J. Austral. Math. Soc. (Series A) **43** (1987), 347-365

CONVOLUTIONS OF DISTRIBUTIONS WITH EXPONENTIAL AND SUBEXPONENTIAL TAILS

DAREN B. H. CLINE

(Received 10 August 1985; revised 24 September 1986)

Communicated by T. C. Brown

Abstract

Distribution tails $\bar{F}(t) = F(t, \infty)$ are considered for which $\bar{F}(t-u) \sim e^{\alpha u}\bar{F}(t)$ and $\overline{F * F}(t) \sim 2d\bar{F}(t)$ as $t \to \infty$. A real analytic proof is obtained of a theorem by Chover, Wainger and Ney, namely that

$$d = \int e^{\alpha u} F(du).$$

In doing so, a technique is introduced which provides many other results with a minimum of analysis. One such result strengthens and generalizes the various known results on distribution tails of random sums.

Additionally, the closure and factorization properties for subexponential distributions are investigated further and extended to distributions with exponential tails.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 60 E 05.
Keywords: convolution tails, exponential tails, subexponential tails.

1. Introduction

Throughout this work we will use distribution tails and denote them, for example, $\bar{F}(t) = F(t, \infty)$, where F is a finite nonnegative measure. For convenience, the distributions will have all their support on $[0, \infty)$. In application, these may be probability distributions (that is, total measure equal to 1) but we will not always assume so. Our convention is that integrals will exclude (include) the lower (upper) endpoint. The exception to this rule is when the lower endpoint is zero or for $\bar{F}(0) = F[0, \infty)$. If a function $A(t)$ is regularly varying with exponent ρ, we

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

write $A \in RV_\rho$. Also, we will signify $A_1 \sim kA_2$ and $A_1 \lesssim kA_2$ ($A_1 = o(A_2)$ for $k = 0$) when

$$\lim_{t\to\infty} \frac{A_1(t)}{A_2(t)} = k \quad \text{and} \quad \limsup_{t\to\infty} \frac{A_1(t)}{A_2(t)} \leqslant k, \quad \text{respectively.}$$

Of course, $kA_2 \gtrsim A_1$ is the same as $A_1 \lesssim kA_2$.

The purpose of this paper is to consider these relations ($\sim$ and $\lesssim$), applied to distribution tails and to their Stieltjes convolution $\overline{F * G}(t) = \iint_{x+y>t} F(dx)G(dy)$. Multiple convolutions of F with itself will be denoted F^{*n} and F^{*0} is the distribution with unit mass at zero. The discussion is limited to the following classes of distributions.

The class L_α, $\alpha > 0$, consists of all distributions F such that $\bar{F}(t-u) \sim e^{\alpha u}\bar{F}(t)$ for each u. One may easily see that $F \in L_\alpha$ if and only if $\bar{F}(\ln t) \in RV_{-\alpha}$. When $\alpha > 0$, F is said to have *exponential tail.*

F is called *convolution equivalent* ($F \in S_\alpha$) if $F \in L_\alpha$ and $\overline{F * F} \sim 2d\bar{F}$ for some finite positive d. If $F \in S_0$ ($\alpha = 0$), it is called *subexponential.* For example, $\bar{F} \in RV_{-\rho}$ implies $F \in S_0$.

The classes S_0 and S_α have received attention with applications to branching processes (Chistykov (1964), Chover, Wainger and Ney (1973a, b)), to renewal theory (Embrechts and Goldie (1982)), to infinite variance time series (Davis and Resnick (1985b) and to stable attraction of sums of products (Cline (1986)). References to further applications in queueing theory, random walks and infinite divisibility may be found in Embrechts and Goldie (1982). Conditions sufficient for $F \in S_\alpha$ are given in Cline (1986).

