SOME DISTRIBUTION PROBLEMS OF ORDER STATISTICS FROM EXPONENTIAL AND POWER FUNCTION DISTRIBUTIONS[1]

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Summary. This paper gives alternative straightforward and simpler proofs of some of the results of Laurent's [10], and Likes' [11], [13]. The derivation of the results is simplified by using the theory of Dirichlet's multiple integral and the transformation used to derive this multiple integral. Some applications of Dirichlet's transformation to order statistic theory from gamma, and normal populations, have been already given by Kabe [7].

1. Introduction. A considerable amount of research work has recently been done in the distribution theory of linear functions, or of ratios of linear functions, of ordered statistics from one or more exponential populations, which sometimes may be truncated. The exponential model is nowadays widely used in Failure Theory, see e.g., Epstein and Sobel [1], Epstein [2], Epstein and Tsao [3], Laurent [10], and Holla [6], and its properties are thoroughly explored either from a mathematical point of view, see e.g., Tanis [19], Hogg and Tanis [5], Likes [11], [12], or from a mixed mathematical and applied point of view, see e.g., Saleh [17], Likes [13], and also Epstein et al. The mathematical tools employed, in the derivation of the distribution problems, by above authors are varied and sometimes complicated, see e.g., Laurent [10], and Likes [11], [12], [13], [14]. A common tool that appears to be frequently used by some of the above authors is a theorem of Epstein and Sobel's [1] which states that $x_{(i)} - x_{(i-1)}$ is distributed as a χ^2 variate with two degrees of freedom, where $x_{(i)}$ is i-th smallest observation in an ordered sample of size N from an exponential population. Dirichlet's transformation which indeed is a very powerful tool for dealing with a substantial part of applied order statistic theory from the exponential and function distributions appears to have been neglected. Our purpose in this paper is to use this tool to give simple, elegant, and straightforward proofs of some of the results of Laurent's [10], and Likes [11], [13]. Alternative proofs of some of the results of other authors may be developed on similar lines.

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Some results which are found useful in the sequel are stated in the next section. We assume that all the integrals occurring in this paper are evaluated over appropriate ranges of the variables of integration.

2. Some useful results. The Dirichlet's multiple integral, Gibson ([4], p.492), states that

$$(1) \qquad \int_R y_1^{\alpha_1-1} y_2^{\alpha_2-1} \cdots y_N^{\alpha_N-1} f(y_1+\cdots+y_N)dy_1 \cdots dy_N$$

$$= (\Gamma\alpha_1 \Gamma\alpha_2 \cdots \Gamma\alpha_N / \Gamma(\alpha_1+\cdots+\alpha_N))\, \theta_N^{\alpha_1+\cdots+\alpha_N-1} f(\theta_N),$$

where the region R of integration is determined by the conditions, $y_i \geq 0$, $i = 1, 2, \ldots, N$; $y_1 + y_2 + \cdots + y_N = \theta_N$. Here f is a suitable function and the integral is to be understood as a part of the volume integral over the range, $y_i \geq 0$, $i = 1, \ldots, N$; $\theta_N < y_1 + \cdots + y_N < \theta_N + d\theta_N$. In case f is a suitable density function of the y's, then obviously the right-hand side of (1) is the density function of the variate θ_N. The integral (1) is then evaluated by using the (Dirichlet's) transformation, Gibson ([4], p.147)

$$(2) \qquad \sum_{i=1}^{j} y_i = \theta_j \theta_{j+1} \cdots \theta_N, \quad j = 1, 2, \ldots, N.$$

The Jacobian J of the transformation from the y's to the θ's is known to be, Gibson ([4], p. 147)

$$(3) \qquad J = \theta_2 \theta_3^2 \cdots \theta_{N-1}^{N-2} \theta_N^{N-1}.$$

We next require a result of Patil and Wani's [15], which may be stated as follows. Let $x_1, x_2, \ldots, x_N$ be a random sample of size N from the distribution $F(x, \theta) = p\{x \leq x\}$, and let $t(x_1, x_2, \ldots, x_N)$ be a complete and sufficient statistic for θ. Then $p\{x_1 < a | t\}$ is the unbiased minimum variance estimate (UMVE) of $F(a, \theta)$, where a is a known constant. It may be mentioned that in view of Tukey's theorem [21], Patil and Wani's result also holds good for truncated populations. Further, it follows that $p\{x_1 < a_1, x_2 < a_2, \ldots, x_k < a_k | t\}$ is UMVE for the product $F(a_1, \theta) F(a_2, \theta) \ldots F(a_k, \theta)$, which shows that powers of $F(a, \theta)$ may be estimated and have UMVE's.

