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A PROBLEM OF EXPRESSIBILITY IN SOME AMALGAMATED PRODUCTS OF GROUPS

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Abstract

Let S be a subset of a group G such that $S^{-1} = S$. Denote by $\mathrm{gr}(S)$ the subgroup of G generated by S, and by $l_S(g)$ the length of an element $g \in \mathrm{gr}(S)$ relative to the set S. Suppose that V is a finite subset of a free group F of countable rank such that the verbal subgroup $V(F)$ is a proper subgroup of F. For an arbitrary group G, denote by $\overline{V}(G)$ the set of values in G of all the words from the set V. In the present paper, for amalgamated products $G = A *_H B$ such that $A \neq H$ and the number of double cosets of B by H is at least three, the infiniteness of the set $\{l_S(g) \mid g \in \mathrm{gr}(S)\}$, where $S = \overline{V}(G) \cup \overline{V}(G)^{-1}$, is established.

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Let G be an arbitrary group and let S a subset of G such that $S^{-1} = S$. Denote by $\mathrm{gr}(S)$ the subgroup of G generated by S. We say that the width of the set S is *finite* if there is $k \in \mathbb{N}$ such that any element g of $\mathrm{gr}(S)$ is representable in the form

$$g = s_1 s_2 \cdots s_n, \quad \text{where} \quad s_i \in S \quad \text{and} \quad n \leq k. \tag{1}$$

The minimal k with this property is called the *width* of the set S in G, and we denote it by $\mathrm{wid}(S, G)$. If for each $k \in \mathbb{N}$ there is an element $g_k \in \mathrm{gr}(S)$ that cannot be expressed in the form (1), we say that the width of S in G is infinite. Many papers are devoted to investigating widths of various subsets: see [1–3, 6–9].

In this paper we consider widths of verbal subgroups. Specifically, let V be a finite subset of the free group F of countable rank. We say that V is *proper* if $V(F)$ is a proper subgroup of F. By the width of the verbal subgroup $V(G)$ of an arbitrary group G we mean the width of the set $\overline{V}(G) \cup \overline{V}(G)^{-1}$ in G. Widths of verbal subgroups have been investigated in a series of papers (see [2, 8, 9] and references therein). In the present paper, for amalgamated products $G = A *_H B$ such that $A \neq H$ and the number of double cosets of B by H is at least three, the infiniteness of the width of the verbal subgroup $V(G)$ is established.

DEFINITION 1. A *quasicharacter* of a semigroup S is a real-valued function f on S such that the set $\{f(xy) - f(x) - f(y) \mid x, y \in S\}$ is bounded.

DEFINITION 2. By a *pseudocharacter* of a semigroup S (or group S) we mean a quasicharacter f satisfying the following condition: $f(x^n) = nf(x)$ for all $x \in S$ and all $n \in \mathbb{N}$ (and all $n \in \mathbb{Z}$ if S is group).

The set of quasicharacters of a semigroup S is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers) which will be denoted by $KX(S)$. The subspace of $KX(S)$ consisting of pseudocharacters will be denoted by $PX(S)$, and the subspace consisting of real additive characters of S will be denoted by $X(S)$.

DEFINITION 3. By a *quasicharacter* of a semigroup S with involution $*$ we mean a quasicharacter φ such that $\varphi(v^*) = -\varphi(v)$ for all $v \in S$.

The set of quasicharacters of a semigroup S with involution will be denoted by $KX(S, *)$.