Chover, Wainger and Ney (1973a) first defined the class S_α in terms of the probability masses on the lattice of nonnegative integers:

$$f_n = F\{n\} \quad \text{and} \quad f_n * f_n = F * F\{n\}, \qquad n = 0, 1, 2, \ldots.$$

That is, $F \in S_\alpha$ if $f_n \sim e^\alpha f_{n+1}$ and $f_n * f_n \sim 2df_n$. They gave another, similar, definition for densities of an absolutely continuous F. In their first paper, they use a Banach algebra approach to demonstrate

$$\text{(1.1)} \qquad \text{If } F \in S_\alpha \text{ then } d = m, \text{ where } m = \int_0^\infty e^{\alpha u} F(du).$$

Chover, Winger and Ney (1973a, b) also verify (1.1) for the "global" version, where S_α is as we have defined it, in terms of distribution tails. Rudin (1973) and Embrechts (1983) have given a real analytic proof for the lattice version of (1.1). In this paper, Theorem 2.9 provides an elementary real analytic proof for the distribution tails version. Embrechts' lattice version is a special case. However, the version for densities is not (see the Conclusion).

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

In the course of pursuing this objective we devise a technique to calculate asymptotic relationships between convolution tails, which enables us to obtain several other known or partially known results with ease. In particular, Theorem 2.13 states in its most complete form a result for the measure $H = \sum_0^\infty \lambda_n F^{*n}$, namely that with appropriate conditions on λ_n, the following are equivalent.

(i) $F \in S_\alpha$.

(ii) $\overline{H} \sim c\overline{F}$ where $c = \sum_0^\infty n\lambda_n m^{n-1}$.

(iii) $H \in S_\alpha$ and $\overline{F} \neq o(\overline{H})$.

That (i) implies (ii) is the main objective in Chover, Wainger and Ney (1973a, b). Embrechts, Goldie and Veraverbeke (1979) and Embrechts and Goldie (1982) prove the equivalency for special cases of λ_n. Rudin (1973) and Embrechts (1983) also prove (i) implies (ii) for lattice F.

Embrechts and Goldie (1980) consider the question, if $F_1 \in S_0$ and $F_2 \in S_0$ then is $F_1 * F_2 \in S_0$? The question remains unanswered, but they show that $F_1 * F_2 \in S_0$ is equivalent to $\overline{F_1 * F_2} \sim m_2\overline{F}_1 + m_1\overline{F}_2$, where $m_j = \overline{F}_j(0)$, and that each is equivalent to $a_1F_1 + a_2F_2 \in S_0$, where $a_1 > 0$, $a_2 > 0$. In Theorem 3.4, we extend this result to the class S_α (with $m_j = \int_0^\infty e^{\alpha u}F_j(du)$) and strengthen it in two important respects. First, if $F_1, F_2 \in L_\alpha$ then either $F_1 \in S_\alpha$ or $F_2 \in S_\alpha$ is a consequence of $F_1 * F_2 \in S_\alpha$. Second, if $F_1 \in S_\alpha$ and either $F_2 \in S_\alpha$ or $\overline{F}_2 = o(\overline{F}_1)$, the statement $\overline{F_1 * F_2} \sim m_2\overline{F}_1 + m_1\overline{F}_2$ may be weakened to $\overline{F_1 * F_2} \sim a_1\overline{F}_1 + a_2\overline{F}_2$ for some a_1, a_2, and it will still retain its power to imply $F_1 * F_2 \in S_\alpha$.

Finally, we will provide actual conditions for which $\overline{F_1 * F_2} \sim m_2\overline{F}_1 + m_1\overline{F}_2$. The well-known result of Feller (1971, page 278) states that this holds for distributions with regularly varying tails. Embrechts and Goldie (1980) show that if $F_1 \in S_0$, then $\sup_t \overline{F}_1(t)/\overline{F}_2(t) < \infty$ suffices. In fact, for $F_j \in S_\alpha$, it is sufficient to have $\overline{F}_1/\overline{F}_2 \in RV_\rho$ for some ρ. This and a more general condition appear in Theorem 3.5.

2. Convolution equivalency

We state at the outset that the reader should keep Theorem 2.9 in mind when considering results 2.5 through 2.8 in this section. Once Theorem 2.9 has been proved we will restate these in a more precise form. Many of these have been proven elsewhere, but only *after* Theorem 2.9 was verified. Our approach is to work in the other direction.

We start with several lemmas.

