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Logarithms and the Reciprocals of Numbers.—The writer when introducing his pupils to the use of Logarithmic Tables is accustomed to give them exercise with the number 1·00023. This number, he tells them, without explaining how he obtains it, is such that $1{\cdot}00023 \times 1{\cdot}00023 \times \ldots 10{,}000$ times would give a number which read to the 3rd decimal place is equal to 10. He makes this the basis of practical exercises in approximations, and builds on it lessons in Indices, and in the use of Logarithmic Tables, with particular explanation of the Mean Difference Columns.

An instructive exercise for the more advanced pupil might be given in this form :—

Employ the Table of Reciprocals of Numbers to prove that

$$1{\cdot}00023 \textit{ is approximately} = 10^{\frac{1}{10{,}000}}.$$

Starting with 1 and multiplying successively by 1·00023 we obtain 1, 1·00023, $(1{\cdot}00023)^2$... Each power is approximately equal to an infinite number of numbers, which are all considered of equal value to a specified degree of approximation. Any number between 1 and 10 is in this sense of approximation either equal to one or other of these powers, or lies between two successive powers. Any number between 1 and 10 thus corresponds to a particular number of times for which the multiplication by 1·00023 has been performed to give any number equal in value to it.

Let us consider to which powers correspond the numbers 1·1, 1·2, 1·3..., that is, the numbers between 1 and 10 progressing by a common Arithmetical difference of ·1.

When 1 is multiplied by 1·00023, the multiplication has added to 1 the value $\cdot 00023 \times 1$.

When $1{\cdot}1$ is multiplied by $1{\cdot}00023$, the multiplication adds to $1{\cdot}1$ the value $\cdot 00023 \times 1{\cdot}1$.

$\therefore$ On an average each multiplication between 1 and $1{\cdot}1$ is adding on $\cdot 00023\left(\frac{1+1{\cdot}1}{2}\right)$.

$\therefore$ To change 1 into $1{\cdot}1$ the multiplication has to be performed $\dfrac{\cdot 1}{\cdot 00023\left(\frac{1+1\cdot 1}{2}\right)}$ times.

This is approximately equal to $\dfrac{\cdot 1}{\cdot 00023}\left(\dfrac{1+\frac{1}{1\cdot 1}}{2}\right)$.

Similarly to change $1{\cdot}1$ to $1{\cdot}2$ the multiplication has to be performed $\dfrac{\cdot 1}{\cdot 00023}\left(\dfrac{\frac{1}{1\cdot 1}+\frac{1}{1\cdot 2}}{2}\right)$ times, and so on till 10 is reached.

$\therefore$ The number of times for which the multiplication has to be performed to reach 10

$$=\frac{\cdot 1}{\cdot 00023}\left(\frac{1+\frac{1}{1\cdot 1}}{2}+\frac{\frac{1}{1\cdot 1}+\frac{1}{1\cdot 2}}{2}+\frac{\frac{1}{1\cdot 2}+\frac{1}{1\cdot 3}}{2}+\ldots+\frac{\frac{1}{9\cdot 9}+\frac{1}{10}}{2}\right)$$

The part within the Bracket may be found by Simple Addition from any Table of Reciprocals to be equal to $23{\cdot}004$ to 3 places.

$\therefore$ The number of times the multiplication has to be performed to change 1 to $10=\dfrac{\cdot 1}{\cdot 00023}\times 23{\cdot}004$

$=10{,}000$ approximately.

If $1{\cdot}0001$ be taken as the multiplier instead of $1{\cdot}00023$, then $(1{\cdot}0001)^{23004}$ is approximately $=10$,

or $\qquad \left\{(1{\cdot}0001)^{10{,}000}\right\}^{2\cdot 3004}=10.$

$(1{\cdot}0001)^{10{,}000}$ is an approximate value for the Napierian base e.

It will be noticed that in thus treating the series $1,\ 1+r,\ (1+r)^2 \ldots (1+r)^{n-1}$ where $r=\cdot 00023$ and $(1+r)^{n-1}=1{\cdot}1$ we have employed approximations. Thus we have taken

$n-1$ to be $=\dfrac{\cdot 1}{\cdot 00023}\left(\dfrac{2}{2\cdot 1}\right)$ which is $=414$ to the nearest integer.

Let us test the approximation by finding the value of $(1+r)^{n-1}$ when $\overline{n-1}$ is taken $=414$.