Let $G = A *_H B$ be the amalgamated product of two nontrivial groups A and B. Let $B = H \cup (\bigcup_{i \in I} Hb_iH)$ be the decomposition of the group B into double cosets. We assume that $|I| \geq 2$. Let $\mathscr{F}$ be the free monoid with free generators $X = \{x_i \mid i \in I\}$. Define $\xi : B \setminus H \to X$ by $\xi(b) = x_i$ whenever $b \in Hb_iH$. The mapping $b \mapsto b^{-1}$ on B induces a permutation $*$ of order two on the set X, as follows: if $b_i^{-1} \in Hb_jH$ we set $x_i^* = x_j$. Now we extend $*$ to an involution on the entire semigroup $\mathscr{F}$, that is, for $v = x_{i_1}x_{i_2}\cdots x_{i_n}$ set $v^* = x_{i_n}^* \cdots x_{i_2}^* x_{i_1}^*$. Let $A_0 = A \setminus H$ and $B_0 = B \setminus H$.

DEFINITION 4. Let $g \in G \setminus H$. By a *canonical* (or *reduced*) *form* of the element g we mean an expression of the form

$$g = c_1c_2\cdots c_k, \tag{2}$$

where $c_i \in A_0 \cup B_0$ and $c_ic_{i+1} \notin A_0 \cup B_0$.

Given a canonical form as above, we put $\dot{g} = c_1$ and $\ddot{g} = c_k$. Now we define $\xi : G \to \mathscr{F}$ as follows. If $g \in A$ we set $\xi(g) = 1$. If $g \notin A$ and (2) is a canonical form for g, we set $\xi(g) = \xi(c_1)\xi(c_2)\cdots\xi(c_k)$. It is clear that the mapping ξ is well defined. Now for each word v from the semigroup $\mathscr{F}$ we introduce the set of 'beginnings' $H(v)$ and the set of 'endings' $K(v)$ as follows. If $v \in X$, we put $H(v) = K(v) = \emptyset$. If $v = x_{i_1}x_{i_2}\cdots x_{i_n}$, where $i_j \in I$ and $n > 1$, we set

$$H(v) = \{x_{i_1}, x_{i_1}x_{i_2}, \ldots, x_{i_1}x_{i_2}\cdots x_{i_{n-2}}, x_{i_1}x_{i_2}\cdots x_{i_{n-1}}\},$$
$$K(v) = \{x_{i_2}\cdots x_{i_{n-1}}x_{i_n}, x_{i_3}\cdots x_{i_{n-1}}x_{i_n}, \ldots, x_{i_{n-1}}x_{i_n}, x_{i_n}\}.$$

It is evident that $H(w) \cap K(w) = \emptyset$ if and only if $H(w^*) \cap K(w^*) = \emptyset$.

DEFINITION 5. Two elements v and w of $\mathscr{F}$ are called *conjugate* if either $v = w$ or there exist elements a and b in $\mathscr{F}$ such that $v = ab$ and $w = ba$. The conjugacy relation will be denoted by $\sim$.

Denote by P^+ the set of words w of length at least two in the alphabet X with the property that $H(w) \cap K(w) = \emptyset$ and $w \not\sim w^*$. For any word $w \in P^+$ and any $v \in \mathscr{F}$ denote by $\psi_w(v)$ the number of occurrences of w in v. Note that two occurrences of w in v cannot overlap, since the overlapping portion would lie in $H(w) \cap K(w)$. For each pair of elements x, y from $\mathscr{F}$ we define a measure $\mu_{x,y}$ on P^+ as follows: we set $\mu_{x,y}(w) = 1$ if $w = ab$ for some $a \in K(x) \cup \{x\}$ and $b \in H(y) \cup \{y\}$; otherwise we set $\mu_{x,y}(w) = 0$. It is easy to verify that $\psi_w(xy) - \psi_w(x) - \psi_w(y) = \mu_{x,y}(w)$ for all $w \in P^+$ and $x, y \in \mathscr{F}$. Now for $w \in P^+$ and $v \in \mathscr{F}$ we put

$$\eta_w(v) = \psi_w(v) - \psi_{w^*}(v).$$

Let $\Delta_{u,v}(w) = \mu_{u,v}(w) - \mu_{u,v}(w^*)$. We obtain the following equality

$$\eta_w(uv) - \eta_w(u) - \eta_w(v) = \Delta_{u,v}(w).$$

It is obvious that the relations $\eta_w(v^*) = -\eta_w(v)$ and $|\Delta_{u,v}(w)| \leq 1$ hold; hence the function η_w is an element of the space $KX(\mathscr{F}, *)$.

