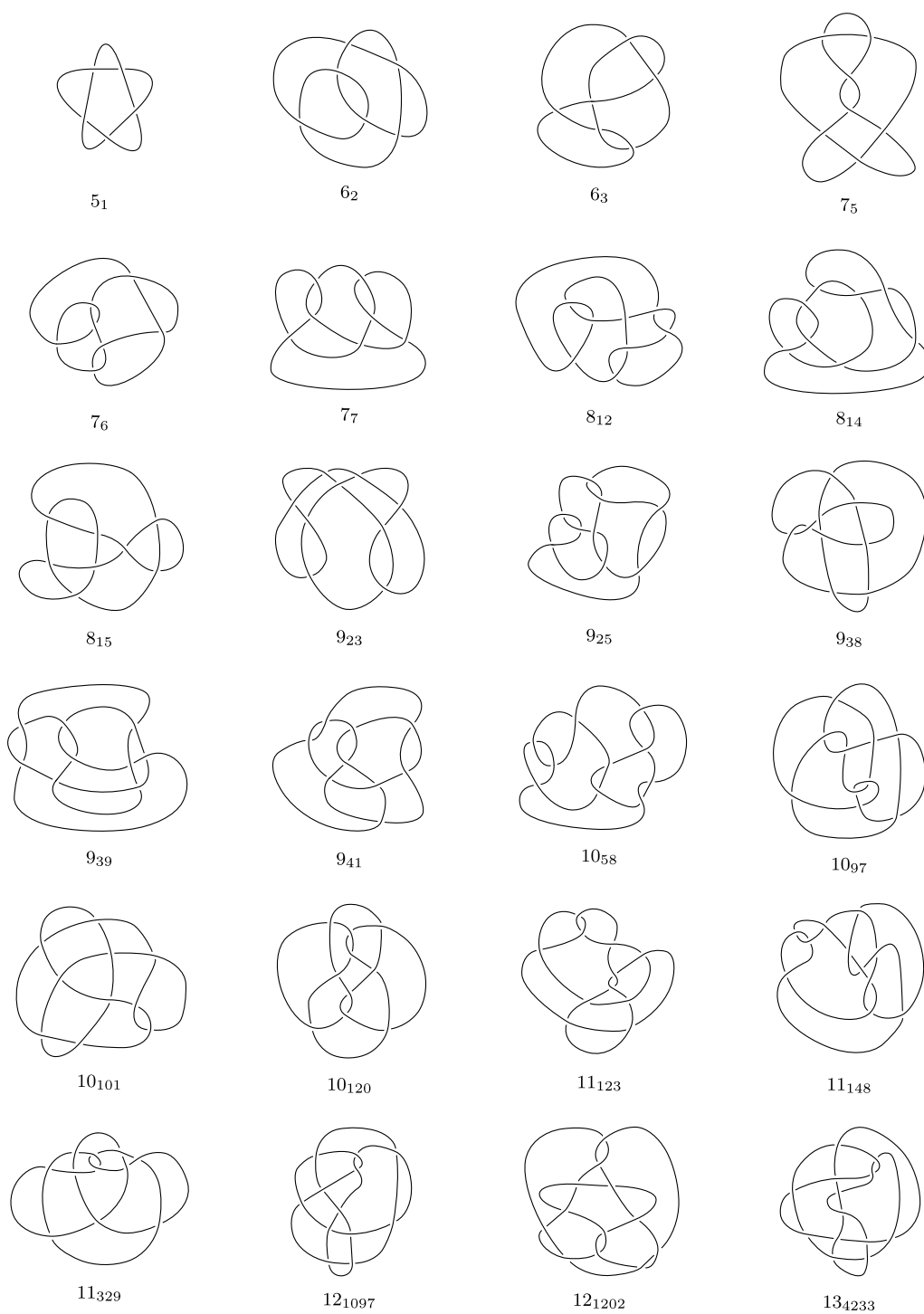


FIGURE 5. *The 24 prime genus 2 generators.*

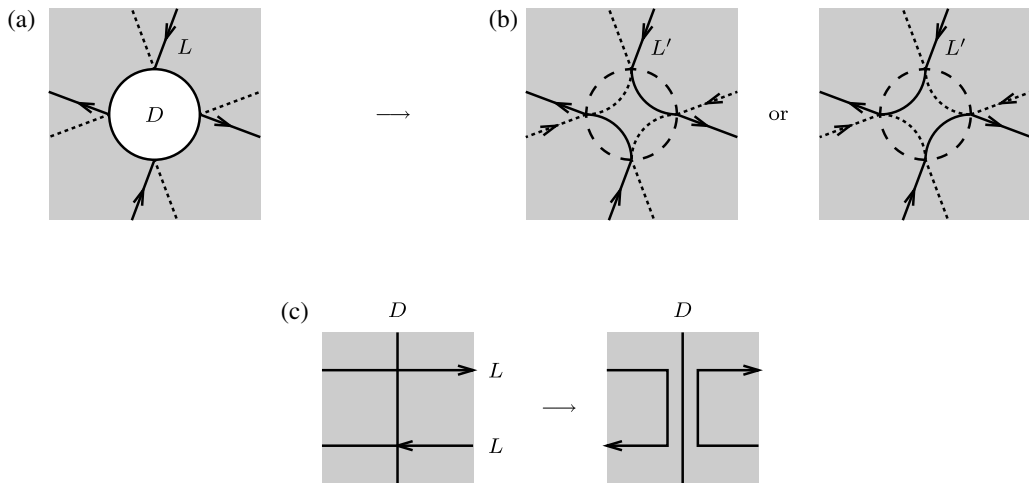


FIGURE 6. Disk decomposition.

We describe some basic operations on sutured manifolds (L, H) .

A *decomposition disk* D is a disk with $P = \partial D \subset \partial H$, properly embedded in the complement of H (that is, $D \cap H = P$). We require that D is not parallel to $S = \partial H$, and satisfies $P \cap F \neq \emptyset$. We assume also that the intersection of P and F is transversal, so that it is a collection of points. Every disk D can be thought of as lying in a handle in the complement of H , and so represents a ‘hole’ of H . Let us call an *empty hole* a hole not intersecting the suture.

Assume that D does not lie in an empty hole. Since $L = \partial F$ is separating on S , the intersection $D \cap L = P \cap F$ is an even number of points, and the orientation of L at the intersection points is alternating (with respect to the orientation of the loop P). See diagram (a) in Figure 6 (where the grayed regions indicate the bicolor).

A *disk decomposition* on D is the operation from part (a) to part (b) in Figure 6. It is shown in the case that $|D \cap L| = 4$, but applies analogously if $|D \cap L| > 4$. It is a transformation of (L, H) into a sutured manifold (L', H') , where L' is obtained by splicing L in such a way that no equal pairs of arcs of the suture are connected on both sides, and gluing $D^2 \times I$ into H along a neighborhood $N(P) \simeq S^1 \times I$ of P on S , so that ‘a hole is filled’. Clearly there are two choices for joining the arcs, which we call *orientations*.

In the case that $|D \cap L| = 2$ there is only one choice for reconnecting the arcs. Part (c) in Figure 6 shows that splicing as seen from inside the hole. This special case is sometimes also called a *product decomposition* (see [16]).

DEFINITION 2.12. We define a sutured manifold (L, H) to be *disk decomposable* as follows.

- (1) Assume H is a ball and L is a (single) loop on ∂H . Then (L, H) is disk decomposable.
- (2) If (L', H') is obtained from (L, H) by a disk decomposition (along some decomposition disk D), and (L', H') is disk decomposable, then so is (L, H) .

In a similar way we can define *product decomposable* by replacing ‘disk decomposition’ by ‘product decomposition’ in the second condition above.

THEOREM 2.10 (Gabai [14, 17]). *If $C(F)$ is disk decomposable, then F is a minimal genus surface for $L = \partial F$. Moreover, F is a fiber surface for L if and only if $C(F)$ is product decomposable (for proper choice of decomposing disks).*

This is a very efficient practical test to detect minimal genus and fibering.

3. *Fibering and minimality for canonical genus 2*

As outlined, the mathematical arguments proving Theorems 1.1 and 1.2 will build on a computation based on the description of canonical genus 2 surfaces (see Theorem 2.9). We used several computer programs that disk decompose the occurring families of diagrams and seek identifications and reductions between the various families. There still remain a small number (about 25) of special cases the programs could not handle (completely), that must be checked by hand.

In the following proofs we will mention some sizes of sets of diagrams. These are provided to give an idea of the extent of calculation and efficiency of the different parts of the argument. The numbers cannot be, however, considered binding, in the sense that they depend on the procedure chosen to a degree of technical detail which is unreasonable to present here.

We remark that in Theorem 1.2 the same statement can be proved for genus 1 links of two components; see Theorem 3.1. Here the amount of computation and case-by-case analysis for canonical surfaces can be reduced significantly. (The check of many generators can be avoided, though the exclusion for two linked \sim -equivalence classes in Lemma 3.11 is no longer valid.) For general fiber surfaces, the opposite situation may again be displayable by adapting Morton's construction, but some details remain to be settled (see Remark 3.2).

One more remark on methodology. Of course, beside Gabai's (geometric) methods, there are copious other (algebraic) criteria for estimating genera and testing fiberedness, involving most prominently the Alexander polynomial Δ , and more modernly its various twisted relatives. All these methods, however, do not fit so well into the setting of 'generators and twisting', which is developed for canonical genus.

While the Alexander polynomial does admit a good description in terms of its skein relation, this shows that it is often unreliable: its obstructions may vanish even for very high numbers of twists. The standard example are the pretzel knots with trivial polynomial. It is only under twisting in restricted types of diagrams that Δ behaves 'nicely' (see [43]). However, in such cases it is indeed useful sometimes, and we will apply it to handle some of the diagram families left over by disk decomposition for Theorem 1.2. (These families are small enough to allow us to conclude that only finitely many monic Alexander polynomials occur infinitely many times, without that we identify exactly which of the knots are fibered.)