LEMMA 2.1. *Let $A_j(t)$ be positive functions.*

(i) *If $A_3 \lesssim A_1 \lesssim cA_2$, then $A_1 + A_2 \lesssim A_3 + A_4$ implies $A_2 \lesssim A_4$.*

(ii) *If $A_3 \sim A_1 \lesssim cA_2$, then $A_1 + A_2 \sim A_3 + A_4$ implies $A_2 \sim A_4$.*

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

(iii) $a_1A_1 + a_2A_2 \sim b_1A_1 + b_2A_2$ *implies* $a_1 = b_1$ *or*

$$A_1 \sim \frac{b_2 - a_2}{a_1 - b_1} A_2.$$

(iv) *If* $A_1 \lesssim A_3$, $A_2 \lesssim A_4$, *then* $A_1 + A_2 \sim A_3 + A_4$ *implies one of the following must hold.*

$$\begin{aligned} &A_1 \sim A_3, \quad A_2 \sim A_4, \\ &A_1 \sim A_3, \quad A_4 = o(A_1), \text{ or} \\ &A_2 \sim A_4, \quad A_3 = o(A_2). \end{aligned}$$

PROOF. Straightforward.

LEMMA 2.2. *Suppose* $F_1, F_2 \in L_\alpha$.

(i) (Embrechts and Goldie, 1982) *The transforms* $m_j(\gamma) = \int_0^\infty e^{\gamma u} F_j(du)$ *have their singularity at* $\gamma = \alpha$.

(ii) *If* $m_j = m_j(\alpha)$, *then* $\overline{F_1 * F_2} \gtrsim m_2\overline{F}_1 + m_1\overline{F}_2$.

In particular, if $\overline{F_1 * F_1} \sim 2d_1\overline{F}_1$ *with* $d_1 < \infty$, *then* $m_1 \leqslant d_1$.

PROOF. (ii) Although this has been observed by several authors, we repeat it because it is such a basic result. Write

$$\overline{F_1 * F_2}(t) = \int_0^{t/2} \overline{F}_1(t-u)F_2(du) + \int_0^{t/2} \overline{F}_2(t-u)F_1(du) + \overline{F}_1(t/2)\overline{F}_2(t/2).$$

Fatou's lemma gives

$$\liminf_{t\to\infty} \int_0^{t/2} \frac{\overline{F}_1(t-u)}{\overline{F}_1(t)} F_2(du) \geqslant \int_0^\infty e^{\alpha u} F_2(du) = m_2.$$

The second term is handled similarly and the third is ignored.

Lemma 2.2(ii) holds for *all* distributions when m_j is replaced with $\overline{F}_j(0) = F_j[0,\infty)$. In fact, it may be shown that if $\overline{F_1 * F_2} \sim \overline{F}_2(0)\overline{F}_1 + \overline{F}_1(0)\overline{F}_2$ and $\overline{F}_2 \lesssim c\overline{F}_1$ then $F_1 \in L_0$. (See Chistykov (1964) for a proof when $F_1 = F_2$.)

Throughout the paper, we will use $m_j = \int_0^\infty e^{\alpha u} F_j(du)$ whenever $F_j \in L_\alpha$ and many of our theorems will assume this parameter to be finite. When referring to distributions F, G or H, the parameter will be denoted $m(= m_F)$, m_G or m_H.

The next lemma shows that L_α is closed under convolutions.

LEMMA 2.3. (i) (*Embrechts and Goldie*, 1980). *If* $F_j \in L_\alpha$, *then* $F_1 * F_2 \in L_\alpha$.

(ii) *Assume* $\lambda_n \geqslant 0$ *and* $\sum_0^\infty \lambda_n(\overline{F}(0))^n < \infty$. *If* $F \in L_\alpha$ *and* $H = \sum_0^\infty \lambda_n F^{*n}$, *then for* $u > 0$, $\overline{H}(t-u) \lesssim e^{\alpha u}\overline{H}(t)$. *In particular, if* $\alpha = 0$ *then* $H \in L_0$.