By the Binomial Theorem

$$(1{\cdot}00023)^{414} = 1 + 414({\cdot}00023) + \text{etc.} = 1{\cdot}0999 \text{ to 4 places.}$$
$$= 1{\cdot}1 \text{ to 3 places.}$$

We also substituted $\dfrac{1+\dfrac{1}{1{\cdot}1}}{2}$ for $\dfrac{1}{\dfrac{1+1{\cdot}1}{2}}$

i.e. $\dfrac{2{\cdot}1}{2{\cdot}2}$ for $\dfrac{2}{2{\cdot}1}$

i.e. $4{\cdot}41$ for $4{\cdot}4$.

The other parts may be tested in a similar manner, when it will be observed that the approximation is as close as that obtainable in Four-figure Logarithmic Tables.

In changing 1 into any value N of the series 1, 1·1, 1·2... the

$$\text{corresponding power} = \frac{{\cdot}1}{{\cdot}00023}\left(\frac{1}{2} + \frac{1}{1{\cdot}1} + \frac{1}{1{\cdot}2} + \dots + \frac{\frac{1}{\mathrm{N}}}{2}\right).$$

In changing 1 into 10 the power is 10,000.

$$\therefore \quad \log_{10}\mathrm{N} = \left(\frac{1}{2} + \frac{1}{1{\cdot}1} + \dots \frac{\frac{1}{\mathrm{N}}}{2}\right) \div 23.$$

It follows that if M and N are two terms of the series, then

$$\frac{1}{\mathrm{M}} + \dots + \frac{1}{\mathrm{N}} = \left(\frac{1}{2} + \frac{1}{1{\cdot}1} + \dots + \frac{\frac{1}{\mathrm{N}}}{2}\right) - \left(\frac{1}{2} + \frac{1}{1{\cdot}1} + \dots + \frac{\frac{1}{\mathrm{M}}}{2}\right) + \frac{\frac{1}{\mathrm{M}} + \frac{1}{\mathrm{N}}}{2}$$

$$= 23(\log_{10}\mathrm{N} - \log_{10}\mathrm{M}) + \frac{\frac{1}{\mathrm{M}} + \frac{1}{\mathrm{N}}}{2}.$$

By means of this equation the sum of a number of terms in Harmonical Progression may be approximately obtained in certain cases.

Thus $\frac{1}{1\cdot04}+\frac{1}{1\cdot08}+\frac{1}{1\cdot12}+\ldots+\frac{1}{1\cdot32}$

$$=\frac{(\log 1\cdot32-\log 1\cdot04)\times 23}{\frac{\cdot04}{\cdot1}}+\frac{\frac{1}{1\cdot04}+\frac{1}{1\cdot32}}{2}$$

$$=5\cdot957+\cdot4807+\cdot3787=\underline{6\cdot8164}$$

The result correct to 3 places obtained from Tables $=\underline{6\cdot822}$.

We may find the sum of

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\ldots+\frac{1}{10{,}000}$$

It is $=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}+\frac{1}{10}\left(\frac{1\cdot}{1\cdot1}+\frac{1}{1\cdot2}+\frac{1}{1\cdot3}+\ldots+\frac{1}{1000}\right)$.

But $1+\frac{1}{2}+\ldots+\frac{1}{10}$ from Table of Reciprocals $=2\cdot92897$, and the remaining part $=\frac{1}{10}\left\{23\log 1000-23\log 1\cdot1+\frac{1}{2}\left(\frac{1}{1\cdot1}+\frac{1}{1000}\right)\right\}$

$$=\frac{1}{10}(68\cdot5028)=6\cdot85028.$$

$\therefore$ The sum of the series $=2\cdot92897+6\cdot85028=9\cdot77925$.

This exceeds $\log_e 10000$ or $9\cdot21$ by $\cdot57$.

(Euler's constant or $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log_e n$ when n is infinite $=\cdot57721\ldots$).

C. M'Leod.

Certain Processes in the Theory of Equations illustrated Geometrically.—

The Derived Function: In what follows by $f(x)$ is meant a rational integral function of x of degree n with the coefficient of x^n unity. By the "roots of $f(x)$" is meant the roots of $f(x)=0$; by $f'(x)$ is meant $df(x)/dx$.

If $$f(x)=(x-a_1)(x-a_2)\ldots(x-a_n),$$

then $$f'(x)=\frac{f(x)}{x-a_1}+\frac{f(x)}{x-a_2}+\ldots+\frac{f(x)}{x-a_n},$$

which may be written shortly

$$\prod_{s=1}^{s=n}(x-a_s)\left\{\sum_{r=1}^{r=n}\frac{1}{x-a_r}\right\}.$$