The modern non-commutative relatives of Δ appear, though, even less helpful for us. For specific examples they can do much better than their classic ancestor. However, their calculation for single knots is already sometimes not easy, and their behaviour seems completely uncontrollable even under such a simple diagrammatic operation as a t'_2 move. In contrast, disk decomposition can be easily 'stabilized' with respect to twisting, and so it will become one main ingredient of our proofs.

3.1. *Minimal genus*

Every diagram D decomposes (possibly trivially) as a Murasugi sum, and its Murasugi sum factors as a connected sum. Recall (Definition 2.3) that a Murasugi atom [39] is a prime factor of a block of D .

DEFINITION 3.1. We consider the *complexity function* $\kappa(D)$ on a diagram D , which is given by an increasingly lexicographically ordered 4-tuple. Let n_1 be the total number of crossings of all the Murasugi atoms of D which are not reversely oriented $(2, k)$ -torus links (k even). Let n_2 be the number of Murasugi atoms. Let $n_3 = t(D)$ be the number of \sim -equivalence classes, and n_4 the number of trivial \sim -equivalence classes. Then the complexity of a diagram is $\kappa(D) = (n_1, -n_2, n_3, -n_4)$.

Thus, first, diagrams are simpler in complexity when they have a *smaller* (total) crossing number of all the Murasugi atoms, which are not reversely oriented $(2, k)$ -torus links. Where

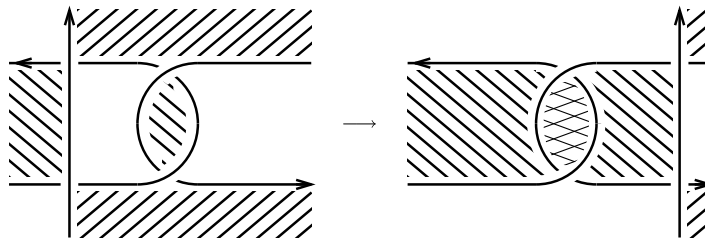


FIGURE 7. A hidden Hopf band. Note that the vertical strand may also pass on the bottom and the clasp may be turned around, or a $\vec{\tau}_2$ twist may be applied at it.

this crossing number is equal, we regard diagrams as simpler if they have *more* Murasugi atoms, and for a fixed number of Murasugi atoms, if they have *fewer* \sim -equivalence classes, or equally many but *more* trivial ones.

The proof of Theorem 1.1, as well as the following proofs, will largely go like this. We generate a finite list of representatives \tilde{D} of the possible occurring diagrams D (using Theorem 2.9), and either show that the canonical surface of D is disk decomposable, or find an isotopy of D to a genus 1 diagram, or a diagram of smaller complexity κ . The difficulty here is that we have to design tests on \tilde{D} so that they extend to all D represented by \tilde{D} .

We will have to examine how κ behaves with respect to various operations. (These operations in fact let to the present design of κ .) We start with two easy but important properties.

LEMMA 3.1. *Let $\kappa(D) < \kappa(D')$ and \tilde{D}, \tilde{D}' be obtained from D , respectively D' , by the same number of $\vec{\tau}_2$ moves. Then $\kappa(\tilde{D}) < \kappa(\tilde{D}')$.* □

LEMMA 3.2. *If D and D' differ by a flype, then $\kappa(D) = \kappa(D')$.* □

Depending on the situation, we will use several further types of move, which reduce complexity:

- a twist tangle move (3.2) (see Lemma 3.3);
- a hidden Hopf band move on Figure 7 (see Lemma 3.4);
- *some particular types* of (crossing number non-augmenting) wave moves (see Lemma 3.5, 3.7 and below it);
- a $(2, -1)$ tangle move (3.7) (see Lemma 3.8);
- a loop move (3.11) (see Lemma 3.12).

Many of these moves will be tested in large amounts computationally, but some will be observed manually in a few instances. We should point out that even the computational tests are not exhaustive, and we may have retained diagrams admitting such moves (which could have been discarded). Ultimately, we must focus on the final goal of dealing with each diagram in one way or the other, rather than obtaining maximal reduction at each stage.

It should be noted also that the definition of κ , the listed moves, and the arguments (below) of how they affect κ do not restrict to D being a knot diagram. It is the basis of our programs on the (knot) DT notation that will compel us to seek knot diagrams D to compare.

Proof of Theorem 1.1. Start with a genus 2 diagram D . The case that D is composite is easy to handle, thus assume D is prime.

We will now do induction on D with respect to the complexity κ . Clearly the minimal genus property is invariant in going over to another diagram with the same genus. So we can assume that D has minimal complexity among all genus 2 diagrams of the knot.

We now cover all occurring D one by one. We will make heavy use of Theorem 2.5, without mentioning it explicitly each time.

Case 1. D is special.

The procedure here can largely be split into three parts, which in some form will also reappear in later verifications.

Part 1. Generating initial representative diagrams not admitting simple reductions

As cautioned by the KT and Conway knot in [14], the geometric methods we use are very sensitive with respect to mutations. Thus we need to work with the series of multiple (alternating) diagrams \hat{D} of a generator. We must consider at least all diagrams obtained by type A flypes (see Figure 4). Still, type B flypes commute with \bar{t}_2 twists. Then we assume without loss of generality that we flype (by type B flypes) the generator diagram so that \sim -equivalent crossings are twist equivalent.

We separate the series $\langle \hat{D} \rangle$ containing D by $-3, -1, 1, 3$ crossings in a trivial \sim -equivalence class of the generator \hat{D} (where $\pm i$ crossings stands for i crossings of writhe ± 1) and $-4, -2, 2, 4$ crossings in a non-trivial \sim -equivalence class. Let \mathcal{D} be the set of such diagrams, which we call \mathring{D} . Then D is obtained by adding \bar{t}_2 twists at a \sim -equivalence class of at least three crossings in some of the diagrams \mathring{D} .

DEFINITION 3.2. In general we will call classes at which we are allowed to apply \bar{t}_2 -moves *twistable*, and the others *untwistable*. Thus our convention here is that a class is twistable if and only if it has at least three crossings. We will stick to this convention most of the time (also in later proofs), and we will explain when we have to depart from it.

Thus there is a sequence of two transformations

$$\hat{D} \longrightarrow \mathring{D} \longrightarrow D. \tag{3.1}$$

Here the first transformation involves change of crossings and application of at most one \bar{t}_2 move per \sim -equivalence class. The second transformation involves applications of some number of \bar{t}_2 moves at \sim -equivalence classes of at least three crossings.

Our attitude is, again, that we will manage only the (finitely many) diagrams \mathring{D} as representatives of all D . By Lemma 3.1 we need to care about the complexity of the diagrams \mathring{D} alone, not of all its twisted versions D .

We put the following *warning* in advance. The structure (3.1) is not invariant under several of the moves listed in the beginning of §3.1. Thus in general a symbol like \mathring{D}' will mean $(\mathring{D})'$, a diagram obtained from \mathring{D} by some move. This does not need to be equal to ${}^\circ(D')$, or to \mathring{E} for any diagram E whatsoever. Some moves alter \sim -equivalence classes on the global level severely. We do not care about this, though; we use \mathring{D}' just to show that all D represented by \mathring{D} can be handled.

We discard (with Lemma 3.2) flype equivalent duplicates \mathring{D} and flype crossings in a \sim -equivalence class of at least three crossings so that the crossings are all twist equivalent. This is a preparation for disk decomposition.

There are two more fragments that we can exclude in diagrams \mathring{D} . We formulate them as separate lemmas.

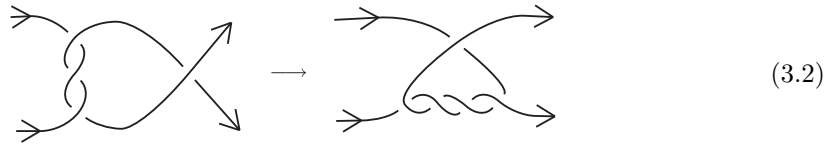
LEMMA 3.3. *If the generator \hat{D} of D has two \approx -equivalent crossings $p \approx q$, then (in the first transformation in (3.1)) in \mathring{D} both crossings p, q must be twisted at; otherwise \mathring{D} can be discarded.*

Proof. Assume that in \mathring{D} , one of p or q is not twisted at. Then we can always flype it to form a parallel clasp, or a $(1, \pm k)$ -tangle with $k > 0$ odd. (If $k = \pm 1$, then it is a clasp.)

If a parallel clasp is trivial (that is, $k = -1$, or in other words both crossings in it have opposite sign), then the surface of \mathring{D} (and hence D) is obviously not of minimal genus.

If the clasp is non-trivial, it can be turned by isotopy (commuting with $\mathring{D} \rightarrow D$) into a $(2, -1)$ -tangle, whose clasp is reverse and gives a Hopf band Murasugi atom. We argue that in the new diagram \mathring{D}' , we have $\kappa(\mathring{D}') < \kappa(\mathring{D})$. Note that the move creates an extra crossing, $c(\mathring{D}') = c(\mathring{D}) + 1$, but because two crossings in \mathring{D}' give a new $(2, \pm 2)$ -torus link Murasugi atom, we have $n_1(\mathring{D}') = n_1(\mathring{D}) - 1$. So we found a simpler diagram \mathring{D}' than \mathring{D} , which we excluded.

A $(1, \pm k)$ -tangle for $k > 1$ odd turns similarly into a $(-1, 1 \pm k)$ -tangle



and a reverse $(2, 1 \pm k)$ -torus link deplumbs with a similar contradiction (to κ -minimality of \mathring{D}). □

The other type of fragment is a ‘hidden Hopf band’, pointed out to me by M. Hirasawa, and shown in Figure 7.

