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(p, r)-CONVEX FUNCTIONS ON VECTOR LATTICES

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We study certain convexity-type properties of homogeneous functions on topological vector lattices, focusing on a concept of 0_+-convexity, and using some probabilistic inequalities.

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1. Introduction

In this paper, we study properties of positive homogeneous functions in topological vector lattices. Our interest in such a topic, besides the existence of definite links to the operator and vector lattices theory, stems from the integration with respect to vector measures with values in linear metric spaces that are not necessarily locally convex (cf. [**12**, Chapter 3] for a general overview). In general, one cannot apply the vast technology wrapped around the Hahn–Banach theorem. Instead, of primary interest are the shapes of neighbourhoods of zero. More precisely, the local boundedness, which by the Aoki–Rolewicz Theorem [**1, 11**], is equivalent to the existence of a p-homogeneous metric, for some $p \in (0, 1]$, is a desired property of a metric linear space from the view point of one working with vector measures.

In this paper we consider vector lattices that are topologized with the help of positive homogeneous functions (analogs of Δ-norms, as appear in [**5**]). We define a two-parameter ((p, r), $0 < p \leqq r \leqq \infty$) analog of convexity of such functions, which in particular (for $p = r \leqq 1$) contains the notion of a p-homogeneous norm. The notion of p-convexity used here is exactly Krivine's one [**6**], transplanted from the environment of Banach lattices to that of relatively uniformly complete lattices. This notion is close yet slightly stronger than the same seen in the literature, with no regard to the order structure (cf. [**5**]). Additionally, we introduce a concept of 0_+-convexity, which is precisely a sufficient and necessary property that turns a locally bounded metric vector lattice into a normed lattice, subject to applying a convexification procedure. Since such a procedure is invertible, many derivatives of the Hahn–Banach theorem, allowing use of continuous functionals, can be easily carried over to a class of non-locally convex vector lattices, despite their poor dual structure.

We are interested in relations between various type of convexities. The related formulas can be conveniently expressed with the help of comparison inequalities for Pareto or stable distributions. The idea of using such distributions is very old, and can be traced back to Paul Lévy. However, in the specific context of this paper, we follow some concepts from [**10**, **9**], where Pareto or stable random variables were employed to obtain interpolation-type results for distinct types of convexities of a norm in a Banach lattice.

If homogeneous expressions

$$\left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p},\quad \left(\prod_{i=1}^{n}x_i\right)^{1/n},$$

are given a meaning in a vector lattice, assuming that the first term is just $\bigvee_{i=1}^{n}|x_i|$ if $p=\infty$, then a monotone positive homogeneous real function $\|\cdot\|$ on a vector lattice is said to be (p,r)-*convex* if there is a constant $C^{(p,r)}=C^{(p,r)}(\|\cdot\|)$ such that

$$\left\|\left(\sum_i|x_i|^r\right)^{1/r}\right\|\leqq C^{(p,r)}\left(\sum_i\|x_i\|^p\right)^{1/p},$$

for all finite families $\{x_i\}\subset\mathbb{L}$. Clearly, we must have $p\leqq r$. When $p=r$, we replace the pair "p,p" by "p".

It is seen that no lattice structure is needed for defining $(p,1)$-convex functions $(0<p\leqq 1)$, which are often called p-norms (one can observe in the literature quite a selection of prefixes, like *semi-*, *quasi-*, *pseudo-*, *para-*, etc.).

Say that a postiive function $\|\cdot\|$ is 0^+-*convex* if

$$\left\|\left(\prod_{i=1}^{n}x_i\right)^{1/n}\right\|\leqq C^{(0^+)}\left(\prod_{i=1}^{n}\|x_i\|\right)^{1/n},\quad \{x_i\}\subset\mathbb{L}_+.$$

The paper is organized as follows. In Section 2 we introduce homogeneous functions with values in relatively uniformly complete vector lattices. In Section 3, which might also be of independent interest, we prove the aforementioned comparison inequalities for Pareto and stable distributions. Section 4 presents further relations between convexities, among which an interpolation property is the most important. This section also contains a construction of a locally bounded F-lattice (with a p-homogeneous metric, $0<p<1$) that does not admit a convexification turning it into a Banach space (again, 0_+-convexity is exactly the property that allows such convexification).

2. Homogeneous functions

Throughout this paper $\mathbb{L}$ denotes a real relatively uniformly complete vector lattice and $\|\cdot\|:\mathbb{L}\to\mathbb{R}_+$ is a positive homogeneous monotone function (see the definitions below). For an auxiliary information pertaining to vector lattices we refer to one of basic monographs, e.g., [**8**, **13**]. For a background in F-spaces, we refer to [**12**, **5**].

It is known that, in a Banach lattice, one can devise counterparts of continuous homogeneous functions $f:\mathbb{R}^n\to\mathbb{R}$ (see [**7**, pages 40–41], [**6**], and references there). In function spaces, this can be done pointwise. If a vector lattice is isomorphically embeddable in such a space, then the construciton can be carried over. For example, every Archimedean vector lattice can be identified, via a suitable order and linear isomorphism, with a space of extended (i.e. admitting values $\pm\infty$) continuous function on a certain topological space (the Johnson–Kist spectral representation theorem, cf. [**8**, Theorem 44.4]). However, the newly defined homogeneous functions need not take values in the original vector lattice. In richer structures, like Banach lattices, this problem can be resolved, but almost a verbatim construction can be repeated requiring much weaker properties of $\mathbb{L}$ than that of being a Banach lattice.

A sequence (x_n) of vectors from $\mathbb{L}$ is said to be *relatively uniformly* (r.u.) convergent to a vector $x \in \mathbb{L}$ if $|x_n - x| \leqq \varepsilon_n u$ for some positive decreasing null sequence (ε_n) (cf. [**8**, Chapter 2.16]). $\mathbb{L}$ is called *relatively uniformly complete*, (r.u.c., in short), if it is complete with respect to the relative uniform convergence.

A function $\|\cdot\|: \mathbb{L} \to \mathbb{R}_+$ is said to be *homogeneous* if $\|cx\| = c\|x\|$, $c \geqq 0$, $x \in \mathbb{L}$, and *monotone*, if there exits a constant $M = M(\|\cdot\|) > 0$ such that, for every $x, y \in \mathbb{L}$, $|x| \leqq |y| \Rightarrow \|x\| \leqq M\|y\|$. Note that the continuity of a monotone function $\|\cdot\|$ with respect to the r.u. topology means exactly that the mapping $\mathbb{R} \ni t \mapsto \|x + ty\| \in \mathbb{R}_+$ is continuous for every $x, y \in \mathbb{L}_+$. Further, any homogeneous monotone function is r.u.-continuous.

With the help of a homogeneous monotone function one can introduce a topology on $\mathbb{L}$, using the sets $\{\|x - x_0\| < r\}$ as a base of neighbourhoods at a point x_0. However, in general, vector or lattice operations may be not continuous. On the other hand, we note a simple positive result, which follows immediately from [**4**, Theorem 1.2].