Next we define a metric $d(\cdot, \cdot)$ on the semigroup $\mathscr{F}$. By an *elementary transformation* of a word v in the alphabet X we mean an insertion or deletion of some $a \in X$. It is clear that any two words u and v from $\mathscr{F}$ can be connected by some sequence of elementary transformations; we define the distance $d(u, v)$ between u and v to be the minimal number of elementary transformations required to connect them. It is clear that the function $d(u, v)$ is a metric, and that it is preserved by the left and right actions of $\mathscr{F}$.

LEMMA 1. *Suppose that $u, v \in \mathscr{F}$ with $d(u, v) \leq 1$. Then there exists a set of at most three pairs of elements $z_i, s_i \in \mathscr{F}$ such that for all $w \in P^+$ we have*

$$\eta_w(u) - \eta_w(v) = \sum_i \Delta_{z_i,s_i}(w).$$

Furthermore, $|\eta_w(u) - \eta_w(v)| \leq 2$.

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PROOF. We may assume that v is obtained from u by the insertion of one letter. Thus we have $u = u_1u_2$ and $v = u_1au_2$, where $a \in X$. Hence, $\psi_w(v) = \psi_w(u_1au_2) = \psi_w(u_1) + \psi_w(a) + \psi_w(u_2) + \mu_{u_1,au_2}(w) + \mu_{a,u_2}(w)$ and $\psi_w(u) = \psi_w(u_1) + \psi_w(u_2) + \mu_{u_1,u_2}(w)$. Therefore,

$$\begin{aligned}\psi_w(v) - \psi_w(u) &= \psi_w(a) + \mu_{u_1,au_2}(w) + \mu_{a,u_2}(w) - \mu_{u_1,u_2}(w) \\ &= \mu_{u_1,au_2}(w) + \mu_{a,u_2}(w) - \mu_{u_1,u_2}(w).\end{aligned}$$

It is easy to see that $\mu_{u_1,au_2}(w) + \mu_{a,u_2}(w) \in \{0, 1\}$ and so it follows that $\psi_w(v) - \psi_w(u) \in \{0, \pm 1\}$. Since the same holds for w^*, we have $\eta_w(v) - \eta_w(u) \in \{0, \pm 1, \pm 2\}$. Moreover,

$$\begin{aligned}\eta_w(v) - \eta_w(u) &= \psi_w(v) - \psi_{w^*}(v) - \psi_w(u) + \psi_{w^*}(u) \\ &= \psi_w(v) - \psi_w(u) - \psi_{w^*}(v) + \psi_{w^*}(u) \\ &= \mu_{u_1,au_2}(w) + \mu_{a,u_2}(w) - \mu_{u_1,u_2}(w) \\ &\quad - \mu_{u_1,au_2}(w^*) - \mu_{a,u_2}(w^*) + \mu_{u_1,u_2}(w^*) \\ &= \Delta_{u_1,au_2}(w) + \Delta_{a,u_2}(w) - \Delta_{u_1,u_2}(w),\end{aligned}$$

and the lemma is proved. □

COROLLARY 1. *Let $u, v \in \mathscr{F}$ with $d(u, v) \leq k$. There exists a set of at most $3k$ pairs of elements $z_i, s_i \in \mathscr{F}$ such that for all $w \in P^+$ we have*

$$\eta_w(u) - \eta_w(v) = \sum_i \Delta_{z_i,s_i}(w).$$

Furthermore, $|\eta_w(u) - \eta_w(v)| \leq 2k$.

DEFINITION 6. Let $g \in G$ with $g \neq 1$. By subdivision of g we mean an expression the form

$$g = g_1g_2 \cdots g_m,$$

where the g_i are canonical forms for each i, and $\ddot{g}_i\dot{g}_{i+1} \notin A_0 \cup B_0$.