Now we proceed with the applications of the results of this section.

3. An Exponential Analog of Thompson's Distribution. Thompson [20] in his studies of rejection of outlying observations from a normal population showed that the variate $(x_i - \bar{x})/s$ has a beta (in fact a symmetrical beta) distribution. Here x_i is the i-th observation $\bar{x}$ is the sample mean and s is the sample standard deviation. Laurent considers a two parameter exponential distribution

$$(4) \qquad f(x;m,\sigma) = (1/\sigma)\exp\{-(x-m)/\sigma\}, \quad x > m, \ \sigma > 0,$$

from which a sample of size N is available with $x_{(1)}$ as the smallest observation and $\bar{x}$ as the mean. If Y denotes $\bar{x}-x_{(1)}$, then it is known that $(x_{(1)}, Y)$ constitutes a complete and sufficient statistic for the pair (m, σ). He finds the distribution of $(\xi - x_{(1)})/Y$ where ξ is any other of the N observations. The distribution of $(\xi - x_{(1)})/Y$ is a beta distribution termed by Laurent as an exponential analog of Thompson's distribution. He uses this distribution to estimate the survivor probability

$$(5) \qquad s(a) = p\{x > a\} = \int_a^\infty f(x;m,\sigma)dx .$$

He also estimates powers of $s(a)$. Laurent uses a very complicated procedure for finding the distribution analogous to Thompson's distribution. We give here an alternative proof. Since the distributions in which we are interested do not depend on m and σ, we take $m = 0$, $\sigma = 1$.

Let $x_{(1)}, x_2, x_3, \ldots, x_N$ be a sample of size N from the density function

$$(6) \qquad f(x) = \exp\{-x\}, \quad 0 < x < \infty,$$

where $x_{(1)}$ denotes the smallest observation. Obviously, the conditional density of $x_2, x_3, \ldots, x_N$, given $x_{(1)}$, is

$$(7) \qquad f(x_2, \ldots, x_N | x_{(1)}) = N^{-1} \exp\{-\sum_{i=2}^{N} (x_i - x_{(1)})\}, \quad x_i > x_{(1)}.$$

Let $\eta_i = x_i - x_{(1)}$, $i = 2, \ldots, k+1$; $\delta_j = x_j - x_{(1)}$, $j = k+2, \ldots, N$. Then we note that

$$(8) \qquad NY = (\eta_2 + \ldots + \eta_{k+1}) + (\delta_{k+2} + \ldots + \delta_N).$$

It follows that the joint density of η's and Y, given $x_{(1)}$, is

$$(9)\quad N^{-1} \int_R \exp\{-(\eta_2 + \dots + \eta_{k+1}) - (\delta_{k+2} + \dots + \delta_N)\}\, d\delta_{k+2} \dots d\delta_N ,$$

where the region R of integration is determined by the condition (8). However, the integral (9) is evaluated by using (1), and we find that

$$(10)\ f(\eta_2, \dots, \eta_{k+1}, Y | x_{(1)}) = \exp\{-NY\}\,(NY - \eta_2 - \dots - \eta_{k+1})^{N-k-2}/\Gamma(N-k-1).$$

Further it is easily shown that

$$(11)\qquad f(Y|x_{(1)}) = \exp\{-NY\}\,(NY)^{N-2}/\Gamma(N-1) .$$

Hence from (10) and (11) we obtain that

$$(12)\ f(\eta_2, \dots, \eta_{k+1} | x_{(1)}, Y) = (N-1)(NY - \Sigma\eta_i)^{(N-k-2)}(NY)^{2-N}/\Gamma(N-k-1).$$