PROOF. (i) We have simplified slightly the proof by Embrechts and Goldie. Fix $u > 0$. For large enough s and $t_0 \geqslant 2s$,

$$(1-\varepsilon)e^{\alpha u}\overline{F}_1(t) \leqslant \overline{F}_1(t-u) \leqslant (1+\varepsilon)e^{\alpha u}\overline{F}_1(t), \qquad t \geqslant s,$$

and

$$(1-\varepsilon)e^{\alpha v}\overline{F}_2(t) \leqslant \overline{F}_2(t-v) \leqslant (1+\varepsilon)e^{\alpha v}\overline{F}_2(t),$$

$$t \geqslant t_0 \text{ and } v = u,\ s \text{ or } u+s.$$

Embrechts and Goldie show that for all $t \geqslant t_0$,

$$\overline{F_1 * F_2}(t-u) \leqslant (1+\varepsilon)e^{\alpha u}\overline{F_1 * F_2}(t). \tag{2.1}$$

Thus,

$$\limsup_{t\to\infty} \frac{\overline{F_1 * F_2}(t-u)}{\overline{F_1 * F_2}(t)} \leqslant e^{\alpha u}.$$

Similarly, for $t \geqslant t_0$,

$$\begin{aligned}\overline{F_1 * F_2}(t-u) &\geqslant (1-\varepsilon)e^{\alpha u}\overline{F_1 * F_2}(t) - \int_{t-u-s}^{t-s} \overline{F}_1(t-u-v)F_2(dv)\\ &\geqslant (1-\varepsilon)e^{\alpha u}\overline{F_1 * F_2}(t) - \overline{F}_1(s-u)\overline{F}_2(t-u-s)\\ &\geqslant (1-\varepsilon)e^{\alpha u}\overline{F_1 * F_2}(t) - (1+\varepsilon)^2e^{2\alpha u}e^{\alpha s}\overline{F}_1(s)\overline{F}_2(t).\end{aligned}$$

Now, $\overline{F}_2 \lesssim 1/m_1\overline{F_1 * F_2}$ by Lemma 2.2(ii). Thus, if $m_1 = \infty$, $\overline{F}_2 = o(\overline{F_1 * F_2})$ and if $m_1 < \infty$, then $e^{\alpha s}\overline{F}_1(s) \to 0$ as $s \to \infty$. In either case,

$$\liminf_{t\to\infty} \frac{\overline{F_1 * F_2}(t-u)}{\overline{F_1 * F_2}(t)} \geqslant e^{\alpha u},$$

which proves (i).

(ii) Apply (2.1) recursively, with $F_1 = F$, $F_2 = F^{*n-1}$. Thus

$$\overline{F}^{*n}(t-u) \leqslant (1+\varepsilon)e^{\alpha u}\overline{F}^{*n}(t), \quad \text{all } n, \text{all } t \geqslant t_0.$$

Hence $\overline{H}(t-u) \leqslant (1+\varepsilon)e^{\alpha u}\overline{H}(t)$ and

$$\limsup_{t\to\infty} \frac{\overline{H}(t-u)}{\overline{H}(t)} \leqslant e^{\alpha u}.$$

In case $\alpha = 0$, then $H \in L_0$ follows because $\overline{H}$ is nonincreasing.

Although we have not given sufficient conditions here for H to be in L_α (except when $\alpha = 0$), we will later show that under certain assumptions $F \in S_\alpha$ implies $H \in S_\alpha$. The next lemma, a new result, is the basis for all our later results.

LEMMA 2.4. *Suppose $F_j \in L_\alpha$ and $m_j < \infty$, $j = 1,2,3,4$.*

(i) *If $\overline{F}_1 \lesssim \overline{F}_3$ and $\overline{F}_2 \lesssim \overline{F}_4$, then $\overline{F_1 * F_2} + m_4\overline{F}_3 + m_3\overline{F}_4 \lesssim \overline{F_3 * F_4} + m_2\overline{F}_1 + m_1\overline{F}_2$.*

(ii) *If $\overline{F}_1 \sim \overline{F}_3$ and $\overline{F}_2 \sim \overline{F}_4$, then $\overline{F_1 * F_2} \sim \overline{F_3 * F_4} + (m_2 - m_4)\overline{F}_3 + (m_1 - m_3)\overline{F}_4$.*

PROOF. We will prove only (i). The proof for (ii) uses a double application of (i) and an appeal to Lemma 2.1(ii).