LEMMA 3.4. *If \mathring{D} has a hidden Hopf band, it can be discarded.*

Proof. By the isotopy of Figure 7 we find a diagram \mathring{D}' . We observe that $g(\mathring{D}') = g(\mathring{D})$ and $c(\mathring{D}') = c(\mathring{D})$. However, in \mathring{D}' , one \sim -equivalence class becomes a separate $(2, \pm 2)$ -torus link Murasugi atom. Thus $n_1(\mathring{D}') < n_1(\mathring{D})$. (Note that, in the transformation $\mathring{D} \rightarrow D$ in (3.1), the two non-clasp crossings in the diagrams of Figure 7 are not affected by \bar{t}_2 moves, and thus the isotopy applies in D as well.) □

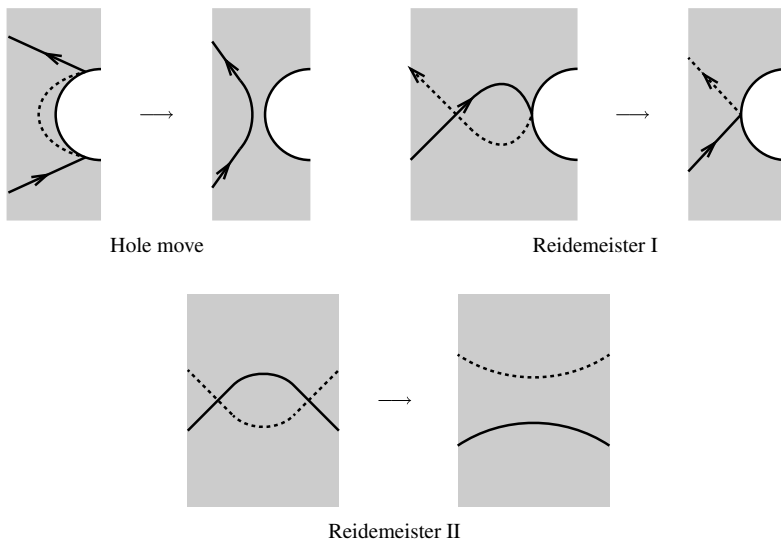
The reduction of the set \mathcal{D} using the previous two lemmas led to a size of approximately 121 000 diagrams \mathring{D} .

Part 2. Stable disk decomposition

The next step is to apply disk decomposition on the diagrams of \mathcal{D} . Note, however, that most of these diagrams are representatives of an infinite family of diagrams, obtained by adding \bar{t}_2 twists at a \sim -equivalence class of at least three crossings. Therefore, we must design the decomposition to become ‘stable’ under these twists.

This is accomplished as follows. We consider the bicolor of the Seifert surface of a special diagram, which is a handlebody of genus $2g$, where g is the genus of the surface (in our case $g = 2$). Through the whole process the position of the handlebody is kept fixed, and we move the suture. We allow except for disk decomposition along some of the $2g + 1$ holes of the handlebody (the infinite region counted) three types of isotopies of the suture on the surface of the handlebody to make it look simpler (see Figure 8).

- Hole move: pulling a loop around a hole. This means a suture segment that goes through a hole X , *does not get into any crossing*, and gets back through the same hole X , *without going around another hole*, to be moved through the hole X , so that its two intersections disappear.
- Reidemeister I move: removing a self-intersection near a hole; suture passes above, goes on the lower side through a hole and passes the same crossing below, without going through any other crossings and around other holes in between.
- Reidemeister II move: removing a pair of self-intersections forming a clasp; one arc on upper side, one on lower side. (There should be no empty hole inside a region bounded by the two arcs.)

FIGURE 8. *Suture moves.*

The diagrams in \mathcal{D} were generated so that all \sim -equivalent crossings, if there are more than two, are twist equivalent. Now if we apply a twist at a group of at least three crossings, then after disk decomposing along one of the neighbored holes, we get from the two new crossings two extra pairs of Reidemeister I moves and a hole move. So the result would be the same as if we did not apply the twisting. In a similar way one sees that a hole move is not affected by twists. For Reidemeister I one must notice that if a twisted crossing is involved in a Reidemeister I, then by hole-moves and Reidemeister I one can reduce all the crossings in the twist, and so Reidemeister I commutes, too, with twisting.

However, the situation is different for Reidemeister II moves. In this case one can undo a certain number of twists, and the behaviour under subsequent disk decompositions can be changed. To remedy this problem we must consider a restricted type of suture isotopy. We put a mark from the beginning on each crossing in $D \in \mathcal{D}$ which belongs to a \sim -equivalence class of three or four crossings, and prohibit the application of Reidemeister II moves in which a marked crossing is involved. (Marked crossings can still be removed using Reidemeister I moves.)

Note that for holes that intersect the suture more than twice, we have two orientations (see Figure 6); also the result of disk decomposition may depend on the order in which holes are dealt with. We used the full capacity of the algorithm and tried all possible orders. If, however, there are at least two holes which do not intersect the suture (empty holes), or the plane projection of the suture becomes split, then all continuations of the procedure will fail, and so we can spare ourselves the work.

From about 121 000 diagrams \mathring{D} in \mathcal{D} , this stable disk decomposition failed only on 778.

Part 3. Rigid wave move reductions

Now we apply a different program that uses wave moves to find reductions (with respect to our complexity function) of such \mathring{D} that commute with the \bar{t}'_2 twists.

In this vein, we take one crossing in a \sim -equivalence class of \mathring{D} of three or four crossings to be rigid. Here ‘rigid’ means that we do not allow any strand at this crossing to be removed by a wave move. Said more clearly, it means that the crossing must not occur as one of the explicitly visible crossings on the left hand sides of the two (types of) moves in Figure 2. We call this a *rigid wave move*.

LEMMA 3.5. *Let \mathring{D} be transformable by a sequence of rigid wave moves into a diagram \mathring{D}' of (a) genus 1, or (b) genus 2 and fewer crossings than \mathring{D} , or (c) a non-special genus 2 diagram \mathring{D}' of the same number of crossings as \mathring{D} . Then \mathring{D} can be discarded.*

Proof. Rigid wave moves were designed to commute with the \bar{t}'_2 twists at rigid crossings (which accomplishes $\mathring{D} \rightarrow D$ in (3.1)). Thus case (a) is obvious, and it is enough to see that, in cases (b) and (c), we have $\kappa(\mathring{D}') < \kappa(\mathring{D})$.

Since \mathring{D} is special, and obviously not a reverse $(2, k)$ -torus link diagram, we have $n_1(\mathring{D}) = c(\mathring{D})$. Moreover, $n_1(\mathring{D}') \leq c(\mathring{D}')$. Thus if $c(\mathring{D}') < c(\mathring{D})$, we have $n_1(\mathring{D}') < n_1(\mathring{D})$, and case (b) is clear. For case (c) just note that $n_1(\mathring{D}') \leq n_1(\mathring{D})$, but $n_2(\mathring{D}') > 1 = n_2(\mathring{D})$. \square

Then our program applies wave moves that keep the rigid crossings, seeking a reduction to a diagram of the sort of Lemma 3.5.

If the program did not find reductions, then it was designed to report the diagram to which it could minimize the input diagram; this allows us to identify input series as equivalent, and to discard duplicate copies of equivalent series, even if the series is not reducible. (Note that in these diagrams we must still keep the information regarding which crossings are rigid.)

It turned out, though, that in fact the program reduced all 778 diagrams, and so we are done.

Case 2. D is not special.

In this case the argument depends somewhat on the specifics of the corresponding generators. One general helpful tool is as follows.

LEMMA 3.6. (a) *Assume the (p, q, r) -pretzel surface, for p, q, r odd, is not of minimal genus. Then some of p, q, r are 1 and -1 .*

(b) *If the (p, q, r) -pretzel surface, for p, q, r even, is not of minimal genus, then two of p, q, r are 0.*

Proof. (a) The (p, q, r) -pretzel must be an unknot, and this occurs only if some of p, q, r are 1 and -1 .

(b) We must have $\chi \geq 1$, so that a minimal genus surface is disconnected. Using linking numbers, one easily sees that this can (and then indeed does) occur only if two of p, q, r are 0. \square

Some cases are easy to handle. If a reverse $(2, k)$ -torus link or $(2, 2, 2)$ -pretzel link occurs as a Murasugi sum factor of \mathring{D} , it must be non-trivial in D . Therefore, we need to consider only the cases that some other Murasugi sum factor/Murasugi atom is not of minimal genus.

A few of the generators are dealt with first.

Case 2.1. 8_{12} : These are 2-bridge knots, and can be immediately discarded.

Case 2.2. 6_2 : One Murasugi sum factor is a (p, q, r, s) pretzel, p, \dots, s odd; the other factor is a Hopf link (which is non-trivial by the assumption that the diagram is in the 6_2 series). If some of p, \dots, s are ± 1 but none is ∓ 1 , then we can deplumb the pretzel surface into Hopf bands, so if this is not the case, then either (a) $1, -1 \in p, \dots, s$, or (b) $|p|, \dots, |s| \geq 3$. In case (a), we have a reduction to a genus 1 diagram. If (b) occurs, then the surface is a minimal genus surface. One way of seeing this is to assume the contrary. Then the link bounds an annulus. So it is either a reverse $(2, k)$ -torus link or its satellite. The former is excluded by the classification of Montesinos links, and the latter by the description of their incompressible surfaces (in particular tori; see [35]).

Case 2.3. 6_3 : This is the Murasugi sum of two trefoils. Under twist they give two (p, q, r) -pretzel surfaces, with p, q, r odd. If D is not genus-minimizing, for one of them some of p, q, r are 1 and -1 . Then one exhibits the reduction to a genus 1 diagram directly.

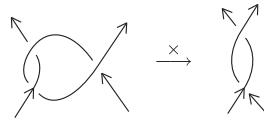
Case 2.4. 7_6 : This is the Murasugi sum of a trefoil and two Hopf bands. If the trefoil has a trivial clasp, one exhibits the reduction to a genus 1 diagram directly.

Case 2.5. 9_{41} : The same argument as for 7_6 applies.

Case 2.6. 12_{1202} has two $(2, 2, 2)$ -pretzel links (as Murasugi sum factors), so it is done.

Case 2.7. All other non-special generators except 9_{39} have the following feature: each of their diagrams \hat{D} turns into a special one \hat{D}'' by operations we call an \times -move. We will need to analyze the move at several places, so we formalize its definition.