Lemma 2.1. *If the addition (or the operation "$\vee$") is continuous at* 0, *that is, for some* $C > 0$, *for every* $x, y \in \mathbb{L}$,

$$\|x\| \leqq 1, \|y\| \leqq 1 \Rightarrow \|x + y\| \leqq C \quad (\textit{or } \|x + y\| \leqq C \max(\|x\|, \|y\|),$$

then the topology is locally bounded and metrizable (cf. [**4**]. *In particular, for some* $p \in (0, 1]$ $(p = \ln 2/\ln C)$, $\|\cdot\|$ *is* $(p, 1)$*-convex.*

Proposition 2.2. *Let* $\mathbb{L}$ *be a vector lattice equipped with a r.u. topology. If* $\|\cdot\|$ *is monotone, homogeneous, and complete, in the sense*

$$\|x_n - x_m\| \to 0 \Rightarrow \textit{there is a unique } x \in \mathbb{L} \textit{ such that } \|x_n - c\| \to 0,$$

then $\mathbb{L}$ *is r.u.c.*

Proof. It is shown in [**8**, Thm 42.2] that $\mathbb{L}$ is r.u.c. if and only if one of the following equivalent conditions holds:

(r.u.c.) $0 \leqq x_n \leqq \lambda_n x$ *where* $x_n, x \in \mathbb{L}, (\lambda_\alpha) \in l^1_+$, *implies* $\sum_n x_n$ *is order convergent*;

(r.u.c.′) $0 \leqq s_n \leqq u$ *where* $s_n, u \in \mathbb{L}, (s_n)$ *is increasing, implies* $\sup_n s_n$ *exists.*

We will use (r.u.c.′). Let (s_n) be a nonnegative increasing sequence majorized by

$u \in \mathbb{L}_+$. Hence, by monotonicity and completeness of the functional $\|\cdot\|$, $\|s_n - s\| \to 0$ for some $s \in \mathbb{L}_+$. By continuity, if $s_n \leqq v$ for all n, $\|s_n \wedge v - s \wedge v\| \to 0$ but also $\|s_n \wedge v - s\| = \|s_n - s\| \to 0$. Hence, by the uniqueness of the limit, $s = s \wedge v$, i.e., $s \leqq v$. In other words, $s = \sup_n s_n$. □

The r.u.-completeness does not ensure the completeness of a norm (e.g., $C[0,1]$ with $\|f\| = \int_0^1 |f(t)|\,dt$). Note also, that the uniqueness in the definition of a complete homogeneous function is ensured, if the function induces a vector topology.

For two quantities $A = A(\theta)$ and $B = B(\theta)$ depending on some parameter θ, we will write A B ($A \preccurlyeq B$, respectively) if there is a constant $c > 0$ such that $c^{-1}A \leqq B \leqq cA$ (respectively, $A \leqq cB$), uniformly for θ. It will be clear from the context whether or not the constant c depends on additional parameters, like dimension or some index α.

Symbol $\mathbf{1}_{\{\cdot\}}$ stands for the $\{0,1\}$-valued indicator function of a set or a logic statement. For $\alpha > 0$ and $t \in \mathbb{R}$, we denote $t^{\alpha} \stackrel{\mathrm{df}}{=} \operatorname{sgn}(t)|t|^{\alpha}$.

Random variables appearing in this paper are defined on a common probability space $(\Omega, \mathscr{F}, \mathbf{P})$ which is assumed to be sufficiently rich to carry all needed sequences of independent random variables. For a positive random variable X, we will write $\mathbf{E}X = \int_{\Omega} X(\omega)P(d\omega)$, $\mathbf{E}[X; A] = \int_A X\,dP$, and $\|X\|_p = (\mathbf{E}X^p)^{1/p} (p > 0)$. Define

$$\|X\|_{0^+} \stackrel{\mathrm{df}}{=} \exp\{\mathbf{E} \ln X\},$$

and note that, if X^q, for some $q > 0$, and $\ln X$ are integrable (neither condition implies the other), then $\|X\|_{0^+} = \lim_{q \to 0} (\mathbf{E}X^q)^{1/q}$.

Let $\mathscr{H}_n$ be the vector sublattice of the vector lattice $C(\mathbb{R}^n)$ of continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ spanned by the "projections" $f_i(t_1, \ldots, t_n) = t_i, i = 1, \ldots, n$. Functions f from $\mathscr{H}_n$ are continuous and homogeneous of degree 1, i.e.

$$f(\lambda t_1, \ldots, \lambda t_n) = \lambda f(t_1, \ldots, t_n), \qquad \lambda \geqq 0.$$

The completion $\widehat{\mathscr{H}_n}$ of $\mathscr{H}$ with respect to the norm $\|f\|_H = \sup\{|f(\underline{t})| : \max_i |t_i| = 1\}$ can be identified with the Banach lattice of all continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ which are homogeneous of degree 1.

Theorem 2.3. *Let $\mathbb{L}$ be r.u.c., $n \geqq 1$, and $\underline{x} = (x_i)_{i=1}^n \in \mathbb{L}^n$. Then there exists a unique linear mapping $\tau = \tau_{\underline{x}}$: $\widehat{\mathscr{H}_n} \to L$ which preserves lattice operations and such that $\tau f_i = x_i$, $i = 1, \ldots, n$. Moreover,*

$$|\tau_{\underline{x}} f| \leqq \|f\|_H \bigvee_i |x_i|. \tag{2.1}$$

Proof. First, consider the case when $\mathbb{L} = \mathbb{C}(\mathbb{S})$, where S is a compact space. Let $\phi_1, \ldots, \phi_n \in C(S)$ and put $\phi_0 = \sup_i |\phi|$. Define a linear mapping $\tau_0: \widehat{\mathscr{H}_n} \to C(S)$ by the formula $\tau_0 f(s) = f(\phi(s), \ldots, \phi(s))$, $f \in \widehat{\mathscr{H}_n}$. Clearly, $\tau_0 f = \tau_0 g$ whenever $f = g$, $\tau f \vee g = \tau f \vee \tau g$, and $|\tau_0 f| \leqq \|_H |\phi_0|$, $f \in \widehat{\mathscr{H}_n}$. The latter inequality means that, for every $f \in \widehat{\mathscr{H}_n}$, $\tau_0 f$ belongs to the ideal $I(\phi_0) = \{\phi \in C(S) : |\phi| \leqq \lambda |\phi_0|$ for some $\lambda > 0\}$.

Consider the general case. Let $x_0,\dots,x_n\in\mathbb{L}$ and put $x_0=\bigvee_i|x_i|$. Denote by $I(x_0)$ the ideal in $\mathbb{L}$ generated by x_0. The completion $\widehat{I(x_0)}$ with respect to the M-norm on $I(x_0)$, $\|x\|=\inf\{\lambda:|x|\leqq\lambda x_0\}$, is order isometric to $C(S)$ for some compact space S (cf. [7, I.b.6]). Viewing $x_0,x_1,\dots,x_n$ as continuous functions on S, we can define a linear mapping $\tau:\mathscr{H}_n\to I(x_0)\subset C(S)$. More formally, if $\phi:\widehat{I(x_0)}\to C(S)$ denotes the aforementioned order isometry, Ψ denotes the mapping ϕ^{-1} restricted to $I(\phi(x_0))$, and $i_{\mathbb{L}}$ is the embedding of $I(x_0)$ into $\mathbb{L}$, then $\tau=i_{\mathbb{L}}\circ\Psi\circ\tau_0$. Obviously, τ preserves lattice operations and

$$|\tau f|\leqq\|f\|_H|x_0|,\qquad f\in\mathscr{H}. \tag{2.2}$$

Now, we will extend τ to the entire space $\widehat{\mathscr{H}_n}$. Let $\|f_n-f\|_H\to 0$. Then the sequence (τf_n) is relatively uniformly Cauchy, and hence, by the completeness assumption, r.u.-converges to a unique element in $\mathscr{L}$. The estimate (2.2) continues to hold on the entire $\widehat{\mathscr{H}_n}$. □

In the sequel, we will write simply $f(\underline{x})=\tau_{\underline{x}}f$, $f\in\widehat{\mathscr{H}_n}$. In particular, for every pair $f,g\in\widehat{\mathscr{H}_n}$

$$f(\underline{t})\leqq g(\underline{t}),\quad \underline{t}\in\mathbb{R}^n \Rightarrow f(\underline{x})\leqq g(\underline{x}),\quad \underline{x}\in\mathbb{L}^n. \tag{2.3}$$

Let $x_1,\dots,x_n\in\mathbb{L}$, $0<p<\infty$, $p_i\geqq 0$, $i=1,\dots,n$, $\sum_i p_i=1$. Then the expressions

$$\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}\quad\text{and}\quad |x|^{p_1}\cdots|x_n|^{p_n}, \tag{2.4}$$

are well defined in $\mathbb{L}$, and, by virtue of (2.3), any inequalities or identities, which are valid in the real case, carry over to the lattice context. Hölder's inequality is a typical example.