LEMMA 2. *Suppose that the elements $g, t \in G$ satisfy $\ddot{g}\dot{t} \notin H$. Then*

$$d(\xi(gt), \xi(g)\xi(t)) \leq 3.$$

PROOF. It is clear that $\xi(gt) = \xi(g)\xi(t)$ unless $\ddot{g}$ and $\dot{t}$ are both in B. Writing $g = g_1b_1$ and $t = b_2t_1$, where $b_1 = \ddot{g}$ and $b_2 = \dot{t}$, we see that $\xi(g)\xi(t) = \xi(g_1)x_1x_2\xi(t_1)$, where $x_1 = \xi(b_1)$ and $x_2 = \xi(b_2)$ are elements of X. Furthermore, $\xi(gt) = \xi(g_1)x_3\xi(t_1)$, where $x_3 = \xi(b_1b_2) \in X$. Now $\xi(g)\xi(t)$ can be transformed into $\xi(gt)$ by deleting x_1 and x_2 and inserting x_3. □

COROLLARY 2. *Suppose that the elements $g, t \in G$ satisfy $\ddot{g}\dot{t} \notin H$. Then*

$$|\eta_w(\xi(gt)) - \eta_w(\xi(g)\xi(t))| \leq 6$$

for all $w \in P^+$. Further, there is a set of at most nine pairs of elements z_i, s_i from $\mathscr{F}$ such that

$$\eta_w(\xi(gt)) - \eta_w(\xi(g)\xi(t)) = \sum_i \Delta_{z_i, s_i}(w)$$

for all $w \in P^+$. Since $\eta_w(\xi(g)\xi(t)) - \eta_w(\xi(g)) - \eta_w(\xi(t)) = \Delta_{\xi(g),\xi(t)}(w)$, there is a set of at most ten pairs of elements z_i, s_i such that

$$\eta_w(\xi(gt)) - \eta_w(\xi(g)) - \eta_w(\xi(t)) = \sum_i \Delta_{z_i, s_i}(w).$$

For each $w \in P^+$ we define a function $\rho_w : G \to \mathbb{R}$ by $\rho_w(g) = \eta_w(\xi(g))$ for all $g \in G$. Our next result shows that ρ_w is a quasicharacter.

PROPOSITION 1. *For any x, y from G there exists a set of at most twelve pairs of elements g_i, t_i from G such that the relation*

$$\rho_w(xy) = \rho_w(x) + \rho_w(y) + \sum_i \Delta_{\xi(g_i),\xi(t_i)}(w)$$

holds for all $w \in P^+$. Hence, we have the following estimate

$$|\rho_w(xy) - \rho_w(x) - \rho_w(y)| \leq 12.$$

PROOF. If $\ddot{x}\dot{y} \notin H$ the result follows immediately from Corollary 2 above. So we may assume that $\ddot{x}\dot{y} \in H$. Now let $x = gz_1$ and $y = z_2t$ be subdivisions of x and y such that $z_1z_2 \in H$ and $\ddot{g}z_1z_2\dot{t} \notin H$. Then $\xi(x) = \xi(g)\xi(z_1)$, $\xi(y) = \xi(z_2)\xi(t)$ and $\xi(z_2) = \xi(z_1)^*$. Hence, we obtain the following relations

$$\begin{aligned}
\eta_w(\xi(x)) &= \eta_w(\xi(g)\xi(z_1)) = \eta_w(\xi(g)) + \eta_w(\xi(z_1)) + \Delta_{\xi(g),\xi(z_1)}(w),\\
\eta_w(\xi(y)) &= \eta_w(\xi(z_2)\xi(t)) = \eta_w(\xi(t)) + \eta_w(\xi(z_2)) + \Delta_{\xi(z_2),\xi(t)}(w),\\
\eta_w(\xi(x)) + \eta_w(\xi(y)) &= \eta_w(\xi(g)) + \eta_w(\xi(t)) + \Delta_{\xi(g),\xi(z_1)}(w) + \Delta_{\xi(z_2),\xi(t)}(w).
\end{aligned}$$