DEFINITION 3.3. The \times -move move on \hat{D} consists of two steps. First, we deplumb a Hopf band in a rational tangle $T = (2, 1)$, in which the clasp is reverse. Then, after deplumbing the Hopf band, we can do a clasping (2.2) at the remaining third (non-clasp) crossing of T , as shown below.



The \times -move extends to a move between twisted diagrams $\times : \hat{D} \mapsto \hat{D}''$ and $D \mapsto D''$, where a tangle $T = (k, 1)$ is modified ($k \neq 0$ even). We call the class of $|k|$ crossings \times -class. The \bar{t}_2 -twisting in the \times -class of D (and \hat{D}) is lost in D'' (\hat{D}'').

Then, by applying an \times -move on \hat{D} , we obtain a new generator knot diagram \hat{D}'' of genus 2, which becomes special. Also, genus minimality of D is equivalent to that of D'' (regardless of the lost twisting in the \times -class). For $\langle 7_7 \rangle$ and $\langle 10_{58} \rangle$ we need to apply two \times -moves to obtain a special D'' .

With this move we can recur the check of these non-special generators \hat{D} to the special ones \hat{D}'' . For example, if $D \in \langle 10_{58} \rangle$ (with the notation in Definition 2.11), then $D'' \in \langle 8_{15} \rangle$. Also D'' may land in different series depending on the size of \sim -classes in D : if $D \in \langle 7_7 \rangle$, then $D'' \in \langle 5_1 \rangle$ or $D'' \in \langle 7_5 \rangle$ are both possible.

We stress that the reason we use D'' is technical: our (stable) disk decomposition program was designed only for special diagrams. Note also that the map $\hat{D} \mapsto \hat{D}''$ and $D \mapsto D''$ is not injective.

Furthermore, $\kappa(D'') > \kappa(D)$. Thus \times -moves provide no recurrence to Case 1 as it is. Essentially, the special generators check must be redone, but the following modifications are necessary.

Part 1

We again separate the series containing D'' by distinguishing diagrams \hat{D}'' with $-3, -1, 1, 3$ crossings in a trivial \sim -equivalence class of the generator ($-i$ crossings are i crossings of writhe -1) and $-4, -2, 2, 4$ crossings in a non-trivial one. However, since the plumbing/clasping may alter a fragment of a hidden Hopf band, we must also consider (as input to disk decomposition) diagrams \hat{D}'' with a hidden Hopf band. The same problem occurs with tangles $(1, \pm k)$. Hence, diagrams \hat{D}'' having these tangles cannot be discarded. Contrarily, we must now discard all diagrams \hat{D}'' that have no parallel clasp.

After sorting out duplicates by (type B) flypes, we have a list \mathcal{E} of about 6000 diagrams \hat{D}'' .

Clarification: the diagrams we have rendered here are thus, *a priori*, $\circ(D'')$. It was cautioned that this is not the same as $(\dot{D})''$. However, \times -moves do not augment sizes of \sim -equivalence classes. (This observation is important and will reappear below.) Thus the set of all $(\dot{D})''$ is a subset of the set of all $\circ(D'')$. Still, our understanding will be to maintain a list of diagrams originally generated as $\circ(D'')$, and we will use $\dot{D}'' = \circ(D'')$ in this meaning below. This way, at each stage, our list contains at least all necessary (although possibly also some extraneous) diagrams.

Part 2

Disk decomposition fails for 249 of the diagrams \dot{D}'' .

Part 3

We subject them to a similar wave move reduction. This part requires the most delicate modifications.

The reduction of $\dot{D}'' \rightarrow \dot{D}_1''$ using (rigid) wave moves must keep (along with a crossing in each \sim -equivalence class of at least three crossings) crossings in parallel clasps rigid, and this is enough, since the restoring of the non-special diagram \dot{D} involves a local move at these clasps. Thus $\dot{D}'' \rightarrow \dot{D}_1''$ gives rise to a (rigid) wave move (sequence) $\dot{D} \rightarrow \dot{D}_1$, and we have the following commutative diagram.

$$\begin{array}{ccc}
 \dot{D} & \xrightarrow{\times} & \dot{D}'' \\
 \downarrow \text{wave} & & \downarrow \text{wave} \\
 \dot{D}_1 & \xrightarrow{\times} & \dot{D}_1''
 \end{array} \tag{3.3}$$

Here we continue our cautionary tale, this time with regard to reading the above diagram. There is an ambiguity of the transformation $\dot{D} \rightarrow \dot{D}''$. It is, as a map, neither injective, nor, in fact, well defined: there are diagrams D with the same \dot{D} giving D'' with different \dot{D}'' . This happens when a twistable class becomes untwistable under an \times -move; we will discuss this situation below (see the use of Lemma 3.11 in Case 2.6 in the proof of Theorem 1.2).

This ambiguity means that (3.3) should be read thus. We start by fixing a pair (\dot{D}, \dot{D}'') , so that (at least) *some part* of the diagrams D represented by \dot{D} turn under \times -moves into diagrams D'' represented by \dot{D}'' . The wave move on the right will be chosen so as to make its left sibling discredit the κ -minimality of this *same part* of diagrams D represented by \dot{D} . Thus for some \dot{D} we will deal with its diagrams in pieces (by discarding various \dot{D}'' in various ways).

As outlined, we want to choose the wave move reduction $\dot{D}'' \rightarrow \dot{D}_1''$ so that $\kappa(\dot{D}_1) < \kappa(\dot{D})$. This is not the same and not ascertained by $\kappa(\dot{D}_1'') < \kappa(\dot{D}'')$, as we will see. However, Lemma 3.5 can still be made to work.

LEMMA 3.7. Assume $\dot{D}'' \rightarrow \dot{D}_1''$ so that \dot{D}_1'' conforms, instead of \dot{D}_1 , to one of the three cases of Lemma 3.5, where \dot{D} is replaced by \dot{D}'' . Then $\kappa(\dot{D}_1) < \kappa(\dot{D})$ (for whatever \dot{D} the diagram \dot{D}'' corresponds to), and consequently \dot{D}'' can be discarded.

Proof. Since $g(\dot{D}'') = g(\dot{D})$, case (a) (from Lemma 3.5) is obvious.

For case (b) note that when \dot{D}'' is obtained from \dot{D} by one \times -move, then

$$n_1(\dot{D}'') - n_1(\dot{D}) = n_1(\dot{D}_1'') - n_1(\dot{D}_1) \in \{1, 3\} \tag{3.4}$$

is one less than the size of the \times -class in \dot{D} and \dot{D}_1 . Also,

$$n_2(\dot{D}'') - n_2(\dot{D}) = n_2(\dot{D}_1'') - n_2(\dot{D}_1) = -1. \tag{3.5}$$

The differences are 2, 4 or 6 in (3.4), respectively -2 in (3.5), when two \times -moves are applied, for $\langle 7_7 \rangle$ and $\langle 10_{58} \rangle$. Important in (3.4) and (3.5), though, are the equalities on the left, and that these remain when all ring accents are removed.

Then the argument in the proof of Lemma 3.5 can be modified. □

Thus if our program finds a reduction of \mathring{D}'' with regard to genus or number of crossings, or to a non-special diagram \mathring{D}_1'' of the same number of crossings, then \mathring{D}'' could again be discarded.

Note that, so far, we have not made use of the inclusion of $n_3(D) = t(D)$ into the order $\kappa(D)$ (recall Definitions 2.9 and 3.1). This is now exploited here, for one further partial disposal of diagrams \mathring{D}'' using rigid wave moves.

Assume a sequence of such moves turns \mathring{D}'' into a special genus 2 diagram \mathring{D}_1'' with the same number of crossings. Then $n_1(\mathring{D}) = n_1(\mathring{D}_1)$ and $n_2(\mathring{D}) = n_2(\mathring{D}_1)$. Our program tries to recognize a reduction of $\mathring{D}'' \rightarrow \mathring{D}_1''$ with respect to the number of \sim -equivalence classes. As cautioned, though, the resulting move from $\mathring{D} \rightarrow \mathring{D}_1$ may no longer be a reduction, that is, $t(\mathring{D}_1) < t(\mathring{D})$ may not hold. To account for this, we observe that

$$t(\mathring{D}_1'') \geq t(\mathring{D}_1) \quad \text{and} \quad t(\mathring{D}'') \leq t(\mathring{D}) + q(\mathring{D}'')$$

where $q(\mathring{D}'')$ is the number of \sim -equivalence classes of \mathring{D}'' which are linked with the two crossings in one parallel clasp, and no other crossing. Then it is legitimate to (and what we did is) test

$$t(\mathring{D}_1'') < t(\mathring{D}'') - q(\mathring{D}''). \tag{3.6}$$

Thus if a wave move $\mathring{D}'' \rightarrow \mathring{D}_1''$ satisfies (3.6), we can ascertain that $\mathring{D} \rightarrow \mathring{D}_1$ is a κ -reduction, and \mathring{D}'' can also be discarded.

The twelve diagrams \mathring{D}'' that are left over all have a parallel clasp included in a (non-alternating) $(2, -1)$ rational tangle.

Then we can argue with the following lemma, which we will also apply later.

LEMMA 3.8. *If \mathring{D}'' has no parallel clasp, or has a parallel clasp involved in a $(2, -1)$ tangle, then any (corresponding) \mathring{D} is not complexity minimal, and we can discard \mathring{D}'' .*

Proof. Clearly \mathring{D}'' must have a parallel clasp by construction. Assume it has a $(2, -1)$ tangle. If the parallel clasp in that tangle is not changed when recovering \mathring{D} , then the $(2, -1)$ tangle changes into a reverse clasp by a $(2, -1)$ tangle move as shown below.



This move reduces n_1 (when \mathring{D}'' is special), and \mathring{D} (and D) would admit the same κ -reduction. Otherwise, \mathring{D} will have a trivial reverse clasp, and by construction, we need not consider such \mathring{D} . □

Case 2.8. There remains the final case $D \in \langle 9_{39} \rangle$. Here also one Hopf band deplumbs, but it is not in a $(2, 1)$ -tangle. Consequently, we cannot use wave moves to reduce the diagrams. However, by claspings (2.2) the remaining special 2-component link diagram D' , we get a diagram D'' in the series of 8_{15} .