We may write

$$\begin{aligned}(2.2)\qquad \overline{F_1 * F_2}(t) = &\int_0^s \overline{F}_1(t-u)F_2(du) + \int_0^s \overline{F}_2(t-u)F_1(du)\\ &+ \int_s^{t-s} \overline{F}_1(t-u)F_2(du) + \overline{F}_1(s)\overline{F}_2(t-s).\end{aligned}$$

$$\begin{aligned}(2.3)\qquad \overline{F_3 * F_4}(t) = &\int_0^s \overline{F}_3(t-u)F_4(du) + \int_0^s \overline{F}_4(t-u)F_3(du)\\ &+ \int_s^{t-s} \overline{F}_4(t-u)F_3(du) + \overline{F}_4(s)\overline{F}_3(t-s).\end{aligned}$$

For large enough s, $\overline{F}_1(u) \leqslant (1+\varepsilon)\overline{F}_3(u)$ and $\overline{F}_2(u) \leqslant (1+\varepsilon)\overline{F}_4(u)$, all $u \geqslant s$. Thus for $t > 2s$,

$$\begin{aligned}(2.4)\qquad &\int_s^{t-s} \overline{F}_1(t-u)F_2(du) + \overline{F}_1(s)\overline{F}_2(t-s)\\ &\leqslant (1+\varepsilon)\left[\int_s^{t-s} \overline{F}_3(t-u)F_2(du) + \overline{F}_3(s)\overline{F}_2(t-s)\right]\\ &= (1+\varepsilon)\left[\int_s^{t-s} \overline{F}_2(t-u)F_3(du) + \overline{F}_2(s)\overline{F}_3(t-s)\right]\\ &\leqslant (1+\varepsilon)^2\left[\int_s^{t-s} \overline{F}_4(t-u)F_3(du) + \overline{F}_4(s)\overline{F}_3(t-s)\right].\end{aligned}$$

Again for large s,

$$\left|\int_0^s e^{\alpha u}F_j(du) - m_j\right| < \varepsilon \min_i m_i, \qquad j = 1,2,3,4.$$

Fixing s, we may choose t large enough so that

$$\left|\int_0^s \frac{\overline{F}_i(t-u)}{\overline{F}_i(t)}F_j(du) - \int_0^s e^{\alpha u}F_j(du)\right| < \varepsilon m_j, \qquad i,j = 1,2,3,4,\ i \neq j.$$

From (2.2), (2.3) and (2.4) we thus have

$$\begin{aligned}&\overline{F_1 * F_2}(t) - (1+2\varepsilon)\big(m_2\overline{F}_1(t) + m_1\overline{F}_2(t)\big)\\ &\qquad \leqslant (1+\varepsilon)^2\big[\overline{F_3 * F_4}(t) - (1-2\varepsilon)\big(m_4\overline{F}_3(t) + m_3\overline{F}_4(t)\big)\big].\end{aligned}$$

This implies

$$(1-4\varepsilon^2)\left[\overline{F_1 * F_2}(t) + m_4\bar{F}_3(t) + m_3\bar{F}_4(t)\right] \leqslant (1+\varepsilon)^2\left[\overline{F_3 * F_4}(t) + m_2\bar{F}_1(t) + m_1\bar{F}_2(t)\right].$$

Since ε is arbitrary, then

$$\overline{F_1 * F_2} + m_4\bar{F}_3 + m_3\bar{F}_4 \lesssim \overline{F_3 * F_2} + m_2\bar{F}_1 + m_1\bar{F}_2.$$

The power of Lemma 2.4 can be seen in the next three corollaries.

COROLLARY 2.5. *If $F_1 \in L_\alpha$, $m_1 < \infty$ and $\bar{F}_1 \sim \bar{F}_2$, then*

$$\overline{F_1 * F_2} \sim \overline{F_1 * F_1} + (m_2 - m_1)\bar{F}_1 \quad \text{and}$$
$$\overline{F_2 * F_2} \sim \overline{F_1 * F_1} + 2(m_2 - m_1)\bar{F}_1.$$

In particular, if $\overline{F_1 * F_1} \sim 2d_1\bar{F}_1$, then $\overline{F_2 * F_2} \sim 2(m_2 + d_1 - m_1)\bar{F}_2$.