Now by using (12) and the result of Patil and Wani's we conclude that

$$(13)\ \int f(\eta_2, \dots, \eta_{k+1} | x_{(1)}, Y)\, d\eta_2 \dots d\eta_{k+1} = (1 - \sum_{j=1}^{k} (a_j - x_{(1)})/NY)^{N-2},$$

is an UMVE of the survivor probability product $S(a_1) \dots S(a_k)$. The range of integration (13) is $\eta_i > (a_{i-1} - x_{(1)})$, $i = 2, \dots, k+1$. Incidentally note that Laurent's results ([10], p.653, equation (6), and p.655, equation (11)) are in error. In case $k = 1$, then (12) gives

$$(14)\quad f(\eta_2 | x_{(1)}, Y) = (N-2)(NY - \eta_2)^{N-3}(NY)^{2-N},\ 0 < \eta_2 < NY .$$

The distribution (14) is termed by Laurent as an exponential analogue of Thompson's well known beta distribution. This, of course, is a simple property of the exponential population which states that if x_1 and x_2 have exponential (or gamma) distribution then the conditional distribution of x_1, given $x_1 + x_2$, is a beta distribution. The multivariate generalizations of this property are also known, and a number of such generalizations have been given by Laurent in a series of papers, abstracts, technical reports, and memorandas.

4. Distribution of reduced i-th order statistic. This section illustrates how we may proceed to obtain the distributions of the ratios of linear functions of ordered statistics from an exponential population. As an example we consider the distribution of the reduced i-th order

statistic, see Laurent ([10], p. 656, equation 20). Since the distributions in which we are interested do not depend on the original parameters, we assume that our model is given by equation (6). Let $x_{(1)} < x_{(2)} < \ldots < x_{(N)}$ be the order statistics. Set

$x_{(j)} = \sum_{i=1}^{j} y_i$, $j = 1, 2, \ldots, N$. Note that the Jacobian of the transformation from the x variates to the y variates is unity, and that the y variates are unordered. Now we are interested in finding the distribution of the statistic

$$(15) \qquad W = (x_{(i)} - x_{(1)})/NY = \left(\sum_{j=2}^{i} y_j / \sum_{j=2}^{N} (N-j+1)y_j\right) = (u/v),$$

where u and v denote, respectively, the numerator and the denominator of third member on the right-hand side of (15). Obviously the joint characteristic function (c.f.) $\phi\ (it_1, it_2)$ of the variates u and v is

$$(16) \quad \phi(it_1, it_2) = N! \int \exp\{it_1 \sum_{j=2}^{i} y_j + it_2 \sum_{j=2}^{N} (N-j+1)y_j - \sum_{j=2}^{N} (N-j+1)y_j\}\, dy_1\, dy_2 \ldots dy_N$$

$$= \{(N-1)!/(N-i)!(1-it_2)^{N-i}\}\{\prod_{k=2}^{i} [(N-k+1)(1-it_2) - it_1]\}^{-1}$$

$$= \sum_{k=1}^{i-1} \frac{(N-1)!\,(-1)^{k-1}\,(1-it)^{2-N}}{(N-i)!\,(k-1)!\,(i-1-k)!\,[(N-i+k)(1-it_2) - it_1]} .$$

On inverting the c.f. (16) we find that

$$(17) \qquad f(u, v) = \sum_{k=1}^{i-1} \frac{(N-1)!\,(-1)^{k-1} \exp\{-v\}\,(v-(N-i+k)u)^{N-3}}{(N-i)!\,(N-3)!\,(i-1-k)!\,(k-1)!}$$

By using (17) we may easily show that

$$(18) \qquad f(W) = \sum_{k=1}^{i-1} \frac{(N-1)!\,(N-2)(-1)^{k-1}(1-(N-i+k)W)^{N-3}}{(N-i)!\,(k-1)!\,(i-1-k)!} ,$$

where the range of values of W is $W > 0$, and that the expression in the bracket containing W is positive. The result (18) has been recently obtained by Likes [14] in a different way.