Let us analyze the minimal genus property of D' . It is obtained from the $(2, 2, 2)$ -pretzel by one claspings, with intermediate result we call D_1 , and then \bar{t}_2' twists. Let a, b be the reverse clasps of D_1 (those that remain from the $(2, 2, 2)$ -pretzel) and c the parallel clasp (obtained

after clasping). The same notation holds in D' , though c may have been turned into a (k, l) -pretzel tangle with k, l odd. Note that c is not a clasp in D , since it is created by the Hopf deplumbing.

Since a, b are not trivial (that is, have non-zero number of crossings) in D , they are not trivial in D_1 ; however, c may be trivial in D_1 and D' . If c is of the form $(\pm 1, k)$, $k \neq \mp 1$, in D' , then we can deplumb a reverse $(2, k \pm 1)$ -torus link and see that D' is of minimal genus. So if D is not of minimal genus (equivalently, D' is not of minimal genus), then c in D' is either trivial, that is, $c = (\pm 1, \mp 1)$, or both crossings in c in D_1 must be $\frac{1}{2}$ twisted at when recovering D' , that is, c is in D' a pretzel tangle (k, l) with $|k|, |l| \geq 3$.

If c is trivial in D' , we easily find an isotopy of D to a pretzel diagram (p, q, r) (with p, q, r odd), of genus 1. It thus remains to check the case when both crossings of c in D_1 are twisted at. This gives (under clasping at one of the crossings in a or b) diagrams D'' of the series $\langle 8_{15} \rangle$ with exactly one parallel clasp, which is non-trivial and lives in an alternating $(2, 1)$ -tangle. Since both crossings of c are twisted at, the diagrams D'' are not Montesinos.

These remaining possibilities were ruled out by going over again to \check{D}'' and direct inspection of the 249 diagrams \check{D}'' in Case 2.7, on which disk decomposition failed, and which remained before the use of wave move reductions. This verification showed that here no diagrams are relevant: there are 24 diagrams in the series $\langle 8_{15} \rangle$, and all have a parallel clasp in a $(2, -1)$ tangle, making Lemma 3.8 applicable. Thus we are also done here, and therefore with the proof of Theorem 1.1. □

Proof of Corollary 1.1. Since canonical Seifert surfaces are free (their complement is a handlebody), Theorem 1.1 shows that if $\tilde{g}(K) \leq 2$, then K has a minimal genus free surface S . However, it is well known that for wrapping number zero satellites K , every minimal genus Seifert surface S can be moved into the companion (solid) torus T : the part of S outside T must be some collection of disks and annuli, which can be successively eliminated by isotopy. Thus S is not free. □

3.2. Alexander polynomial of canonical fiber surfaces

Proof of Theorem 1.2. We examine now how D must look such that its canonical surface (of genus 2) is a fiber surface.

Again the fiber property is invariant under going over to another diagram of the same genus (by Neuwirth–Stallings’ theorem). We thus use induction with respect to the same complexity as before (see Definition 3.1).

The central criterion to exclude fibering we use is again based on disk decomposition.

LEMMA 3.9 (see Gabai, [17, Lemma 2.6]). *Assume that H is a bicolar of a Seifert surface S and D a circle on the boundary of H (bounding a disk in $S^3 \setminus H$) such that $|D \cap S| \geq 4$. Decompose H along D with both orientations to get bicolars K, L . If one can reduce both K, L by disk decomposition to a ball with a circle, then the surface S in H is not a fiber surface. Also, if one can reduce a bicolar H' to H by disk decomposition, then the Seifert surface in H' is not a fiber surface either.*

Apart from this we will also use Theorem 2.7, saying that (in our case) the Conway polynomial coefficient $v_4 = [\nabla]_4$ is $v_4(D) = \pm 1$ for a fiber surface.

The following is a test (again geometrically proved), which is useful in some cases.

LEMMA 3.10 (see Gabai [17, Theorem 6.7, case I]). *There are only finitely many pretzel surfaces (p, q, r) , (p, q, r, s) , all p, q, r, s odd (with twist classes of p, \dots, s reverse) which are fibers. The only infinite sequence of pretzel surfaces (p, q, r) with p, q, r even is for (up to permutation) $p = 2, q = -2, r$ even.*

The disk decomposition we apply now is a variant of the one for the proof of Theorem 1.1. The main difference is that we need to succeed twice, instead of once, with the proper orientations at the holes, as described in Lemma 3.9.

Again we need to separate the series $\langle \hat{D} \rangle$ containing D by distinguishing $-3, -1, 1, 3$ crossings in a trivial and $-4, -2, 2, 4$ crossings in a non-trivial \sim -equivalence class, and consider repeated twisting at a \sim -equivalence class of three or more crossings. Discarding duplicates we gain a set \mathcal{D} of diagrams to examine.

Note, however, the following: a \sim -equivalence class of n crossings produces in general not more than $n - 1$ intersections with the disks in the neighboring holes. Since for a fibering check it is relevant whether the intersections of the suture with the loop are two or more, for the fiberedness test it is invariant under \bar{t}_2 -twists only from four crossings on, but possibly not from three. Thus it may occur that the disk decomposition as in the proof of Theorem 1.1 fails to show non-fibering, but might be able to do so after we add a twist at some class(es) of three crossings. This is, in a way, good news, because we gain a further non-fibering test. We call it the *extended non-fibering test*.

Now we also use the Alexander polynomial. The following are two simple cases where it can do some work.

LEMMA 3.11. *Twisting in one \sim -equivalence class (group) gives either one Alexander polynomial infinitely many times, or infinitely many Alexander polynomials one time. Twisting at two linked \sim -equivalence classes gives at most one Alexander polynomial infinitely many times.*

Proof. When we twist at one group, the Alexander polynomials form an arithmetic progression. It is either constant, or not. Thus the claim is obvious.

Now consider two linked twist groups, parametrized by integers a, b . To see that all polynomials but one are realized finitely many times, it is both necessary and sufficient to look at v_2 . According to the explanation at the end of § 2.4, the invariant has an expression in the form $v_2 = C_1 + C_2a + C_3b + C_4ab$, where C_i are constants. In case the \sim -equivalence classes we twist at are linked, it can be concluded that the constant C_4 is non-zero (see for example the remarks on v_2 in [22, § 12] or its Gauß sum formula in [37]). It is easy to see that such a form realizes any integer only finitely many times, except at most one. The a, b correspond to factorizations of $v_2/C_4 + C$ for some constant C , so if this expression is non-zero, only finitely many (a, b) can occur. \square

In our context this lemma says that we can discard all $\hat{D} \in \mathcal{D}$ with at most one, or two linked, \sim -equivalence classes of three or four crossings.

Case 1. D is special.

We use again disk decomposition with Gabai's non-fibering criterion Lemma 3.9, so that this procedure is stable under twists at \sim -equivalence classes of three or four crossings. We stick to the accent nomenclature in (3.1).

About 3000 diagrams \hat{D} failed to be shown non-fibered. Wave move reduction tests reduced this list to 27, and application of Lemma 3.11 to 23.

These were processed by an extended non-fibering test as follows: we twist zero or one times at each \sim -equivalence class of (exactly) three crossings. Then the resulting diagrams were processed by a version of disk decomposition in which we mark crossings in groups of at least four crossings. Additionally we apply Lemma 3.11 and discard all input diagrams in which only one \sim -equivalence class has at least four crossings, or there are (exactly) two such classes, and they are linked. (The understanding here is that we twist only at such classes.)

This procedure reduced the list of diagrams to 17. These diagrams are given below in DT notation (see § 2.2). For technical reasons, we will fix the identifier (the second integer) in DT notation displayed throughout this section to indicate the subscript/index of the underlying generator, as numbered in Figure 5 (for example, 120 if $\tilde{D} \in \langle 10_{120} \rangle$).

- 14 120 6 -14 24 28 18 16 -4 -22 10 -26 -12 2 -20 8
- 16 120 6 -14 28 32 20 18 -4 -26 -24 10 -30 -12 -16 2 -22 8
- 16 120 6 28 -16 32 -18 -4 26 24 -10 -8 -30 2 14 12 -22 -20
- 20 1097 8 -16 -28 36 -22 -20 30 -4 -34 -32 -10 -40 -38 -6 14 12 -18 2 -26 -24
- 17 329 8 -16 26 18 34 24 -28 -4 -2 10 -32 -30 6 -14 -12 -22 -20
- 22 1097 8 -22 -20 38 -28 -26 32 -42 -40 -6 -4 -36 -34 -10 -44 14 12 -24 2 -18 -16 -30
- 16 1097 8 14 -26 16 -20 28 4 2 24 -10 -32 -30 -6 12 -18 -22
- 20 1097 8 16 -32 -18 26 -36 -34 4 2 -30 -28 -38 10 40 -20 -6 -14 -12 -24 -22
- 19 329 8 18 -28 -20 38 36 -26 30 4 2 -12 -34 -32 -6 16 14 -24 -22 10
- 19 329 8 18 -28 -20 38 36 -26 30 4 2 -12 34 32 -6 16 14 24 22 10
- 19 329 8 18 -30 -20 38 22 -28 32 4 2 12 10 36 34 -6 16 14 26 24
- 21 329 8 20 -32 -22 42 40 24 -30 34 4 2 14 12 38 36 -6 18 16 28 26 10
- 18 120 8 30 -16 -14 36 22 2 -6 -28 -26 12 -34 -32 -18 4 20 -24 10
- 18 120 8 30 -16 -32 36 22 20 -6 28 26 12 -34 -14 18 4 2 -24 10
- 21 4233 8 30 -16 -32 42 40 22 -6 28 36 34 12 -38 18 4 2 -14 20 -26 -24 10
- 21 4233 8 32 -14 -34 42 22 -6 30 28 38 36 10 -40 18 16 4 2 -12 20 -26 -24
- 23 4233 8 34 -16 -36 46 44 24 -6 32 30 40 38 12 -42 20 18 4 2 -14 22 -28 -26 10

On these diagrams, disk decomposition still rules out \tilde{t}_2 twists at some groups of three or four crossings. For the remaining groups, we use the Alexander polynomial. Let $a, b, c, d \geq 0$ be variables that indicate the number of \tilde{t}_2 twists to apply at these remaining groups of three or four crossings. (In all diagrams we have at most four such groups.) Then both v_2 and v_4 depend polynomially on a, b, c, d . Moreover, these polynomials are square-free and of (total) degree at most 2, respectively 4 (cf. the end of § 2.4). The polynomials can be easily determined by applying zero or one twists at each group of three or four crossings and calculating the Conway polynomials of the resulting diagrams.