If convexity or monotonicity constants of a homogeneous function $\|\cdot\|$ are finite, then one can always find an equivalent (by means of the relation $\asymp$) function such that the underlying constants are equal to 1. More precisely:

(i) If $\|\cdot\|$ is a monotone function on $\mathbb{L}$ then $M(\|\cdot\|')=1$ and $1/M(\|\cdot\|)\|x\|\leqq\|x\|'\leqq\|x\|$, where

$$\|x\|'=\sup\{\|y\|:|y|\leqq|x|\};$$

(ii) If $\|\cdot\|$ is (p,r)-convex, $0<r<p\leqq\infty$, then $C^{(p,r)}(\|\cdot\|')=1$ and $1/C^{(p,r)}(\|\cdot\|)\|x\|\leqq\|x\|'\leqq\|x\|$, where

$$\|x\|'\overset{\mathrm{df}}{=}\inf\left\{\left(\sum_i\|x_i\|^r\right)^{1/r}:\left(\sum_i|x_i|^p\right)^{1/p}=x,\,x_i\in\mathbb{L}\right\};$$

(iii) If $\|\cdot\|$ is 0^+-convex, then $C^{(0^+)}(\|\cdot\|')=1$ and $1/C^{(0^+)}(\|\cdot\|)\|x\|\leqq\|x\|'\leqq\|x\|$, where

$$\|x\|' \overset{\mathrm{df}}{=} \inf\left\{\prod_i \|x_i\|^{p_i}: \prod_i |x_i|^{p_i}=x, x_i\in\mathbb{L}, p_i\geqq 0, \sum_i p_i=1\right\}.$$

We omit a routine argument.

3. Some uses of Pareto random variables

For $0<\alpha<\infty$, let X_α denote a positive α-Pareto random variable, i.e., with the density $\alpha x^{-1-\alpha}\mathbf{1}_{\{x\geqq 1\}}$. Throughout the section, U will stand for a random variable uniformly distributed on $[0,1]$. Notice that we may choose $X_\alpha=U^{-1/\alpha}$. Denote by $Z_{\gamma,i}$, $i\in\mathbb{N}$, independent copies of a positive γ-stable random variables Z_γ, $0<\gamma<1$, i.e., with the Laplace transform $\mathbf{E}\exp\{-tZ_\gamma\}=\exp\{-t^\gamma\}$, $t\geqq 0$. In the sequel, parameters appearing in subscripts may be dropped for the sake of clarity.

Remark 1. The distribution of an α-Pareto random variable, $0<\alpha<1$, belongs to the domain of normal attraction of a positive α-stable distribution, i.e. the probability law of $n^{-1/\alpha}\sum_i X_i$ converges weakly to the positive α-stable law. Moreover, such normalized sums are bounded in L_p for every $p<\alpha$, hence their qth powers, where $q<\alpha$, are uniformly integrable.

Proposition 3.1. *Let $X=X_\alpha$ be an α-Pareto random variable, and $\|\cdot\|$ be a positive homogeneous monotone function on $\mathbb{L}$. Then the following statements hold*

(i)
$$\|\,\|x+Xy\|\,\|_{0^+}\geqq(\|x\|^\alpha+\|y\|^\alpha)^{1/\alpha},\qquad x,y\in\mathbb{L}_+. \tag{3.1}$$

(ii) *If $0^+\leqq q<\alpha<r<\infty$, then there exists a universal constant $b=b(\alpha,q,r)$ such that*

$$\|(s^r+X^rt^r)^{1/r}\|_q\leqq(s^\alpha+bt^\alpha)^{1/\alpha},\qquad s,t\in\mathbb{R}_+.$$

Proof. (i): Assume that $\|x\|=\|y\|=1$. Notice that

$$\alpha\mathbf{E}\ln X_\alpha=\alpha^2\int_1^\infty u^{-1-\alpha}\ln u\,du=1$$

Let $t\leqq 1$. We have

$$\begin{aligned}\alpha\mathbf{E}\ln\|x+tXy\| &\geqq \alpha\mathbf{E}[\ln\|x+tXy\|; tx>1]\\ &\geqq \alpha\mathbf{E}[\ln(tX); tX>1]\\ &= t^\alpha\alpha\mathbf{E}\ln X\\ &\geqq \ln(1+t^\alpha).\end{aligned}$$

Now let $t>1$.

$$\begin{aligned}\alpha \mathbf{E} \ln \|x+tXy\| &\geqq \alpha \mathbf{E}[\ln(tX)]\\ &=\ln(t^{\alpha}\exp\{\alpha \mathbf{E}\ln X\})\\ &\geqq \ln(t^{\alpha}(\alpha \mathbf{E}\ln X+1))\\ &\geqq \ln(1+t^{\alpha}).\end{aligned}$$

To prove (ii), it is enough to consider the function $(\mathbf{E}(1+t^rX^r)^{q/r})^{1/q}$, $t>0$, where $r>\alpha$. Substitute $\beta=\alpha/r<1$, $p=q/r<\beta$, $u=t^{\alpha}$ (hence $t^r=u^{1/\beta}$), $X_\beta=X^r=U^{-1/\beta}$. Thus it suffices to prove that $(\mathbf{E}(1+u^{1/\beta}X_\beta)^p)^{\beta/p}\leqq 1+bu$ for $u>0$, where $0<p<\beta<1$. First, we consider the case $u\leqq 1$. The left hand side of the above inequality can be estimated as follows.

$$\begin{aligned}(\mathbf{E}(1+u^{1/\beta}Y_\beta)^p)^{\beta/p} &=\left(1+u\int_{u^{1/\beta}}^{\infty}\frac{(1+v)^p-1}{v^{1+\beta}}\beta\,dv\right)^{\beta/p}\\ &\leqq 1+u\int_{0}^{\infty}\frac{(1+v)^p-1}{v^{1+\beta}}\beta\,dv\Bigg)^{\beta/p}\\ &=(1+uc')^{\beta/p},\end{aligned}$$

where $c'=\int_0^\infty((1+v)^p-1/(v^{1+\beta}))\beta\,dv$. Since $\beta/p>1$, the function $u\mapsto(1+uc')^{\beta/p}$, $u\in[0,1]$, is convex. Therefore the estimate

$$(1+uc')^{\beta/p}\leqq 1+b'u,\qquad u\in[0,1],$$

holds if we take $b'=(1+c')^{\beta/p}-1$.

Consider then the case $u\geqq 1$. The left hand side of the above inequality can be estimated as follows. Since $p<1$, we have

$$(\mathbf{E}(1+u^{1/\beta}Y_\beta)^p)^{\beta/p}\leqq(1+u^{p/\beta}\mathbf{E}Y_\beta^p)^{\beta/p}=(1+u^{p/\beta}c'')^{\beta/p},$$

where $c''=\mathbf{E}Y_\beta^p=\beta/(\beta-p)$. Since $p/\beta<1$. the function $u\mapsto(1+u^{p/\beta}c'')^{\beta/p}$, $u\in[1,\infty]$, is concave. Therefore the estimate

$$(1+u^{p/\beta}c'')^{\beta/p}\leqq 1+b''u,\qquad u\in[1,\infty]$$

holds if we take $b''=(1+c'')^{\beta/p}-1$.

Finally, we put $b=\max(b',b'')$. Notice that c' and c'' do not majorize each other (hence b' and b'' do not, either). Indeed, $c'\to\infty$ when $\beta\to 1$ while c'' remains bounded. On the other hand $c'\to 0$ while $c''\to 1$, as $p\to 0$. □

The following result contains estimates of moments of combinations of Pareto random variables. It is a strengthened version of a classical relation (cf., e.g., [7, Lemma I.f.8] or [9, Proposition 1.1]). Besides the use of a general function $\|\cdot\|$, the key point lies in the augmentation of the standard family of L_p-norms, $p>0$, by the functional $\|\cdot\|_{0^+}$. This requires an exact evaluation of the appropriate constants.