Likes [13] considers the statistics

$$(19) \qquad \psi = \psi(r, s; p, q) = (x_{(s)} - x_{(r)})/(x_{(q)} - x_{(p)}) ,$$

$$1 < p < r < s < q < N ,$$

in his studies of testing outliers from an exponential population. We note that the density of ψ does not depend on the original parameters. Since ψ is a ratio of linear functions of exponential populations its distribution is of a beta type, in fact a finite series of beta distributions of the type

$$(20) \quad f(x) = a^m b^n \{B(m, n)\}^{-1} x^{m-1}(1-x)^{n-1}/(ax+b(1-x))^{m+n}, \quad 0 < x < 1.$$

However, Likes [13] obtains them in a disguised integral representation form. The density of ψ may be easily obtained by the method of c.f.s as we have done in this section. The distribution of the reduced i-th order statistic or the distribution of ψ may also be obtained for the truncated exponential population by using the method of c.f.s as indicated in the next section.

5. Truncated Exponential Model. In case the exponential model is truncated, then the method of characteristic function may be applied to study the distribution theory of linear functions of ordered sample values. Now consider the model

$$(21) \qquad f(x) = (1 - \exp(-x_o))^{-1} \exp\{-x\} , \quad 0 < x < x_o .$$

The c.f. of x is

$$(22) \qquad (1 - \exp(-x_o))^{-1} (1 - \exp\{-(1-it)x_o\})(1 - it)^{-1} .$$

Thus the c.f. of $\delta = x_{k+1} + x_{k+2} + \ldots + x_N$ is

$$(23) \quad (1 - \exp(-x_o))^{-(N-k)}(1 - \exp\{-(1-it)x_o\})^{N-k}(1 - it)^{-(N-k)} .$$

On expanding the middle term of (23) by the binomial theorem and inverting (23), we readily find that

$$(24) \qquad f(\delta) = (1 - \exp(-x_o))^{-(N-k)}((N-k))^{-1} \sum_{r=0}^{N-k} \binom{N-k}{r}$$

$$\exp\{-\delta\} (\delta - rx_o)^{N-k-1}(-1)^r ,$$

where $(\delta - rx_o)$ is positive, hence r really runs from 0 to $[\delta/x_o]$.

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By setting $k = 0$ in (24) we get the density of the variate $t = x_1 + \ldots + x_N$. Thus using (21) and (24) we find the joint density of $x_1, x_2, \ldots, x_k$, and δ to be

$$(25)\quad f(x_1, x_2, \ldots, x_k)\, f(\delta) = f(x_1, x_2, \ldots, x_k, \delta)$$

$$= (1 - \exp(-x_o))^{-N} ((N-k))^{-1} \exp\{-(x_1 + \ldots + x_k + \delta)\}$$

$$\sum_{r=0}^{N-k} \binom{N-k}{r} (-1)^r (\delta - rx_o)^{N-k-1} .$$

Further setting $\delta = t - x_1 - \ldots - x_k$, and using (24) and (25) we have that

$$(26)\quad f(x_1, \ldots, x_k | t)$$

$$= \frac{N \sum_{r=0}^{N-k} \binom{N-k}{r}(-1)^r (t - x_1 - \ldots - x_k - rx_o)^{N-k-1}}{(N-k) \sum_{r=0}^{N} \binom{N}{r} (-1)^r (t - rx_o)^{N-1}} .$$

Thus $p\{x_1 > a_1, x_2 > a_2, \ldots, x_k > a_k | t\}$, which can be explicitly evaluated, is UMVE of the survivor probability product $s(a_1) \ldots s(a_k)$. Thus, e.g., if t is held fixed in $0 < t < x_o$, then $s(a_1) \ldots s(a_k)$ is estimated by $(t - a_1 - \ldots - a_k)^{N-1}/t^{N-1}$. In case $k = 1$, the result (26) agrees with the one given by Holla ([6], p. 334, equation (8)). Holla's result (9) is wrong.