It is then a matter of simple algebra, and sometimes short case-by-case study, to restrict the possible values of a, b, c, d . We explain only a few cases to show how this works. Following Definition 2.9, let $\{p\}$ stand for the group to which the p -th crossing (in the given DT notation) belongs. We will abbreviate by $d \sim \{p\}$ that d is the number of twists to apply at the group $\{p\}$.

As a first example, consider the 15th diagram. Disk decomposition restricts us to further treat only diagrams where we have no \tilde{t}_2 twists at $\{13\}$, but do have at least one twist at $\{10\}$. With $d \sim \{10\}$, we then find that $(d + 1) \mid v_4$, so $v_4 = \pm 1$ never occurs for $d > 0$.

Sometimes it is easier if we look at v_2 . In the 6th diagram, disk decomposition rules out twists at $\{7\}$, and we have

$$v_2 = a + 2b + 4c + ca + cb + d + ab + dc$$

with $a \sim \{2\}$, $b \sim \{8\}$, $c \sim \{5\}$, $d \sim \{12\}$. Since v_2 is thus bounded below by any of the four variables, we see that $v_2 \rightarrow \infty$, so each value of v_2 is taken only finitely many times.

Consider the 14th diagram. With $a \sim \{9\}$, $b \sim \{6\}$, $c \sim \{2\}$, and $d \sim \{1\}$, we find

$$v_4 = -ac - bd + abcd \quad \text{and} \quad v_2 = -2 + ab + cd.$$

If one of a or c is zero, then $v_4 = -bd = \pm 1$ implies $b = d = 1$, and only one variable is free. Then by Lemma 3.11 we are done. Similarly one argues if one of b and d is zero. If all a, b, c, d are non-zero, then $v_2 \geq x - 2$ for each x among a, b, c, d . This implies again that $v_2 \rightarrow \infty$.

With this type of argument we can deal with most of the families. (We defer one more such calculation in cases to the part of the proof for non-special diagrams.)

Something more interesting happens in the 9th and 13th diagrams. In this case, we easily restrict the variables (for proper naming) to $b = c = 1$, but under this condition, we

have $v_4 = -1$ independently of a and d , and v_2 is of the form $-5 + a - d$. Then we see that indeed we can realize any monic Alexander polynomial with leading coefficient -1 infinitely many times by twisting at a and d . Here again disk decomposition was necessary. We perform one \bar{t}'_2 twist at the groups of b and c , and apply a non-fibered check, this time keeping marked the crossings in a group of three or four (but not more than four). Again, this decomposition was successful and excluded the fibering.

Case 2. D is non-special.

Case 2.1. $6_2, 6_3, 7_6, 7_7$: Murasugi desum into Hopf bands and trefoils, so only finitely many fiber surfaces occur in their series.

Case 2.2. 8_{12} : This yields the 2-bridge knots; they are immediately ruled out.

Case 2.3. 9_{41} : Murasugi desums into a $(2, 2, 2)$ -pretzel link and trefoil. So there is only one unlimited twist group, and we are done with Lemma 3.11.

Case 2.4. 12_{1202} : Murasugi desums into two $(2, 2, 2)$ -pretzel links, so we may have two twist groups. (Note that in this case, since for $(-2, 2, n)$ -pretzel $\max \text{cf } \Delta = -1$, all such fibers have $\max \text{cf } \Delta = 1$, by multiplicativity of $\max \text{cf } \Delta$ under Murasugi sum; see Theorem 2.6.) To see that only finitely many polynomials are realized infinitely many times, in case the \sim -equivalence classes we twist at are linked, we have Lemma 3.11. If the twist groups are unlinked, then one sees that here v_2 (or equivalently Δ) remains the same under twists at both groups.

Case 2.5. 9_{39} : We repeat the study of 9_{39} in the proof of Theorem 1.1. We use the same notation as there. It remains to see from what D_1 do canonical fibers in D come. The non-minimal genus cases are out, and if the clasp c becomes $(\pm 1, k)$ in D' , with $k \neq \mp 1$, then we can plumb a $(2, k \pm 1)$ -torus link down to a $(2, 2, 2)$ -pretzel family. Then necessarily for fiberedness $k = \pm 3$, and we have by Lemma 3.10 infinite degree of twist only in at most one \sim -equivalence class. Thus we are done by Lemma 3.11. If c is trivial, we saw that D reduces to a genus 1 diagram. Now it remains again to check the case that both crossings of c in D_1 are twisted at. This gives (under clasping at one of the crossings in a or b) diagrams D'' of the 8_{15} series with exactly one parallel clasp, which is non-trivial and lives in an alternating $(2, 1)$ -tangle, and D'' is a non-Montesinos diagram.

For such D'' , we check again our previous calculation with \mathring{D}'' (which, we recall, is used in the sense of $\mathring{D}'' \rightarrow D''$). Note that $\mathring{D}'' \rightarrow D''$, or twisting at non-trivial classes, does not change the Montesinos property of a diagram. Among the 875 diagrams in \mathcal{E} (from Part 1 in Case 2.7 of the proof of Theorem 1.1) for which the non-fibered check fails, there are 103 diagrams \mathring{D}'' in the 8_{15} series. However, all are either Montesinos, or have a parallel clasp in a $(2, -1)$ -tangle, so we are done here.

Case 2.6. All other generators: again we need to examine the diagrams \mathring{D}'' in \mathcal{E} for which the non-fibered check fails.

Note the following important fact: the \times -class of the \times -moves that create D'' out of D cannot be twisted at in the case of fiber surfaces of D , because of (Theorem 2.5 and) the fact that the surface of the reverse $(2, k)$ -torus link is a fiber surface only for $k = \pm 2$.

Disk decomposition and wave moves. We check reduction with wave moves, which keep parallel clasps rigid. We also try to reduce with respect to the number of \sim -equivalence classes for special diagrams of the same number of crossings.

Then 166 diagrams \mathring{D}'' remain. A disposal of diagrams with a parallel clasp near the first crossing (in their DT notation) in a $(2, -1)$ tangle gives 59 diagrams (recall Lemma 3.8).

Applying Lemma 3.11. Next we want to use Lemma 3.11. The upshot is that this lemma reduces the list of 59 to 22. However, there are some caveats in applying the lemma. We must again clarify twistable classes (see Definition 3.2), most importantly before the \times -moves, since it is the (Alexander) polynomial of D we are interested in, not the one of D'' . Again, one method we pursue here is to keep, in D and D'' , the convention to regard a class as twistable if and only if it has three or more crossings. (See Remark 3.1 below for an indication of an alternative.) This leads to the old specification of \mathring{D} and \mathring{D}'' , and the problem discussed below (3.3).

To investigate in which way one can use Lemma 3.11 here, we remark first, by the observation at the beginning of the case, that an \times -move now (for fiber surfaces) does not erase a twistable \times -class; neither does it (in the general case) create any new twistable \sim -equivalence class.

Moreover, an \times -move does not alter the linking status of \sim -equivalence classes (outside the spot it acts). It may alter, though, the size of one such class. There is thus one situation in which the assumption of Lemma 3.11 is changed between \mathring{D} and \mathring{D}'' . This occurs when \mathring{D}'' has a \sim -equivalence class of exactly two elements, and these crossings are linked with a parallel clasp. (In other words, up to flypes, there is then a $(2, 2)$ -rational tangle with a parallel clasp in \mathring{D}'' .) We checked that in the list of 59 there are three such diagrams \mathring{D}'' , as shown below.

$$\begin{array}{r}
 7 \ 5 \ 4 \ 10 \ -12 \ -14 \ 2 \ 8 \ -6 \\
 9 \ 5 \ 4 \ 12 \ -14 \ 16 \ -18 \ 2 \ 10 \ 8 \ 6 \\
 11 \ 23 \ 4 \ 12 \ 16 \ 14 \ 20 \ 2 \ 10 \ 6 \ -22 \ 8 \ -18
 \end{array} \tag{3.8}$$

It is easy to see that all diagrams D'' obtained by twisting at non-trivial classes can be simplified by undoing claspings to (p, q, r) -pretzel diagrams with (p, q, r) odd. Then Lemma 3.10 finishes these three \mathring{D}'' . (With such \mathring{D}'' , there are only finitely many D'' , and hence D , which could be canonically fibered.)

REMARK 3.1. There are ways of getting around (3.8), by making the definition of twistable classes in D more flexible (than just by distinguishing between at least three and less than three elements), and depending on D'' . But the exact description of this would be similarly technical. In any case, Lemma 3.11 requires some thought here.