PROOF. The first two relationships are straightforward applications of Lemma 2.4(ii). The third follows immediately from the second.

We see, therefore, that S_α is closed under asymptotic tail equivalency.

COROLLARY 2.6. *Let $F \in L_\alpha$ and $m = \int_0^\infty e^{\alpha u}F(du) < \infty$.*

(i) *Then $\bar{F}^{*n} \gtrsim m^{n-2}\overline{F * F} + (n-2)m^{n-1}\bar{F} \gtrsim nm^{n-1}\bar{F}$.*

(ii) *If $\overline{F * F} \lesssim 2d\bar{F}$ ($\overline{F * F} \gtrsim 2d\bar{F}$), then $\bar{F}^{*n} \lesssim a_n\bar{F}$ ($\bar{F}^{*n} \gtrsim a_n\bar{F}$), where*

$$a_n = \begin{cases} nm^{n-1}, & \text{if } d = m, \\ \dfrac{(2d-m)^n - m^n}{2(d-m)}, & \text{if } d \neq m. \end{cases}$$

PROOF. (i) By repeated application of Lemma 2.2(ii),

$$\begin{aligned} \bar{F}^{*n} &\gtrsim m\bar{F}^{*n-1} + m^{n-1}\bar{F} \\ &\gtrsim m^2\bar{F}^{*n-2} + 2m^{n-1}\bar{F} \\ \cdots &\gtrsim m^{n-2}\overline{F * F} + (n-2)m^{n-1}\bar{F} \\ &\gtrsim nm^{n-1}\bar{F}. \end{aligned}$$

(ii) Assume $\overline{F * F} \lesssim 2d\bar{F}$. The proof is by induction and is similar when $\overline{F * F} \gtrsim 2d\bar{F}$. Suppose $\bar{F}^{*n} \lesssim a_n\bar{F}$. Then using Lemma 2.4(i) and Lemma 2.1(ii),

$$\begin{aligned} \bar{F}^{*n+1} = \overline{F^{*n} * F} &\leqslant a_n\overline{F * F} + (m^n - a_n m)\bar{F} \\ &\lesssim ((2d-m)a_n + m^n)\bar{F} = a_{n+1}\bar{F}. \end{aligned}$$

Once we establish that $d = m$, we will also be able to show that $\bar{F}^{*n} \sim nm^{n-1}\bar{F}$ is equivalent to $F \in S_\alpha$. (See Corollary 2.11.)

COROLLARY 2.7. *Suppose* $F_1 \in L_\alpha$, $\overline{F_1 * F_1} \sim 2d_1\bar{F}_1$ *and* $\lim_{t\to\infty} \bar{F}_2(t)/\bar{F}_1(t) = k < \infty$. *Then* $\bar{G} = \overline{F_1 * F_2} \sim (m_2 + k(2d_1 - m_1))\bar{F}_1$ *and* $G \in S_\alpha$.

PROOF. The proof is valid whether $k = 0$ or $k > 0$. Since $\overline{F_1 + F_2} \sim (1 + k)\bar{F}_1$, then by Lemma 2.4(ii),

$$\begin{aligned}\overline{F_1 * (F_1 + F_2)} &\sim (1 + k)\overline{F_1 * F_1} + (m_2 - km_1)\bar{F}_1\\ &\sim (2d_1(1 + k) + (m_2 - km_1))\bar{F}_1.\end{aligned}$$

However, we also have $\overline{F_1 * (F_1 + F_2)} = \overline{F_1 * F_1} + \bar{G} \sim 2d_1\bar{F}_1 + \bar{G}$. Applying Lemma 2.1(ii), $\bar{G} \sim (m_2 + k(2d_1 - m_1))\bar{F}_1 = c\bar{F}_1$. By Corollary 2.5, $\overline{G * G} \sim 2(m_G + c(d_1 - m_1))\bar{G}$. Therefore $G \in S_\alpha$.

The proof of the next lemma follows that of a similar lemma in Chover, Wainger and Ney (1973b). (See Lemma 2.12.)