In case only the first r ordered observations are available from (21), then we proceed as has been done by Holla ([6], pp. 334-335). Incidentally note that we may show that Epstein and Sobel's [1] transformation

$$(27)\quad U_i = (N-i+1)(x_{(i)} - x_{(i-1)}), \quad i = 1, \ldots, r; \quad x_{(0)} = 0,$$

shows that U_i's are independently and identically distributed when x_o is infinite as Holla remarks ([6], p.334, last but one line. Since U_i, $i = 1, 2, \ldots, r$, has exactly the density (21), we see that Holla's second result ([6], p.335, equation (13)) can be deduced from Holla's first result ([6], p.334, equation (9)) by simply substituting r for n and $\bar{u}$ for $\bar{x}$.

Similarly result (14) of Holla's may be derived from result (10), see Holla ([6], pp. 334-335, equations (10) and (14)). It thus follows that if $x_{(1)} = 0$, i.e., the population is truncated at 0, rather than at $x_{(1)}$ as assumed by Laurent, then from (13) we note that an UMVE of the product $(s(a_1)s(a_2)\ldots s(a_k))$ is $(1 - \sum_{j=1}^{k} a_j/r\bar{u})^{r-2}$, where only the first $r > k$ ordered statistics are available from (4), with $m = 0$, and $\bar{u}$ is the mean of the U_i's given by (27). Obviously if $k = 1$ and t is fixed between 0 and x_o, then (26) is Laurent's exponential analogue of Thompson's distribution for the truncated exponential distribution.

6. Power function distributed. Let $x_{(1)} < x_{(2)} < \ldots < x_{(N)}$ be an ordered sample of size N from the power function distribution

$$(28) \qquad f(x) = k\beta^{-k} x^{k-1}, \quad 0 < x < \beta.$$

Then Likes [11] obtains some distributions of the powers of the products and quotients of the ordered values. For this purpose he uses the order statistic theory from the exponential population. It might be interesting to obtain these distributions from the first principles. If both β and k are unknown, then the pair $(x_{(N)}, \log x_{(N)} - \Sigma \log x_{(i)}/N)$ constitutes a complete and sufficient statistic for the pair (β, k). The joint density of this pair may be easily obtained by the procedure we shall outline in this section. Similarly we may obtain the distributions of products and quotients of the ordered value. We simply transform the ordered x variates to the unordered θ variates by the relation

$$(29) \qquad x_{(j)} = \sum_{i=1}^{j} y_i = \theta_j \theta_{j+1} \ldots \theta_N, \quad j = 1, 2, \ldots, N.$$

The Jacobian J of this transformation is given by (3). As first example we shall find the covariance of $x_{(i)}$ and $x_{(j)}$ $i < j$, $i = 1, 2, \ldots, N-1$. Obviously the covariance is

$$(30) \qquad \beta^2 \{E(\prod_{i=1}^{j-1} \theta_i \prod_{j=j}^{N} \theta_j^2) - E(\prod_{i=i}^{N} \theta_i) E(\prod_{j=j}^{N} \theta_j)\}$$

$$= \beta^2 \prod_{i=i}^{j-1} E(\theta_i) \{\prod_{j=j}^{N} [E(\theta_j^2) - (E(\theta_j))^2]\},$$

where the distribution of θ's is

$$(31)\quad f(\theta_1, \theta_2, \ldots, \theta_N) = N!\, k^N \prod_{i=1}^{N} \theta_i^{ik-1}, \quad 0 < \theta_i < 1,\ i = 1, \ldots, N.$$

By using (31) we may easily prove that

$$(32)\qquad E(\theta_i^t) = ik/(ik + t), \quad i = 1, 2, \ldots, N.$$

it follows that

$$(33)\qquad \operatorname{cov}(x_{(i)}, x_{(j)}) = \prod_{i=1}^{j-1} \frac{ik}{ik+1} \prod_{j=j}^{N} \frac{jk}{(kj+2)(jk+1)} .$$