Theorem 3.2. *Let $X_i=X_{\alpha,i}$ be independent copies of an α-Pareto random variable X_α, $0<\alpha<\infty$, and $\|\cdot\|$ be a positive homogeneous monotone function on $\mathbb{L}$. Then*

(i) $\|\|\sum_i x_iX_i\|\|_{0^+}\geqq(\sum_i\|x_i\|^\alpha)^{1/\alpha}, \qquad x_i\geqq 0,$

(ii) *if $0^+\leqq q<\alpha<r$, then, for some $b=b(\alpha,q,r)$,*

$$\left\|\left(\sum_i t_i^r X_i^r\right)^{1/r}\right\|_q \leqq b\left(\sum_i t_i^\alpha\right)^{1/\alpha}, \qquad t_i\in\mathbb{R}_+.$$

Proof. (i) We will apply an inductive argument (compare [14]). We will show that

$$\exp\left\{\mathbf{E}\ln\left\|x+\sum_i x_iX_i\right\|\right\}\geqq\left(\|x\|^\alpha+\sum_i\|x_i\|^\alpha\right)^{1/\alpha}, \qquad x_i\geqq 0. \tag{3.2}$$

Let $X_1,\ldots,X_n$ be independent α-Pareto random variables. We may assume that these random variables are defined on a product probability space. Denote by $\mathbf{E}_{n-1}$ (respectively, $\mathbf{E}_n$) the expectation with respect to the probability spaces that the $n-1$ first variables (respectively, X_n) are defined on. Notice that, for positive random variables A and B,

$$(\mathbf{E}(A^\alpha+B^\alpha)^{q/\alpha})^{1/q}\geqq((\mathbf{E}A^q)^{\alpha/q}+(\mathbf{E}B^q)^{\alpha/q})^{1/\alpha}, \tag{3.3}$$

(in other words, l^α-norm is q-convex), hence, by letting $q\to 0$,

$$\exp\{\mathbf{E}\ln(A^\alpha+B^\alpha)^{1/\alpha}\}\geqq(\exp\{\alpha\mathbf{E}\ln A\}+\exp\{\alpha\mathbf{E}\ln B\})^{1/\alpha}. \tag{3.4}$$

By virtue of Fubini's theorem and by (3.4) and (3.1), we have

$$\begin{aligned}\exp\left\{\mathbf{E}\ln\left\|x+\sum_{i=1}^n x_iX_i\right\|\right\}&=\exp\left\{\mathbf{E}_{n-1}\mathbf{E}_n\ln\left\|x+\sum_{i=1}^{n-1}x_iX_i+x_nX_n\right\|\right\}\\&\geqq\exp\left\{\mathbf{E}_{n-1}\ln\left(\left\|\sum_{i=1}^{n-1}x_iX_i\right\|^\alpha+\|x_n\|^\alpha\right)^{1/\alpha}\right\}\\&\geqq\exp\left\{\alpha\mathbf{E}\ln\left\|x+\sum_{i=1}^{n-1}x_iX_i\right\|\right\}+\|x_n\|^\alpha\Big)^{1/\alpha}\end{aligned}$$

It is clear how the induction works. After completing the induction, we put $x=0$.

We will prove (ii) using a more elaborate argument (note that, basically, one needs an inequality similar to (3.3) but, unfortunately, its direction is not appropriate for purposes of this proof). By a suitable substitution, we may reduce the proof to the case $\alpha<1$. Choose any $s<1$, and put

$$\beta=\frac{s\alpha}{r}, \quad p=\frac{sq}{r}, \quad v_i=t_i^{\alpha/\beta}=t_i^{r/s}.$$

Consider $X_{\beta,i}=U_i^{-1/\beta}$. Then the inequality in (ii) takes the form

$$\left\|\left(\sum_i |v_i X_{\beta,i}|^s\right)^{1/s}\right\|_p \leqq \left(b\sum_i v_i^\beta\right)^{1/\beta}. \qquad v_i\in\mathbb{R}_+, \tag{3.5}$$

where $0<p<\beta<s<1$.

By the classical standardization of stable distributions, (3.5) will follow from the inequality

$$\mathbf{E}\left|\sum_i v_i X_{\beta,i} Z_{s,i}\right|^p \preccurlyeq \mathbf{E}\left|\sum_i v_i Z_{\beta,i}\right|^p, \tag{3.6}$$

which, in turn, can be derived from the estimate

$$\mathbf{E}\,|1+vX_\beta Z_s|^p \leqq \mathbf{E}\,|1+cvZ_\beta|^p, \tag{3.7}$$

where a constant $c>0$ does not depend on v.

Since the distributions of Pareto random variables belong to the domain of normal attraction of a stable distribution (cf. [3]), then by virtue of continuity of the function $t\mapsto\|x+ty\|$, in Proposition 3.1 one can replace Pareto by stable random variables (cf. Remark 1 and [2, Theorem 5.4]).

The sought-for estimate (3.7) follows from the inequality

$$\|(1+bv^s X_\beta^s)^{1/s}\|_p \leqq (1+v^\beta)^{1/\beta},$$

which holds for suitable $b=b(\beta,s,p)$. Up to the appropriate adjustment of the constants, this is the inequality appearing in Proposition 3.1(ii). □

Statements (i) in the above proposition and theorem are essentially proved in the case $\mathbb{L}=l^\infty$ and $\|\cdot\|$ being the supremum norm, since $\|\sum_i t_i x_i\| \geqq \max_i t_i \|x_i\|$. Therefore, the lower bounds for combinations of Pareto random variables have a universal meaning. They are independent on the choice of a particular homogeneous function $\|\cdot\|$ or an L^p-norm even though, in some particular cases, one can obtain much better estimates. For instance, when $\mathbb{L}=\mathbb{R}$, $\|x\|=|x|$, and $\alpha>p>1$, then

$$\|1+tX_\alpha\|_p \geqq \|1+tX_\alpha\|_1 = 1+\alpha/(\alpha-1)t, \qquad t \geqq 0,$$

which immediately yields the bound

$$\left\|\sum_i t_i X_{\alpha,i}\right\|_p \geqq \alpha/(\alpha-1)\sum_i t_i, \qquad t_i \geqq 0.$$

As a by-product, the following result allows us to derive a number of universal lower bounds for certain combinations of X_i's and positive vectors from $\mathbb{L}$.

Proposition 3.3. *Let $(X_i : i=1,\ldots,n)=(X_{\alpha,i} : i=1,\ldots,n)$ be a finite sequence of i.i.d. α-Pareto random variables and $0<p<\infty$, Then*

(i) $$\left\|\left\|\left(\sum_i |x_i X_i|^p\right)^{1/p}\right\|\right\|_{0^+} \geqq \left(\sum_i \|x_i\|^\alpha\right)^{1/\alpha}, \qquad x_i \in \mathbb{L}.$$

(ii) *Let $\psi : \mathbb{R}^n \to \mathbb{R}_+$ be a positive homogeneous monotone function. Then, for the $\mathbb{L}$-valued counterpart of ψ, the universal lower estimate*

$$\| \|\psi(x_1 X_1,\ldots,x_n X_n\| \|_{0^+} \succcurlyeq \left(\sum_i \|x_i\|^\alpha\right)^{1/\alpha}, \qquad x_i \in \mathbb{L}. \tag{3.8}$$

holds if and only if

$$\psi(\underline{x}) \succcurlyeq \bigvee_i |x_i|. \tag{3.9}$$

The constant, implicit in relation "$\succcurlyeq$", may depend only on ψ.