Now $\xi(gz_1z_2) = \xi(g)$, since $z_1z_2 \in H$, and so $\eta_w(\xi(xy)) - \eta_w(\xi(x)) - \eta_w(\xi(y))$ equals

$$\eta_w(\xi(gz_1z_2t)) - \eta_w(\xi(gz_1z_2)) - \eta_w(\xi(t)) - \Delta_{\xi(g),\xi(z_1)}(w) - \Delta_{\xi(z_2),\xi(t)}(w).$$

Hence by Corollary 2 (applied with gz_1z_2 in place of g) there is a set of twelve or fewer pairs g_i, t_i such that $\eta_w(\xi(xy)) - \eta_w(\xi(x)) - \eta_w(\xi(y)) = \sum_i \Delta_{\xi(g_i),\xi(t_i)}(w)$. □

DEFINITION 7. An element $v \in \mathscr{F}$ is said to be *simple* if there is no integer $m \geq 2$ such that $v = w^m$ for some $w \in \mathscr{F}$.

Obviously for any $u \in \mathscr{F}$ there is an $n \in \mathbb{N}$ and a simple element w such that $u = w^n$. It is clear that if $u \sim v$, then u is simple if and only if v is simple, and $u^m \sim v^m$ for all $m \in \mathbb{N}$.

In [5] the following result was obtained.

LEMMA 3. *If v is a simple element from $\mathscr{F}$, then there is $w \in \mathscr{F}$ such that $v \sim w$ and $H(w) \cap K(w) = \emptyset$.*

LEMMA 4. *Let $v \in \mathscr{F}$ be an element of length at least two with $v \not\sim v^*$. Then there exist $w \in P^+$ and $n \in \mathbb{N}$ such that $v \sim w^n$.*

PROOF. Suppose that $v = w_1^n$, where w_1 is simple. It is clear that $w_1 \not\sim w_1^*$. Now by Lemma 3 we obtain that there is $w \sim w_1$ such that $H(w) \cap K(w) = \emptyset$. □

In [4] the following result was obtained.

THEOREM 2. *Let S be a semigroup, and f a quasicharacter of S such that $|f(xy) - f(x) - f(y)| < c$ for all $x, y \in S$. Then the function*

$$\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n})$$

is well defined and is a pseudocharacter, with $|\widehat{f}(xy) - \widehat{f}(x) - \widehat{f}(y)| < 4c$ for all $x, y \in S$.

COROLLARY 4. *Let G be a group, and f a quasicharacter of G such that $|f(xy) - f(x) - f(y)| < c$ for all $x, y \in G$. Then the function*

$$\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n})$$

is well defined and is a pseudocharacter, with $|\widehat{f}(xy) - \widehat{f}(x) - \widehat{f}(y)| < 4c$ for all $x, y \in G$.

PROOF. By Theorem 2 it suffices to show that for each $x \in G$ the equality $\widehat{f}(x^{-1}) = -\widehat{f}(x)$ holds. Since $\widehat{f}(x^n) = n\widehat{f}(x)$ for all $x \in G$ and $n \in \mathbb{N}$, we obtain $\widehat{f}(1) = 0$. Hence $|\widehat{f}(1) - \widehat{f}(x) - \widehat{f}(x^{-1})| < 4c$ becomes $|\widehat{f}(x) + \widehat{f}(x^{-1})| < 4c$ for all $x \in G$, whence it follows that

$$n|\widehat{f}(x) + \widehat{f}(x^{-1})| = |\widehat{f}(x^n) + \widehat{f}((x^{-1})^n)| < 4c$$

for all $x \in G$ and $n \in \mathbb{N}$. This is possible only if $\widehat{f}(x^{-1}) = -\widehat{f}(x)$. Now for each $k > 0$ we have $\widehat{f}(x^{-k}) = \widehat{f}((x^k)^{-1}) = -\widehat{f}((x^k)) = -k\widehat{f}(x)$, and the corollary is proved. □

In particular, it follows from Corollary 4 that $\widehat{\rho}_w$ is a pseudocharacter of G whenever $w \in P^+$.