Extended non-fiber test. This means that we can continue working with the list of the aforementioned 22 diagrams. Finally we applied the extended non-fiber test, and Lemma 3.11, where we regard classes in \mathring{D}'' (but not \mathring{D}) as twistable if they have at least four crossings. This restriction disposes us of pathologies such as (3.8). In the end, the list of 22 is reduced to eight.

$$\begin{array}{r}
 15 \ 38 \ 10 \ -18 \ 26 \ -22 \ -20 \ 30 \ -24 \ -28 \ -4 \ -2 \ -8 \ -12 \ 6 \ -16 \ -14 \\
 15 \ 38 \ 10 \ -18 \ 26 \ 22 \ 20 \ 30 \ -24 \ -28 \ -4 \ -2 \ 8 \ -12 \ 6 \ -16 \ -14 \\
 12 \ 15 \ 4 \ 10 \ -16 \ -14 \ 2 \ 20 \ -8 \ -6 \ -24 \ -22 \ 12 \ -18 \\
 12 \ 15 \ 4 \ 10 \ 24 \ -14 \ 2 \ -18 \ -8 \ 22 \ 20 \ -12 \ 16 \ 6 \\
 18 \ 101 \ 4 \ 12 \ -20 \ -30 \ -28 \ 2 \ -26 \ -24 \ 32 \ -6 \ -36 \ -34 \ -14 \ -10 \ -8 \ 18 \ 16 \ -22 \\
 18 \ 101 \ 14 \ -24 \ -22 \ -34 \ 20 \ -28 \ -26 \ 36 \ -30 \ 10 \ 8 \ -4 \ -2 \ -12 \ -18 \ -16 \ -6 \ -32 \\
 18 \ 101 \ 4 \ 12 \ 36 \ 20 \ -28 \ 2 \ 26 \ 24 \ 30 \ 8 \ -34 \ -32 \ 14 \ -10 \ 18 \ 16 \ -22 \ 6 \\
 15 \ 38 \ 8 \ -16 \ 22 \ 20 \ 30 \ 28 \ 24 \ -4 \ -26 \ -2 \ 6 \ 14 \ 12 \ -18 \ 10
 \end{array} \tag{3.9}$$

Case-by-case Vassiliev invariant tests. We number the diagrams in (3.9) by 1 to 8 consecutively and analyze this list below in four cases.

Case 2.6.1. Diagrams number 3 and 4 have a parallel clasp in a $(2, -1)$ tangle, so we can discard them.

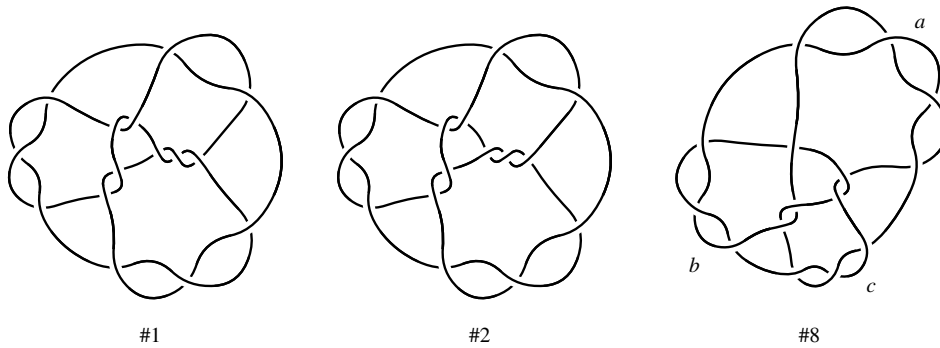


FIGURE 9. Three of the remaining diagrams whose Alexander polynomials must be investigated.

Case 2.6.2. Consider the first two diagrams (see Figure 9). All three twist groups are linked; the leading coefficient v_4 of z^4 in the Conway polynomial is of the form

$$v_4 = C_1abc + C_2ab + C_3ac + C_4bc + \text{linear terms}, \quad (3.10)$$

where (a, b, c) are the number of crossings[†] in the twisted classes, and $C_1 \neq 0$. (The constant C_1 can be obtained by smoothing out a crossing each in the classes of a, b, c and taking the leading coefficient of the Alexander polynomial of the link, which in \hat{D}'' is a Hopf link, and so in D remains a Hopf link, possibly of opposite sign.) If infinitely many diagrams have Alexander polynomial with leading coefficient ± 1 , then some of a, b, c must be bounded. But (a, b, c) are pairwise linked. Then the argument in Lemma 3.11 about two linked groups shows that v_2 is realized only finitely many times.

For the other diagrams it was necessary to calculate and analyze the polynomial expressions for $v_{2,4}$ in the numbers of twists a bit more closely. Note again that v_4 would depend just up to sign on the sign of the Hopf band, but v_2 will differ more significantly. Thus, instead of calculating the exact polynomial expressions twice, we argue by direct observations on \hat{D}'' . In some cases we still used the partial exclusions that were provided by the extended non-fibered test.

Case 2.6.3. Diagram number 8: let c be the lower right, b the lower left, and a the upper group. We use the same variable for the group and for the (even/odd) number of crossings in it. To recover D from D'' , we must smooth out one crossing in the parallel (negative) clasp and plumb a Hopf band. (If the band is negative, we have a non-alternating $(2, -1)$ tangle, that is, D and D'' are isotopic.) Again the coefficient v_4 of z^4 in the Conway polynomial in D is of the form (3.10), with $C_1 = -1/8 \neq 0$. Hence, if $v_4 \equiv \pm 1$ is constant, then some of a, b, c is bounded (above). Now (a, b) and (b, c) are linked. If b is unbounded, then depending on whether both a and c are bounded or one is not, we can argue with either parts of Lemma 3.11. Thus assume b is fixed.

Note that the application of Lemma 3.11 implicitly uses v_2 , which we declared to avoid. However, even though v_2 will change depending on the way (sign and/or spot) of plumbing, once this choice is fixed, the property to form an arithmetic progression is always satisfied. Moreover, the linking status of groups in which \bar{t}_2 twists are applied is not changed between D and D'' . Therefore, at most one polynomial will be realized infinitely many times by the diagrams D regardless of the values of v_2 .

[†]Here we alternatively use the number of crossings in a \sim -equivalence class instead of the number of twists.

In that case, if $b \geq 5$, then one sees that v_4 (as a linear polynomial in a, c) has a non-trivial coefficient in ac . This coefficient is itself some polynomial in b , and thus bounded away from 0 uniformly for all $b \geq 5$. Therefore, if $v_4 \equiv \pm 1$, then one of a or c must be bounded. Then we have only one group free to twist at, and with Lemma 3.11 we are done.

If $b = 3$, then v_4 has no coefficient in ac , and neither has v_2 . Then the linear coefficient of c in $v_2(D)$ is $\pm 1/2$ and in v_4 it is 0, since smoothing out a crossing in c we obtain a Hopf link in \hat{D}' , and so also in \hat{D} . The linear coefficient of a in v_2 is 0, because the linking number of the link L_a obtained by smoothing a crossing in a is 0 (in D'' , and it is not changed when recovering D). The linear coefficient of a in v_4 is $\pm 1/2$, because L_a is a Hopf plumbed $(2, -2, c-1)$ pretzel link (c is odd), which is fibered. Then $|v_2| + |v_4| \rightarrow \infty$ if $\max(a, c) \rightarrow \infty$, and we are done with diagram 8.

Case 2.6.4. The surfaces in the diagrams 5–7 are all Hopf plumbing equivalent; consider number 6. We use the DT notation in the above list. For recovering D we again have two possibilities, depending on the sign of the Hopf band. (These diagrams lie in $\langle 11_{148} \rangle$.)

19 148 14 -24 -22 -34 20 -28 -26 38 -30 10 8 -4 -2 -12 -18 -16 -36 -6 -32
 19 148 14 -24 -22 34 20 -28 -26 38 -30 10 8 -4 -2 -12 -18 -16 -36 6 -32

We calculate v_4 : it does not depend on the site of plumbing, and depends on the sign of the Hopf band only up to sign. Thus, as long as we avoid considering v_2 , we can work for all cases (all three diagrams \hat{D}'' and both signs of Hopf bands plumbed) simultaneously. Let $a, b, c \geq 0$ be the number of additional twists at the groups of crossings 9, 6, 5, and d the number of additional twists at the group of crossing 2 in \hat{D}'' (so that the groups $\{9\}, \{6\}, \{5\}, \{2\}$ have $3 + 2a, 3 + 2b, 3 + 2c, 4 + 2d$ crossings, respectively). Then verification of v_4 on some individual example diagrams shows

$$v_4 = b + 2ba - 3c - 2ca - 2cb + d(b + ab - 2c - ca - cb).$$

We analyze this expression in the following cases.

- If $a = 0$, then $v_4 = b - 3c - 2cb + d(b - 2c - cb)$.
 - * If $c > 0$, then $v_4 \leq -b - 3c \leq -3$, a contradiction.
 - * If $c = 0$, then $v_4 = b + db$, and thus for $v_4 = \pm 1$ we need $b = 1$ and $d = 0$. Then we have no free parameter, and we are done.

Thus $a > 0$.

- If $c = 0$, then $v_4 = b + 2ba + d(b + ab)$.
 - * If $b > 0$, then with $a > 0$ we have $v_4 \geq 2$, so we are done.
 - * If $b = 0$, then $v_4 = 0$, and a contradiction.

We conclude $c > 0$.

- If $b = 0$, then $v_4 = -3c - 2ca + d(-2c - ca)$, and with $c > 0$, we get $v_4 \leq -3$, a contradiction. Therefore, $b > 0$.

Now using the extended non-fibered test on D' , we found that for one twist at crossings 5, 6, 9 the knot is reported (stably) non-fibered.

The proof of Theorem 1.2 is now complete. □

Of course, with some more effort, one could characterize the frequency of the occurring polynomials completely, but we preferred to leave out the technical details, being mainly interested in showing the qualitative difference of canonical fiber surfaces with regard to the Alexander polynomial.

For higher genera, it is known from [49] that infinitely many Alexander polynomials occur for infinitely many canonical fiber surfaces. (Thus Theorem 1.2 does not hold in this sharp form

for polynomials of higher degree.) However, the frequency of such polynomials as compared to all monic polynomials of given degree remains unclear.

There is a partial extension to 2-component links.

THEOREM 3.1. *For almost all monic polynomials Δ of degree $3/2$, there exist only finitely many canonically fibered 2-component links of genus 1 realizing Δ .*

Proof. Let D' be a diagram of such a link L with a canonical fiber surface. By applying a clasping at a crossing involving both components of D' , we obtain a knot diagram D of genus 2 with a canonical fiber surface. We choose D to be κ -minimal (that is, with respect to the order in Definition 3.1) among all diagrams thus obtainable.