Proof. That the constant in the estimate is independent of α (and n) is essential. Statement (i) immediately follows from Theorem 3.2(i) by convexification. That is, it is enough to note that, for $p \in (0,\infty)$,

$$\|(x^p+y^p)^{1/p}\| = \|x \oplus_{1/p} y\|_{(p)}^p, \qquad x, y \in \mathbb{L}_+.$$

In order to prove the sufficient condition in statement (ii), it suffices to approximate, for a fixed n, $\bigvee |x_i|$ by $(\sum_i |x_i|^p)^{1/p}$ as $p \to \infty$. Hence (3.8) follows from the inequality

$$\left\|\left\|\bigvee_i |x_i X_i|\right\|\right\|_{0^+} \geqq \left(\sum_i \|x_i\|^\alpha\right)^{1/\alpha}, \qquad x_i \in \mathbb{L},$$

and (3.9).

In order to prove the necessity, we may confine ourselves to the case $\mathbb{L}=\mathbb{R}$. Inequality (3.8) yields

$$\left\|\bigvee_i X_{\alpha,i}\right\|_{0^+}\psi(t_1,\ldots,t_n)\geqslant\left(\sum_i t_i^\alpha\right)^{1/\alpha},\qquad t_i\in\mathbb{R}_+.$$

However, since $\|V^c\|_{0^+}=\|V\|_{0^+}^c$ and $X_\alpha=X_1^{1/\alpha}$, then

$$\left\|\sup_i X_{\alpha,i}\right\|_{0^+}=\left\|\sup_i X_{1,i}\right\|_{0^+}^{1/\alpha}.$$

Therefore,

$$\left\|\sup_i X_{1,i}\right\|_{0^+}^{1/\alpha}\psi(t_1,\ldots,t_n)\geqslant\left(\sum_i t_i^\alpha\right)^{1/\alpha},\qquad t_i\in\mathbb{R}_+,$$

and it is enough to let $\alpha\to\infty$. Notice that the same necessary condition appears in the case of an arbitrary norm $\|\cdot\|_p$, that might replace $\|\cdot\|_{0^+}$, where $0<p<\alpha$. In the last step of the proof we would rather use the Dominated Convergence Theorem, $\lim_{\alpha\to\infty}\|V^{1/\alpha}\|_p=1$. □

4. Relations between convexities

Recall that $\|\cdot\|$ is a monotone positive homogeneous real function on a relatively uniformly complete vector lattice $\mathbb{L}$.

4.1. Montonicity

It is known that, for $r\geqq s\geqq 1$ and a Banach lattice $\mathbb{L}$, if $\mathbb{L}$ is r-convex then $\mathbb{L}$ is s-convex (cf. [7, Proposition 1.d.5]. We will extend this property to our context.

Theorem 4.1. *Let $0<p\leqq r\leqq\infty$. If $r\leqq r_1\leqq\infty$ and $0<p_1\leqq p$, or if*

$$0<r_1\leqq r, 0<p_1\leqq r_1\quad\text{and}\quad 1/p_1-1/r_1=1/p-1/r$$

then $C^{(p_1,r_1)}\leqq C^{(p,r)}$ for every positive homogeneous monotone function $\|\cdot\|$. In particular, the function $p\mapsto C^{(p)}$ is nondecreasing on its domain $\{p\geqq 0^+:C^{(p)}<\infty\}$. This means that $\|\cdot\|$ is 0^+-convex whenever is p-convex for some $p>0$.

We need two auxiliary results.

Lemma 4.2. *If a monotone function $\|\cdot\|$ on $\mathbb{L}$ is 0^+-convex then*

$$\left\||x_1|^{p_1}\ldots|x_n|^{p_n}\right\|\leqslant\|x_1\|^{p_1}\ldots\|x_n\|^{p_n},(x_i\}\subset\mathbb{L},p_i\geqq 0,\sum_i p_i=1.\tag{4.1}$$

Proof. Suppose $\|\cdot\|$ is 0^+-convex. Then

$$\|x_1^{r_1}\dots x_n^{r_n}\| \leqq (C^{(0^+)})^2\|x_1\|^{r_1}\dots\|x_n\|^{r_n}, \{x_i\}\subset \mathbb{L}_+,$$

for every selection of rational nonnegative numbers $r_i s$ where $\sum_i r_i = 1$. Indeed, write $r_i = k_i/l_i$, and $m_i = \prod_j l_j k_i/l_i$, so $\sum_i m_i = \prod l_i \stackrel{\text{df}}{=} L$. Then

$$\prod_i x_i^{r_i} = \left(\prod_i x_i^{m_i}\right)^{1/L},$$

and the inequality follows by 0^+-convexity.

Now, given, $x_1,\dots,x_n \in \mathbb{L}_+$, put $x_0 = \sup_i x_i$, and, for an arbitrary discrete distribution (p_i), choose nonnegative rational numbers $r_i \leqq p_i$. Put $r = \sum_i r_i$. Then

$$\begin{aligned}\|x_1^{p_i}\dots x_n^{p_n}\| &= \|(x_1^{r_1/r}\dots x_n^{r_n/r})^r(x_1^{(p_1-r_1)/(1-r)}\dots x_n^{(p_n-r_n)/(1-r)})^{1-r}\|\\ &\leqq \|(x_1^{r_1/r}\dots x_n^{r_n/r})^r x_0^{1-r}\|\\ &\leqq C^{(0^+)}\|x_1^{r_1/r}\dots x_n^{r_n/r}\|^r\|x_0\|^{1-r}\\ &\leqq (C^{(0^+)})^{1+r}\|x_1\|^{r_1}\dots\|x_n\|^{r_n}\|x_0\|^{1-r}.\end{aligned}$$

Finally, let $r_i \to p_i, i = 1,\dots,n$. □

For a discrete probability distribution $\underline{p} = (p_1,\dots,p_n)$, define its entropy

$$H_n(\underline{p}) \stackrel{\text{df}}{=} -\sum_i p_i \ln p_i.$$

Lemma 4.3. *Let $\|\cdot\|:\mathbb{L}\to\mathbb{R}_+$ be a monotone homogeneous (p,r)-convex function on $\mathbb{L}, 0<p\leqq r\leqq\infty$, with a constant $C^{(p,r)}$. Let $\underline{p}=(p_1,\dots,p_n)$ be a discrete probability distribution, i.e. $p_i\geqq 0, i=1,\dots,n, \sum_i p_i = 1$. Then*

$$\left\|\prod_{i=1}^n x_i^{p_i}\right\| \leqq C^{(p,r)}\exp\{(1/p-1/r)H(\underline{p})\}, \qquad x_i\geqq 0, \qquad \|x_i\| = 1, i=1,\dots,n.$$

Proof. The following inequalities, involving homogeneous functions, can be verified in the real context, and subsequently carried over to the lattice case. If $\sum_i u_i p_i = 0$ then

$$\begin{aligned}x_1^{p_1}\dots x_n^{p_n} &= ((x_1^r e^{ru_1})^{p_1}\dots(x_n^r e^{ru_n})^{p_n})^{1/r}\\ &\leqq \left(\sum_i (p_i^{1/r}e^{u_i}x_i)^r\right)^{1/r}.\end{aligned}$$

Since $\|\cdot\|$ is monotone and (p,r)-convex, then

$$\|x_1^{p_1}\dots x_n^{p_n}\|\leqq C^{(p,r)}\left(\sum_i p_i^{p/r}e^{pu_i}\|x_i\|)^p\right)^{1/p}.$$

Computing the unique minimum, over the set $\{\underline{u}:\sum_i u_ip_i=0\}$, of the strictly convex function on the right hand side, we obtain the sought-for estimate.

Proof of Theorem 4.1. The part of the statement corresponding to the case $r\leqq r_1\leqq\infty$ and $0<p_1\leqq p$ is trivial.