PROPOSITION 2. *Let $C \lhd G$ and $C \not\subseteq H$. Then there exists a pseudocharacter φ of G such that $\varphi|_C \not\equiv 0$.*

PROOF. By Theorem 1 there is $g \in C$ of the form $g = \alpha_1\beta_1 \cdots \alpha_k\beta_k$ with $\alpha_i \in (A \setminus H)$ and $\beta_i \in (B \setminus H)$ for each i, such that if $v = \xi(g) \in \mathscr{F}$ then $v \not\sim v^*$.

Replacing g by a conjugate of itself if necessary, by Lemma 4 we may assume that $v = w^n$ for some $w \in P^+$ and $n \in \mathbb{N}$. The pseudocharacter $\widehat{\rho}_w$ then has the desired property, since $\rho_w(g^k) = nk$ for all $k \in \mathbb{N}$, and thus $\widehat{\rho}_w(g) = n \neq 0$. This completes the proof. □

LEMMA 5. *Let $\varphi \in PX(G)$, and suppose that $|\varphi(xy) - \varphi(x) - \varphi(y)| < \varepsilon$ for all $x, y \in G$. Then:*

(A) *The inequality $\left|\varphi(x_1x_2\cdots x_{n+1}) - \sum_{i=1}^{n+1}\varphi(x_i)\right| < n\varepsilon$ holds for any positive integer n and any $x_1, x_2, \ldots, x_n \in G$.*

(B) *If φ is a bounded function, then $\varphi \equiv 0$.*

(C) *$\varphi(a^{-1}ba) = \varphi(b)$ for any $a, b \in G$.*

PROOF. Assertion (A) is easily proved by induction on n. Let us prove (B). If δ is a positive number such that $|\varphi(x)| < \delta$ for all $x \in G$, then for any positive integer n we have $n|\varphi(x)| = |\varphi(x^n)| < \delta$. Therefore $\varphi(x) = 0$, as required.

From (A) it follows that $|\varphi(a^{-1}b^na) - \varphi(a^{-1}) - \varphi(b^n) - \varphi(a)| < 2\varepsilon$. Hence $|\varphi(a^{-1}b^na) - \varphi(b^n)| = |\varphi((a^{-1}ba)^n) - \varphi(b^n)| < 2\varepsilon$, and so $n|\varphi(a^{-1}ba) - \varphi(b)| < 2\varepsilon$. Since the latter inequality holds for all $n > 1$, we obtain $\varphi(a^{-1}ba) = \varphi(b)$. The lemma is proved. □

Let i and j be distinct elements of I, and put $w_k = x_i^{3k}x_j^{2k}x_i^kx_j^k$, for each $k \in \mathbb{N}$. Consider the set $\mathscr{M} = \{w_k \mid k \in \mathbb{N}\}$. It can easily be checked that w_l is not a subword of w_k for $k \neq l$, and also that

$$H(w_k) \cap K(w_l) = \emptyset \quad \text{for all} \quad k, l \in \mathbb{N}. \tag{3}$$

Hence $|\mathscr{M} \cap \operatorname{supp} \mu_{u,v}| \leq 1$ for all $u, v \in \mathscr{F}$ and

$$|\mathscr{M} \cap \operatorname{supp} \Delta_{u,v}| \leq 2 \quad \text{for all} \quad u, v \in \mathscr{F}. \tag{4}$$