Now we can use almost the same criteria as in the proof of Theorem 1.2 to restrict the diagrams D that come in question. The non-special generators containing a parallel clasp are $6_2, 6_3, 7_6, 8_{14}$ and 9_{25} , and are again easily done with Lemma 3.10. So assume that D (and hence D') is special.

Then we apply the selection procedure as for the diagrams \tilde{D}'' in Case 2.6 in the previous proof. Now there is the exception that when processing the list of 59 diagrams using Lemma 3.11, we are no longer allowed to exclude diagrams with two linked twist groups. The thought when our new operation $D' \rightarrow D$ changes the conditions of the so modified Lemma 3.11 leads to exactly the same list (3.8), which is discarded in the same way.

With the relaxed (that is, not considering linked twist groups) test in Lemma 3.11, instead of the previous list of 8 diagrams in (3.9), we obtain a larger list of 21, which additionally contains, beside the 8 'old' ones, the following 13 'new' ones.

```

13 38 8 -14 -20 -18 26 -22 -4 -24 -2 -6 -12 -10 -16
13 38 8 -14 -20 -18 26 22 -4 -24 -2 -6 12 10 -16
13 38 8 -14 20 18 26 22 -4 -24 -2 6 12 10 -16
13 38 8 -14 20 18 26 22 -4 -24 2 6 12 10 -16
13 38 8 -14 20 18 26 22 -4 24 -2 6 12 10 16
13 38 8 14 -20 -18 26 -22 4 -24 2 -6 -12 -10 -16
14 101 4 14 -20 -18 -26 -24 2 -22 -8 -6 -28 -12 -10 -16
14 101 4 14 20 18 -26 -24 2 -22 8 6 -28 -12 -10 -16
14 101 4 14 20 18 26 24 2 -22 8 6 -28 12 10 -16
15 38 10 -14 20 -26 -24 2 -22 -30 -28 6 4 -12 -8 -18 -16
15 38 10 -18 26 22 20 30 -24 28 -4 -2 8 -12 6 16 14
16 101 4 10 -18 -26 2 -24 -22 28 -6 -32 -30 -12 -8 16 14 -20
16 101 4 10 18 -26 2 24 22 28 6 -32 -30 12 -8 16 14 -20

```

(Again, as in (3.9), the knot identifier indicates the underlying generator.)

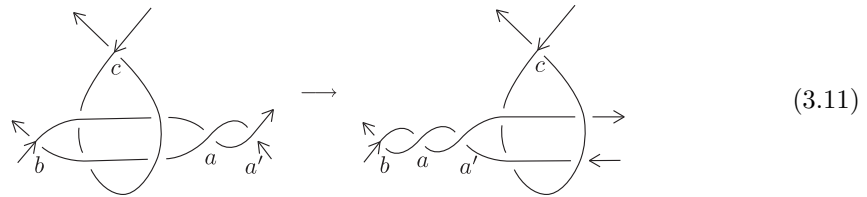
This list is easily analyzed in cases. Let us for simplicity use again the Conway polynomial ∇ . Note that for 2-component links of genus 1, the variation of a monic polynomial is given by the value of $\nabla_1 = [\nabla]_1$, and this is just the linking number of both components.

Case 1. The two old twelve crossing diagrams in $\langle 8_{15} \rangle$ are again discarded as in the previous Case 2.6.1, because they have a $(2, -1)$ -tangle with a parallel clasp. (It is again easily observed that a simpler diagram D would also be possible then.)

Case 2. The three (old) eighteen crossing diagrams in $\langle 10_{101} \rangle$ are dealt with by the previous Case 2.6.4, which made use only of Gabai and ∇_4 (the latter becoming ∇_3 here).

Case 3. Consider the first three (new) thirteen crossing diagrams in $\langle 9_{38} \rangle$, and the three (new) fourteen crossing diagrams \tilde{D} . These diagrams \tilde{D} have a unique parallel clasp, and thus come from only one \tilde{D}' .

We will apply the following *loop move*, an isotopy ‘moving through a loop’, which relates two \sim -equivalence classes.



The convention is that only \bar{t}_2 twists at a, a', b, c are allowed. One may switch the crossing c , as well as *simultaneously* the other four unnamed crossings. We will call these four crossings *loop crossings*. Similarly a, a', b may be switched. The orientation of the strand at c may also be reverted, as well as, *simultaneously*, those of the strands involved in a, a', b . Note that the loop move preserves the property of a diagram to be special. \square

LEMMA 3.12. *Let D' be a special diagram with $\chi(D') < 0$ and admitting a loop move, such that the classes of a, b in (3.11) are non-trivial. Then one can obtain from D' by loop moves a diagram D'_1 with $\kappa(D'_1) < \kappa(D')$.*

Moreover, when a clasp is applied at the same of the four loop crossings in (3.11) to obtain diagrams D, D_1 from D' , respectively D'_1 , then $\kappa(D_1) < \kappa(D)$.

Proof. Since D' is special and $\chi(D') < 0$ (that is, D' is not a reverse $(2, k)$ -torus link diagram), we have $n_1(D') = c(D')$. Thus $n_1(D'_1) \leq n_1(D')$. If D'_1 is not special, then $n_2(D'_1) > 1 = n_2(D')$. If D'_1 is special, then $n_2(D'_1) = n_2(D') (= 1)$. If one of the \sim -equivalence classes of a and b in (3.11) is even, it can be eliminated by (applying loop moves and) absorbing it into the other one, and thus we can have $n_3(D') = n_3(D'_1) + 1$. Otherwise, for both classes of a and b odd, we can make one class trivial, and then $n_3(D') = n_3(D'_1)$ but $n_4(D') = n_4(D'_1) - 1$.

It remains to argue about the effect of the clasp. Since all four diagrams are special (and prime), we have

$$n_1(D') - n_1(D) = n_1(D'_1) - n_1(D_1) = -1,$$

and

$$n_2(D') = n_2(D) = n_2(D'_1) = n_2(D_1) = 1.$$

Also, the clasp changes n_3 and n_4 by the same quantity for $D' \rightarrow D$ and $D'_1 \rightarrow D_1$. \square

We observe that for the six diagrams \mathring{D} designated in the beginning of the case, after undoing the clasp to recover \mathring{D}' , the two twist groups (\sim -equivalence classes) are non-trivial and can be simplified by loop moves. This again contradicts the κ -minimal choice of \mathring{D} (and D).

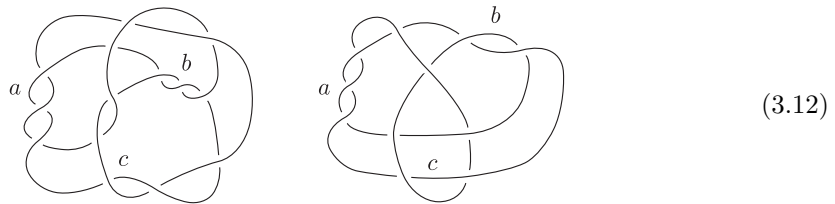
Case 4. For the last three thirteen crossing diagrams \mathring{D} , in D' , both twist groups involve strands of different components, and both with the same sign. Thus for any sequence of diagrams D' obtained by twists we have $\nabla_1 \rightarrow \pm\infty$, and any (monic) polynomial occurs only finitely many times.

Case 5. The two sixteen crossing diagrams have the property that in D' , none of the twist groups involves crossings of different components. Thus the linking number ∇_1 remains constant, and so at most two monic polynomials can occur.

Case 6. There remain five (three old and two new) fifteen crossing diagrams in $\langle 9_{38} \rangle$. For them again direct calculation of $\nabla(D')$ and trivial algebra show that if $\nabla_3 = \pm 1$, only finitely many ∇_1 can occur infinitely many times, except for one case.

This diagram \mathring{D} is the second of the three old fifteen crossing diagrams. Let $a \sim \{8\}$, $b \sim \{4\}$ and $c \sim \{2\}$. Then $\nabla_3 = \pm 1$ easily leads to the restriction $c = 0$, but then $\nabla_1(D') = b - a - 2$,

and so any value could be realized infinitely many times. It needs a bit of hand manipulation using wave moves to see that in this case the left diagram \mathring{D}' below is isotopic to the right one.



(In both diagrams components are oriented so that all clasps are reverse and the diagrams are special.) This isotopy identifies, after a loop move, both twist groups a, b , and again contradicts the κ -minimal choice of \mathring{D} by Lemma 3.12. (We also see that the links are indeed fibered, and their fiber is a Stallings twist of the one of the connected sum of a trefoil and Hopf link.)

With this the proof is complete. \square

Note also the difference between the two alternatives $\max \text{cf } \Delta = \pm 1$. This distinction is very unusual in the eyes of algebraic topology, and makes the normalization condition $\Delta(1) = +1$ for knots important. It became apparent already in the proof of Theorem 1.2: for example, with the 12_{1202} series we have a two-fold twist family with the same polynomial of $\max \text{cf } \Delta = 1$, while for leading coefficient -1 we have no such family.

When considering links, the difference becomes even more drastic. For instance, the Alexander polynomial of fibered links of three components and genus 0 are $\pm(t-1)^2/t$. While the $(2, -2, 2k)$ -pretzel links is an infinite family with Alexander polynomial $-(t-1)^2/t$, for Alexander polynomial $+(t-1)^2/t$, there exist only the connected sums of two positive or two negative Hopf links.

REMARK 3.2. As an extension of Morton's result [28] to links, in [49] a different construction of infinitely many (even canonical) fiber surfaces of every possible Alexander polynomial was given for four or more components. The status of two and three components remains unsettled at this point (even without regard to canonical fiber surfaces). Morton's construction appears to possibly adapt to 2 components. One can deplumb a Hopf band from Burde's knots Morton begins with; a crossing change in these bands unknots the knot, and the Stallings twists Morton uses happen to be outside of this Hopf plumbing spot. The JSJ decomposition argument to distinguish the links would require some extra work, though.

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