Consider the other case. Let $r<\infty$. Put $\alpha=r/r_1$ and let β be such that $1/\beta+1/\alpha=1$, i.e. $\beta=r/(r-r_1)$. Put

$$a_i=\|x_i\|^{p_1/p-1}.$$

Then, using Hölder's inequality, for $x_i\in\mathbb{L}$ we have

$$\begin{aligned}\|(\sum|x_i|^{r_1})^{1/r_1}\|&=\|(\sum|x_ia_i|^{r_1}a_i^{-r_1})^{1/r_1}\|\\&\leqq\|(\sum|x_ia_i|^{\alpha r_1})^{1/\alpha}(\sum a_i^{-\beta r_1})^{1/r_1}\|\\&=\|(\sum|x_ia_i|^{r})^{1/r}\|(\sum a_i^{(-rr_1/(r-r_1))})^{(r-r_1)/(rr_i)}\\&\leqq C^{(p,r)}(\sum\|x_ia_i\|^p)^{1/p}(\sum a_i^{-rr_1/(r-r_1)})^{(r-r_1)/(rr_1)}\\&=C^{(p,r)}(\sum\|x_i\|^{p_1})^{1/p_1}.\end{aligned}$$

Let now $r=\infty$. The above proof works exactly the same way, if we consider $\alpha=\infty$, $\beta=1$ and interpret $(\sum_i|x_i|^r)^{1/r}$ as $\sup_i|x_i|$. Hence the required monotonicity relation between the convexity constants is fulfilled.

(ii) The first part of the statement is obvious. The second part follows immediately from Lemma 4.3 by putting $p=r$. □

In the proof of Lemma 4.3, if one assumed additionally that $\|\cdot\|$ is "$\downarrow$-continuous" (i.e. $x_n\downarrow x\geqq 0\Rightarrow\|x_n\|\downarrow\|x\|$), then, in the inequality defining (p,r)-convexity, one could use positive numerical multipliers $p_i^{1/r}$ of x_i's such that $\sum_i p_i=1$. They would appear in the right hand side in the form $p_i^{p/r}$. Now, using the montonicity proposition, and letting $r\to 0$, the required limit estimate would follow. However, an example following Proposition 4.4 shows a norm lacking this property.

The estimate obtained in Lemma 4.3 is sharp, i.e., the bound is attained for a concrete lattice and function, which can be seen from the following construction.

Example 1. (**An s-norm failing 0^+-convexity**). We will construct an $(\infty,1)$-convex (in particular, $(1,\frac{1}{2})$-convex) positive monotone homogeneous functional which is not 0^+-convex, hence this functional cannot be q-convex for any $q>0$. By convexification, for any $a>0$, we can modify the construction to obtain the corresponding counterexample of (∞, a)-convex (in particular, $(1, a/(1+a)$-convex) functional which is not q-convex for any $q \geqq 0^+$.

Let $\mathbb{L} = \mathbb{R}^n$ with the coordinatewise ordering. Let $\{u_i : 1 \leqq i \leqq n!\}$ be the set of all vectors whose coordinates are permutations of $\{1, \ldots, n\}$. Define

$$U = \bigcup_i [0, u_i].$$

Let $\|\cdot\| = \|\cdot\|_U$ be the gauge function. It is easy to check that $\|\cdot\|$ is positive homogeneous and monotone. We will prove that

$$\sup_i p_i u_i \in U, \qquad p_i \geqq 0, \sum_i p_i = 1 \tag{4.2}$$

and

$$\left(\prod_i u_i\right)^{1/n!} \quad n(1, \ldots, 1). \tag{4.3}$$

Relation (4.3) follows easily, since $(n!)^{1/n} \quad n$ and

$$\left(\prod_i u_i\right)^{1/n!} = (1^{(n-1)!} \cdot 2^{(n-1)!} \ldots n^{(n-1)!})^{1/n!}(1, \ldots, 1) = (n!)^{1/n}(1, \ldots, 1),$$

while $\|(1, \ldots, 1)\| = 1$.

The proof of (4.2) is more elaborate though still elementary. Let $\underline{p}$ be a probability distribution on $\{1, 2, \ldots, n!\}$. Without loss of generality we may assume that $p_1 \geqq p_2 \geqq \ldots \geqq p_{n!}$. We will find a specific enumeration $\bar{u}_1, \bar{u}_2, \ldots$ of the family $\{u_i\}$ such that

$$\max_i p_i u_i \leqq v \overset{\text{df}}{=} \max_i p_i \bar{u}_i.$$

As $\bar{u}_1$, we choose a vector with n on the first coordinate. So, $v_1 = np_1$. Other coordinates will be chosen later. As $\bar{u}_2$, choose a vector with the first two coordinates equal $(1, n)$, and then choose $(n-1)$ as the second coordinate of $\bar{u}_1$. Now, $v_2 = (n-1)p_1 \vee np_2$. As $\bar{u}_3$, choose a vector whose first three coordinates are $(1, 2, n)$, then choose $(n-2)$ as the third coordinate of $\bar{u}_1$ and $(n-1)$ as the third coordinate of $\bar{u}_2$. Hence $v_3 = (n-2)p_1 \vee (n-1)p_2 \vee np_3$.

Continuing, the vector v will have the following coordinates:

$$v_k = (n-k+1)p_1 \vee (n-k+2)p_2 \vee \ldots \vee (n-1)p_{k-1} \vee np_k, \qquad k = 1, \ldots, n.$$

Since $1 \geqq p_1 + \cdots + p_k$, then, by monotonicity, $p_k \leqq 1/k$.

Hence

$$(n-k+i)p_i \leqq \frac{n-k+i}{i} \leqq n-k+1.$$

In other words,

$$v \leqq (n, n-1, \ldots, 2, 1),$$

which proves (4.2).

4.2. Interpolation of convexities

By virtue of the following result, distinct convexities can be interpolated. A Banach lattice version of this result is known ([7, Theorem 1.f.7], see also [**10**, **9**]). Following the same idea, we will also use Theorem 3.2 containing some comparison inequalities for combinations of independent Pareto random variables, and providing sharper constants (we need them) than those appearing in [7].

We will employ some probabilistic techniques. Let $\Psi: \mathbb{R}^n \to \mathbb{R}_+$ be a continuous positive homogeneous function and $\theta_1, \ldots, \theta_n$ be real random variables. Define $\Theta = \Psi(\theta_1 x_1, \ldots, \theta_n x_n)$. By Theorem 2.3, the following $\mathbb{L}$-valued homogeneous functions are well defined

$$(\mathbf{E}\Theta^p)^{1/p}, \qquad \exp\{\mathbf{E}\ln\Theta\}.$$

Proposition 4.4. *Let $\|\cdot\|$ be a positive homogeneous monotone function. Let $p \geqq 0^+$ and $\|\cdot\|$ be p-convex. Then*

$$\| \|\Theta\|_p \| \leqq C^{(p)} \| \|\Theta\| \|_p.$$

Proof. We will prove the statement only in the case $p = 0^+$, which is slightly more difficult than the case $p > 0$, though the proof are similar. For any random event $A \subset \Omega$ we have

$$\begin{aligned}
\|\exp\{\mathbf{E}\ln|\Theta|\}\| &= \|\exp\{\mathbf{E}[\ln|\Theta|; A] + \mathbf{E}[\ln|\Theta|; A^c]\}\| \\
&= \|\exp\{\mathbf{P}(A)\,\mathbf{E}[\ln|\Theta|; A]/\mathbf{P}(A) + \mathbf{P}(A^c)\mathbf{E}[\ln|\Theta|; A^c]/\mathbf{P}(A^c)\}\| \\
&\leqq \|\exp\{\mathbf{E}[\ln|\Theta|; A]/\mathbf{P}(A)\}\|^{\mathbf{P}(A)} \|\exp\{\mathbf{E}[\ln|\Theta|; A^c]/\mathbf{P}(A^c)\}\|^{\mathbf{P}(A^c)},
\end{aligned}$$

by 0^+-convexity. Also, putting $x_0 = \sup_i |x_i|, \theta_0 = \max_i \ln_+ |\theta_i|$, we have

$$\exp\{\mathbf{E}[\ln|\Theta|; A]/\mathbf{P}(A)\} \leqq \psi(x_0)\exp\{\mathbf{E}[\theta_0; A]/\mathbf{P}(A)\}$$