By Proposition 1 it follows that for any pair of elements x, y from G there are at most 24 elements w in $\mathscr{M}$ such that $\rho_w(xy) \neq \rho_w(x) + \rho_w(y)$. For each $g \in G$ and each integer $m \geq 2$, the set

$$O_m(g) = \{ w \in \mathscr{M} \mid \rho_w(g) \not\equiv 0 \pmod{m} \}$$

is finite. Denote by $\gamma_m(g)$ the cardinality of $O_m(g)$. Evidently,

$$O_m(xy) \subseteq O_m(x) \cup O_m(y) \cup \{ w \in \mathscr{M} \mid \rho_w(xy) \neq \rho_w(x) + \rho_w(y) \}.$$

Hence

$$\gamma_m(xy) \leq \gamma_m(x) + \gamma_m(y) + 24. \tag{5}$$

Similar arguments establish the following assertions (for all $x, y \in G$).

(a) There is a set of at most 36 pairs y_i, t_i such that

$$\rho_w(x^{-1}y^{-1}xy) = \rho_w(x^{-1}) + \rho_w(y^{-1}) + \rho_w(x) + \rho_w(y) + \sum_i \Delta_{\xi(y_i),\xi(t_i)}(w),$$

and since $\rho_w(x^{-1}) + \rho_w(y^{-1}) + \rho_w(x) + \rho_w(y) = 0$ we see that there are at most 72 elements $w \in \mathscr{M}$ such that $\rho_w(x^{-1}y^{-1}xy) \neq 0$. So

$$\gamma_m(x^{-1}y^{-1}yx) \leq 72. \tag{6}$$

(b) By Corollary 3, there are at most $12(m-1)$ elements y_i, t_i such that

$$\rho_w(x^m) = m\rho_w(x) + \sum_i \Delta_{\xi(y_i),\xi(t_i)}(w).$$

Now from (4) we obtain that there are at most $24(m-1)$ elements in the set $O(x^m)$. Hence

$$\gamma_m(x^m) \leq 24(m-1). \tag{7}$$

THEOREM 3. *Let V be a finite subset of the free group F such that the verbal subgroup $V(F)$ is a proper subgroup of F. Then the verbal subgroup $V(G)$ of G has infinite width.*

PROOF. Suppose that $V(F) \subseteq F'$. Let $\varphi \in PX(G)$ and choose $r \in \mathbb{R}$ such that $|\varphi(xy) - \varphi(x) - \varphi(y)| \leq r$ for all $x, y \in G$. By Lemma 5

$$|\varphi(x^{-1}y^{-1}xy)| = |\varphi(x^{-1}y^{-1}xy) - \varphi(x^{-1}) - \varphi(y^{-1}xy)| \leq r$$

for all $x, y \in G$. Since V is finite there is an integer l such that each element of V is a product of at most l commutators, and we deduce that $\varphi(g) < (l-1)r$ for all

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$g \in V(G)$. Hence if $\mathrm{wid}(V(G)) < \infty$ it follows that the pseudocharacter φ is bounded on $V(G)$. By Lemma 5 we obtain $\varphi \equiv 0$ on $V(G)$, contradicting Proposition 2.

Now suppose that $V(F) \not\subseteq F'$. Let $Z = \{z_1, z_2, \dots\}$ be a set of free generators of F, and let $V = \{v_1, v_2, \dots, v_k\}$. Then there is positive integer n such that each v_i is uniquely expressible in the form

$$v_i = z_1^{l_{i1}} z_2^{l_{i2}} \cdots z_n^{l_{in}} u_i, \tag{8}$$

where $l_{ij} \in \mathbb{Z}$ and $u_i \in F'$, and each u_i is a word in the alphabet $\{z_1, \dots, z_n\}$. Let m be the highest common factor of the numbers $\{ l_{ij} \mid 1 \le i \le k,\ 1 \le j \le n \}$, and for each i let m_i be the highest common factor of the numbers $\{ l_{ij} \mid 1 \le j \le n \}$. It is clear that m is the highest common factor of the numbers $\{ m_i \mid 1 \le i \le k \}$. Choose integers α_{ij} such that $m_i = \sum_{j=1}^{n} \alpha_{ij} l_{ij}$.