As second example we consider a result of Likes ([11], p. 270), who shows that the statistic

$$(34)\quad 2kN/k = 2kN \log U_2 = 2kN \log(x_{(N)}/(x_{(1)} \cdots x_{(N)})^{1/N}),$$

has a χ^2 distribution with (2N-2) degrees of freedom. Here k represents the maximum likelihood estimator of k. Now we note that the distribution of k does not depend on β, and thus it amounts to finding the distribution of the statistic

$$(35)\qquad z = x_{(N)}/(x_{(1)} \cdots x_{(N)})^{1/N} = \left(\prod_{i=1}^{N-1} \theta_i^{i/N}\right)^{-1},$$

where the θ's have the joint density (31). The moments of z determine its distribution uniquely. The (s-1)-th moment of z, or the Mellin transform of the density function of z, is

$$(36)\qquad E(z^{s-1}) = (Nk)^{N-1}/(Nk + 1 - s)^{N-1}.$$

On inverting the Mellin transform (36), we find that

$$(37)\qquad f(z) = \frac{(Nk)^{N-1}}{(N-1)z} \exp\{-Nk \log z\} (\log z)^{N-2}, \qquad 1 < z < \infty .$$

Thus 2kN log z has a χ^2 distribution with two degrees of freedom.

Dirichlet's multiple integral (and the transformation) has several applications in probability theory and it may not be out of place to show some of its applications to probability theory in this paper. It might perhaps be of at least pedagogical interest.

7. Some other applications. Take for instance the following example considered by Takacs ([18], p. 107, example 5), who

essentially evaluates the integral

$$(38) \qquad \int_R dx_1 dx_2 \ldots dx_N = (x - r\alpha)^N/N!, \quad x > r\alpha,$$

where the range R of integration is determined by the conditions

$$(39) \qquad 0 < x_1 + \ldots + x_N < x, \quad x_i > \alpha, \quad i = 1, 2, \ldots, r.$$

The integral is, of course, evaluated by using (1). Tackas ([18], pp. 107-108) gives an application of integral (38) to the distribution of a linear function of sample values from uniform distribution. The integral (38) may be applied to many classroom type problems in probability and particle counter theory. As an illustration consider the following example, Karlin ([9], p. 264, example 16). If N points are chosen according to the uniform distribution on a line of length L, then the probability that no two points will be closer together than the distance d, $0 < d < L/(N-1)$, is $(L-(N-1)d)^N$. If $0 < x_{(1)} < x_{(2)} < \ldots < x_{(N)} < L$ are the points, then obviously the required probability is

$$(40) \qquad (N!/L^N) \int_R dx_{(1)} \ldots dx_{(N)},$$

where the region R of integration is determined by the conditions

$$(41) \qquad 0 < x_{(1)} < \ldots < x_{(N)} < L, \quad x_{(i)} - x_{(i-1)} > d.$$

By using the transformation $x_{(j)} = \sum_{i=1}^{j} y_i$, $j = 1, \ldots, N$, we find that the integral (40) is equivalent to the integral

$$(42) \qquad (N!/L^N) \int dy_1 \ldots dy_N,$$

the range of integration determined by the condition, $0 < y_1 + \ldots + y_N < L$, and $y_j > d$, $j = 2, \ldots, N$, which is evaluated by using (38) and gives the required probability.

Sometimes the Dirichlet's integral may appear in a disguised form, like the following integral in Parzen ([16], p. 142)

$$(43) \quad G(t) = N! \int_{t_1}^{t-t_2-\ldots-t_N} dx_1 \int_{x_1+t_2}^{t-t_3-\ldots-t_N} dx_2 \int_{x_{N-1}+t_N}^{t} dx_N$$

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$$= N! \int dx_1 \ldots dx_N, \quad 0 < x_1 - t_1 < x_1 - t_1 - t_2 < \ldots < x_N - t_1 - \ldots - t_N < t - t_1 - \ldots - t_N$$

$$= N! \int dz_1 \ldots dz_N, \quad 0 < z_1 < z_2 < \ldots < z_N < t - t_1 - \ldots - t_N$$

$$= N! \int dW_1 \ldots dW_N, \quad 0 < W_1 + \ldots + W_N < t - t_1 - \ldots - t_N, \; W_i > 0, \quad i = 1, \ldots, N$$

which, of course, is a Dirichlet's integral and can be evaluated by using (1).

Some other applications of Dirichlet's integral to particle counter theory may be found in Kabe [8]. Wilks [22] gives some interesting applications of Dirichlet's integral to distribution problems of ordered statistics from an arbitrary continuous distribution.

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