Combining both observations, we obtain

$$\|\exp\{\mathbf{E}\ln|\Theta|\}\| \leqq \|\exp\{\mathbf{E}[\ln|\Theta|; A^c]/\mathbf{P}(A^c)\}\|^{\mathbf{P}(A^c)} \|\psi(x_0)\|^{\mathbf{P}(A)} \exp\{\mathbf{E}[\theta_0; A]\}.$$

Now, for $\varepsilon>0$, we can find a set A and discrete random variables $\theta_i^{(\varepsilon)}$ such that

$$\mathbf{P}(A)\leqq\varepsilon,\quad \mathbf{E}[\theta_0;A]\leqq\varepsilon,\quad \theta_i^{(\varepsilon)}\leqq\theta_i,\theta_i\leqq(1+\varepsilon)\theta_i^{(\varepsilon)}\quad \text{on}\quad A^c,\ i=1,\dots,n,\quad k\in\mathbb{N}.$$

Hence, for a certain numerical quantity $\Delta(\varepsilon)$ that converges to 1 when $\varepsilon\to 0$, we have

$$\begin{aligned}\|\exp\{\mathbf{E}\ln|\Theta|\}\|&\leqq\Delta(\varepsilon)\|\exp\{\mathbf{E}[\ln|\Theta^{(\varepsilon)}|;A^c]/\mathbf{P}(A^c)\}\|^{\mathbf{P}(A^c)}\\ &\leqq\Delta(\varepsilon)\exp\{\mathbf{E}[\ln\|\Theta^{(\varepsilon)}\|;A^c]\}\\ &\leqq\Delta(\varepsilon)\exp\{\mathbf{E}[\ln\|\Theta\|;A^c]\}\\ &\leqq\Delta(\varepsilon)\exp\{\mathbf{E}[\ln\|\Theta\|]\}.\end{aligned}$$

To complete the proof, let $\varepsilon\to 0$. □

If the function $\|\cdot\|$ is monotone continuous (i.e. $0\leqq x_n\uparrow x\Rightarrow\|x_n\|\uparrow\|x\|$, then use of a monotone approximation by simple functions simplifies the proof significantly. However, even a norm need not be monotone continuous. For example, let $\mathbb{L}=l_\infty$, and define a norm

$$\|x_i\|=\limsup_{N\to\infty}N^{-1}\sum_{i=1}^{N}x_i.$$

Let $0\leqq a\leqq b\leqq\infty$. Choose two positive sequences (a_n) and (b_n) such that $b_n\geqq a_n, a_n\to a, b_n\to b$. Define an increasing sequence of vectors

$$x_n=(b_1,b_2,\dots,b_n,a_{n+1},a_{n+2},\dots).$$

Clearly, $\|x_n\|=a$ and $\|\sup_n x_n\|=b$.

Theorem 4.5. *If $\|\cdot\|$ is (p,r)-convex for some p and r, $0<p\leqq r\leqq\infty$, and 0^+-convex, then $\|\cdot\|$ is s-convex for every s, $0<s<p$.*

Proof. By Proposition 4.4 and Theorem 3.2, we have

$$\begin{aligned}\left\|\Big(\sum_i|x_i|^s\Big)^{1/2}\right\|&\leqq\left\|\exp\left\{\mathbf{E}\ln\Big(\sum_i|x_iX_{s,i}|^r\Big)^{1/r}\right\}\right\|\\ &\leqq C^{(0^+)}\exp\left\{\mathbf{E}\ln\left\|\Big(\sum_i|x_iX_{s,i}|^r\Big)^{1/r}\right\|\right\}\\ &\leqq C^{(p,r)}C^{(0^+)}\exp\left\{\mathbf{E}\ln\Big(\sum_i\|x_iX_{s,i}\|^p\Big)^{1/p}\right\}\\ &\leqq b(s,0^+,p)C^{(p,r)}C^{(0^+)}\Big(\sum_i\|x_i\|^s\Big)^{1/s}.\end{aligned}$$

This completes the proof. □

Combining this theorem with Lemma 2.1 we deduce the following:

Corollary 4.6. *If the addition (or the supremum) is continuous with respect to the topology induced by a positive homogeneous monotone 0_+-convex functional $\|\cdot\|$, then this functional is p-convex for some $p\in(0,1]$, which makes the underlying topology metrizable.*

It is easy to see that 0_+-convexity alone may imply none of other types of convexity discussed here. Moreover, the lack of a stronger convexity may be determined by shapes of two-dimensional spaces. The idea is as follows. Let $\mathbb{L}_n=\mathbb{R}^2$ be a sequence of two-dimensional spaces and $\|\cdot\|_n:\mathbb{L}_n\to\mathbb{R}_+$ be positive homogeneous 0_+-convex functions with $C^{(p)}(\|\cdot\|_n)\to\infty$. Let $\psi:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}_+$ be any homogeneous monotone 0^+-convex function. The function $\|\cdot\|:(\mathbb{R}^2)^{\mathbb{N}}\to\mathbb{R}_+$, defined by the formula

$$\|(x_i)_i\|=\psi((\|x_n\|_n)_n),\qquad x_n\in\mathbb{R}^2,\qquad n\geqq 1,$$

generates a vector lattice

$$\mathbb{L}=\{x\in(\mathbb{R}^2)^{\mathbb{N}}:\|x\|<\infty\}.$$

Clearly, $\mathbb{L}$ is r.u.c., and $\|\cdot\|$ is 0^+-convex, but not q-convex for any $q>0$. By Theorem 4.5, $\|\cdot\|$ is not (p,r)-convex for any pair $(p,r), 0<p\leqq r\leqq\infty$.

We give two examples.

1. Let $\mathbb{L}_n=l_2^{p_n}$ endowed with the usual positive homogeneous functionals $\|\cdot\|_{p_n}$, $1>p_n\downarrow 0$.
2. Let $a\geqq 1$ and let $\|\cdot\|_a=\|\cdot\|_{U_a}$ be the gauge function of the set $U_a=\{(x,y)\in[0,a]^2:xy\leqq 1\}$. In other words, denoting $\|(x,y)\|_0=\sqrt{|xy|}$ and $\|(x,y)\|_\infty=|x|\vee|y|$, $\|\cdot\|_a=\|\cdot\|_0\vee(\|\cdot\|_\infty/a)$. It is clearly seen that $C^{0^+}(\|\cdot\|_a)=1$ and $C(\|\cdot\|_a)^{(q)}\geqq(a^q+a^{-q})/2$.

4.3. Convexities via mutually disjoint vectors

In the definition of (p,∞)-convexity, it suffices to consider only mutually disjoint vectors. This is known in the case of a Banach lattice, and the proof remains essentially the same, up to some nuances. Again, for the sake of completeness, we will provide details. We will follow the idea of the proof of [7, Proposition 1.f.6].

Theorem 4.7. *The function $\|\cdot\|$ is (p,∞)-convex if*

$$\left\|\sup_i|x_i|\right\|\preccurlyeq\left(\sum_i\|x_i\|^p\right)^{1/p},\qquad x_i\in\mathbb{L},\quad |x_i|\wedge|x_j|=0,\quad 1\leqq i,j\leqq n,\quad n\in\mathbb{N}.$$

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Proof. The theorem will follow from two observations stated below. The vector lattice $\mathbb{L}$ can be embedded into an order complete vector lattice $\bar{\mathbb{L}}$. However, extensions (not unique, in general) of functions $\|\cdot\|$ on $\bar{\mathbb{L}}$ need not preserve some of the discussed properties (see examples following the proof). Recall that the order completeness (in other words, Dedekind completeness) means that every nonempty set that is order bounded from above has a supremum. Every Archimedean vector lattice has a Dedekind completion (cf. [8, Theorem 32.5]), i.e., there exists an order complete vector lattice $\bar{\mathbb{L}}$ such that $\mathbb{L}$ is lattice and vector isomorphic to a sublattice of $\bar{\mathbb{L}}$ (for simplicity, we say from now on that $\mathbb{L}$ is a sublattice of $\bar{\mathbb{L}}$) and, for every $\bar{x}\in\bar{\mathbb{L}}$,

$$\bar{x}=\sup\{x\in\mathbb{L}:x\leqq\bar{x}\}=\inf\{y:y\geqq\bar{x}\}. \tag{4.4}$$

The following observation belongs to the standard repertoire of the theory of vector lattices (cf. the proof of aforementioned [7, Proposition 1.f.6], or [13, Section II.2]).