We have $u_i = u_i(z_1, z_2, \dots, z_n)$. If t is any element of F and $k_1, \dots, k_n$ are any integers, then $u_i(t^{k_1}, t^{k_2}, \dots, t^{k_n}) = 1$, since $u_i \in F'$. Hence we obtain

$$v_i(t^{\alpha_{i1}}, t^{\alpha_{i2}}, \dots, t^{\alpha_{in}}) = t^{\alpha_{i1} l_{i1}} \cdot t^{\alpha_{i2} l_{i1}} \cdots t^{\alpha_{in} l_{in}} = t^{m_i},$$

and we see that $t^{m_i} \in V(F)$ for any $t \in F$. Now as there are integers $\beta_1, \dots, \beta_k$ such that $\beta_1 m_1 + \cdots + \beta_k m_k = m$ it follows that for all $g \in F$,

$$g^m = g^{\beta_1 m_1 + \cdots + \beta_k m_k} = g^{\beta_1 m_1} \cdots g^{\beta_k m_k} \in V(F).$$

Since $V(F) \neq F$, it follows that $m \ge 2$.

From (8) and (5) we obtain

$$\gamma_m(z_1^{l_{i1}} z_2^{l_{i2}} \cdots z_n^{l_{in}} u_i) \le \sum_{j=1}^{n} \gamma_m(z_j^{l_{ij}}) + \gamma_m(u_i) + 24n. \tag{9}$$

We have $l_{ij} = m p_{ij}$ for some $p_{ij} \in \mathbb{Z}$; hence by (7)

$$\gamma_m(z_j^{l_{ij}}) \le 24(m-1). \tag{10}$$

It is clear that there is $q \in \mathbb{N}$ such that each u_i from (8) is representable as a product of at most q commutators, and then by (6) and (5) we have

$$\gamma_m(u_i) \le 72q + 24(q-1). \tag{11}$$

Now from (9), (10) and (11) we obtain that there is an $l \in \mathbb{N}$ such that for any $u \in \overline{V}(G)$ the relation $\gamma_m(u) \le l$ holds. This implies that if $V(G)$ has finite width, then the function γ_m is bounded on $V(G)$. Indeed, if $\mathrm{wid}\, V(G) = k$, then by (5) for any $g \in V(G)$ we have $\gamma_m(g) \le 24(k-1)l$.

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Let us choose ab_i and ab_j such that $\xi(ab_i) = x_i$, $\xi(ab_j) = x_j$, and consider the elements $g_k = (ab_i)^{3k}(ab_j)^{2k}(ab_i)^k(ab_j)^k$ and $d_k = g_m g_{2m} g_{3m} \cdots g_{km}$, where $k \in \mathbb{N}$. It is clear $\xi(g_k) = w_k$ and $\xi(d_k) = v_k = w_m w_{2m} \cdots w_{km}$. Obviously, $d_k \in V(G)$ for all $k \in \mathbb{N}$. Now from (3) it follows that there is exactly one occurrence of w_i in v_k if $i \in \{m, 2m, \ldots, km\}$, and no occurrence otherwise. It is easy to see that for any $i, j, k \in \mathbb{N}$ the relation $\mu_{w_k, w_j}(w_i^*) = 0$ holds. It follows that w_i^* does not occur in v_k for any value of i. Hence $\rho_w(g_k) = 1$ if $w \in \{w_m, w_{2m}, \ldots, w_{km}\}$, and $\rho_w(g_k) = 0$ for other elements $w \in \mathcal{M}$. So $\gamma_m(g_k) = k$ for all $k \in \mathbb{N}$ and we obtain a contradiction. This completes the proof. □

COROLLARY 5 (See [8]). *Let V be a finite subset of the free group F such that $V(F) \neq F$. Suppose that A and B are nontrivial groups such that the order of B is at least three, and let $G = A * B$ be the free product. Then the width of the verbal subgroup $V(G)$ is infinite.*

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