Observation 1. If $\mathbb{L}$ is an order complete vector lattice then, for every finite set $\{x_i:1\leqq i\leqq n\}\subset\mathbb{L}_+$, there exists a finite set $\{y_i:1\leqq i\leqq n\}\subset\mathbb{L}_+$ such that

$$y_i\wedge y_j=0,\quad 0\leqq y_i\leqq x_i,\quad 1\leqq i,j\leqq n,\qquad \sum_i y_i=\sup_i y_i=\sup_i x_i.$$

Observation 2. The "upper" extension $\|\cdot\|^*$ defined by the formula

$$\|\bar{x}\|^*\stackrel{\mathrm{df}}{=}\inf\{\|y\|:y\geqq\bar{x}\}.$$

preserves (p,r)-convexity for any $p,r(0^+\leqq p\leqq r\leqq\infty)$.

The statement follows routinely. We omit details. □

We may also define the "lower" extension (cf. Section 4.3) of $\|\cdot\|$ on $\bar{\mathbb{L}}$

$$\|\bar{x}\|_*\stackrel{\mathrm{df}}{=}\sup\{y:y\leqq\bar{x}\}.$$

Then any positive monotone extension $\|\cdot\|^\sim$ satisfies $\|\cdot\|_*\leqq\|\cdot\|^\sim\leqq\|\cdot\|^*$. In general, extensions need not be unique nor preserve any type of convexity.

For example, take $\mathbb{L}=C[0,1]$, put $\|x\|=|x(1/2)|$ and $\bar{x}=\mathbf{1}_{[0,1/2]}$. Then $\|\bar{x}\|_+=1$ while $\|\bar{x}\|_*=0$. For $\bar{x}_1=n\mathbf{1}_{[0,1/2]}$, $\bar{x}_2=n\mathbf{1}_{(1/2,1]}$, we have $\|\bar{x}_1\vee\bar{x}_2\|_*=n$ while $\|\bar{x}_1\|_*=\|\bar{x}_2\|_*=0$. For $\bar{x}_1=n^2\mathbf{1}_{[0,1/2]}+\mathbf{1}_{(1/2,1]}$, $\bar{x}_2=\mathbf{1}_{[0,1/2]}+n^2\mathbf{1}_{(1/2,1]}$, we have $\|(\bar{x}_1,\bar{x}_2)^{1/2}\|_*=n$ while $\|\bar{x}_1\|_*=\|\bar{x}_2\|_*=1$. Thus $\|\cdot\|_*$ fails to be (p,r)-convex for any $p,r,0^+\leqq p\leqq r\leqq\infty$. The same may be true for the lower extension of a complete norm, e.g. consider a norm $\|\cdot\|_\infty+\|\cdot\|$ on $C[0,1]$.

4.4. Convexification of vector lattices

Definition 4.4.1. One can extend a procedure which is well known in the case of

Banach lattices (cf. [7, p. 53]). In a vector lattice $\mathbb{L}$, we introduce new vector operations and define an order turning the new structure to another vector lattice. Let $\alpha>0$. Define

$$x \oplus y=(x^{1/\alpha}+y^{1/\alpha})^{\alpha}, \qquad x,y\in\mathbb{L},$$

$$a \odot x=a^{\alpha}\cdot x, \qquad a\in\mathbb{R}, x\in\mathbb{L},$$

$$x \circledleq y \Leftrightarrow x\leqq y, \qquad x,y\in\mathbb{L}.$$

If necessary, we will use a subscript, e.g., $\oplus_{\alpha}, \odot_{\alpha}$. It is easy to check that $(\mathbb{L}a, \oplus, \odot, \circledleq)$, called the *α-convexification* of $\mathbb{L}$, is a vector lattice. Clearly, in the case of a function space, $\mathbb{L}^{(\alpha)}=\{f: f^{\alpha}\in\mathbb{L}\}$. Let $\|\cdot\|$ be a positive homogeneous function on $\mathbb{L}$ and $\alpha>0$. Define a homogeneous function $\|\cdot\|_{(\alpha)}: \mathbb{L}a\to\mathbb{R}_+$ by the formula $\|x\|_{(\alpha)}=\|x\|^{1/\alpha}$.

Proposition 4.8. *Let $\mathbb{L}$ be a r.u.c. vector lattice and $\|\cdot\|: \mathbb{L}\to\mathbb{R}_+$.*

(i) *$\mathbb{L}^{(\alpha)}$ is also a r.u.c. vector lattice, and the identity map constitutes a topological isomorphism between $\mathbb{L}$ and $\mathbb{L}^{(\alpha)}$ endowed with r.u. topologies.*
(ii) *For $\alpha,\beta>0$, we have $(\mathbb{L}^{(\alpha)})^{(\beta)}=\mathbb{L}^{(\alpha\beta)}$, $\mathbb{L}^{(1)}=\mathbb{L}$, $(\mathbb{L}^{(\alpha)})^{(1/\alpha)}=\mathbb{L}$, where the identities are meant in the sense of vector, lattice, and topological isomorphism.*
(iii) *If $\|\cdot\|$ is (p,r)-convex function on $\mathbb{L}$, $0^+\leqq p\leqq r\leqq\infty$, then $\|\cdot\|_{(\alpha)}$ is $(p\alpha,r\alpha)$-convex.*
(iv) *$\|\cdot\|$ is 0^+-convex function on $\mathbb{L}$ if and only if $\|\cdot\|_{(\alpha)}$ is 0^+-convex.*
(v) *If $\|\cdot\|$ is q-convex for some $q>0$ then $\|\cdot\|_{(\alpha)}$ becomes a norm if $\alpha q\geqq 1$.*

Proof. (i) One must show that

$$x_n \overset{\text{r.u.}}{\to} x \text{ in } \mathbb{L} \quad\Leftrightarrow\quad x_n \overset{\text{r.u.}}{\to} x \text{ in } \mathbb{L}^{\alpha},$$

or equivalently, that, for any $\alpha>0$,

$$|x-x_n| \overset{\text{r.u.}}{\to} 0 \quad\Leftrightarrow\quad |x_n^{1/\alpha}-x^{1/\alpha}|^{\alpha} \overset{\text{r.u.}}{\to} 0.$$

The equivalence and the invariance of completeness follow immediately from two inequalities involving homogeneous continuous functions. It is enough to verify the following inequalities in the real context:

$$|y^{1/\alpha}-x^{1/\alpha}|^{\alpha}\leqq|y-x|, \qquad x,y\in\mathbb{L} \quad \text{if } \alpha\geqq 1$$
$$|y^{1/\alpha}-x^{1/\alpha}|^{\alpha}\leqq\alpha^{-\alpha}|y\vee x|^{1-\alpha}|x-y|^{\alpha}, \qquad x,y\in\mathbb{L} \quad \text{if } \alpha\leqq 1.$$

(ii) For example, to prove the first identity, we have to check that

$$(x^{1/\beta}\oplus_{\alpha}y^{1/\beta})^{\beta}=(x^{1/(\alpha\beta)}+y^{1/(\alpha\beta)})^{\alpha\beta}.$$

The identity clearly holds in the real context, and the functions of the two arguments appearing on both sides are homogeneous and continuous. Hence, it can be carried over to the vector lattice case.

All other statements follow immediately. □

Besides 0^+-convexity, there are a number of obvious invariants of convexification, like monotonicity, completeness, continuity, to name but a sample. If a topology τ generated by $\|\cdot\|$ is separable, so is $\tau^{(\alpha)}$.

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