LEMMA 2.8. *Suppose* $F \in L_\alpha$, $\overline{F * F} \sim 2d\bar{F}$, $d < \infty$, *and* $m = \int_0^\infty e^{\alpha u}F(du)$. *Then for each* $\varepsilon > 0$ *and some* K_ε,

$$\bar{F}^{*n}(t) \leqslant K_\varepsilon(2d - m + \varepsilon)^{n-1}\bar{F}(t), \quad \textit{all } n, \textit{ all } t.$$

PROOF. For large enough s and t,

$$\int_0^s \frac{\bar{F}(t-u)}{\bar{F}(t)}F(du) \geqslant m - \varepsilon/4.$$

Fixing s, we may choose $t_0 > s$ so that for $t \geqslant t_0$,

$$\begin{aligned}\frac{\overline{F * F}(t)}{\bar{F}(t)} &= \int_0^{t-s}\frac{\bar{F}(t-u)}{\bar{F}(t)}F(du) + \int_0^s \frac{\bar{F}(t-u)}{\bar{F}(t)}F(du) + \frac{\bar{F}(s)\bar{F}(t-s)}{\bar{F}(t)}\\ &\leqslant 2d + \varepsilon/4.\end{aligned}$$

Thus $\int_0^{t-s}\bar{F}(t-u)F(du) + \bar{F}(s)\bar{F}(t-s) \leqslant (2d - m + \varepsilon/2)\bar{F}(t)$, $t \geqslant t_0$. Choose

$$K_\varepsilon = \max\left[\frac{2}{\varepsilon}\sup_{t\geqslant t_0}\frac{\bar{F}(t-s)}{\bar{F}(t)}, \frac{\bar{F}(0)}{\bar{F}(t_0)}\right] > 1.$$

The conclusion obviously holds for $n = 1$. We continue by induction. For $t \leqslant t_0$,

$$\bar{F}^{*n+1}(t) \leqslant \bar{F}^{*n+1}(0) \leqslant m^n\bar{F}(0)\frac{\bar{F}(t)}{F(t_0)} \leqslant K_\varepsilon(2d - m + \varepsilon)^n\bar{F}(t).$$

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

For $t > t_0$,

$$
\begin{aligned}
\bar{F}^{*n+1}(t) \\
&= \int_0^{t-s} \bar{F}^{*n}(t-u)F(du) + \bar{F}^{*n}(s)\bar{F}(t-s) + \int_0^{s} \bar{F}(t-u)F^{*n}(du) \\
&\leqslant K_\varepsilon(2d-m+\varepsilon)^{n-1}\left[\int_0^{t-s} \bar{F}(t-u)F(du) + \bar{F}(s)\bar{F}(t-s)\right] + m^n\bar{F}(t-s) \\
&\leqslant \left[K_\varepsilon(2d-m+\varepsilon)^{n-1}(2d-m+\varepsilon/2) + K_\varepsilon(\varepsilon/2)m^n\right]\bar{F}(t) \\
&\leqslant K_\varepsilon(2d-m+\varepsilon)^{n}\bar{F}(t).
\end{aligned}
$$

We come now to our first major result, a new proof of a theorem by Chover, Wainger and Ney (1973a).

Theorem 2.9. *If $F \in S_\alpha$, then $\overline{F * F} \sim 2m\bar{F}$ where $m = \int_0^\infty e^{\alpha u}F(du)$.*

Proof. Suppose $\alpha > 0$ and define

$$\bar{F}_0(t) = \int_t^\infty e^{\alpha u}F(du).$$

It is easy to show that $F \in S_\alpha$ implies $F_0 \in S_0$ (see Embrechts and Goldie (1982); they also show that the converse is not true), that $m_0(0) = m_F(\alpha) = m$ and that $\overline{F_0 * F_0} \sim 2d\bar{F}_0$. From these we see that it suffices to prove the theorem for the case $\alpha = 0$.

From Lemma 2.2, we know that $d \geqslant m$. For $\lambda < 1/m$, define

$$H_\lambda = \sum_0^\infty \lambda^n F^{*n}, \qquad m_\lambda = \int_0^\infty H_\lambda(du) = \frac{1}{1-m\lambda}.$$

By Lemma 2.3(ii), $H_\lambda \in L_0$. If $\lambda < 1/(2d-m)$, then Corollary 2.6(ii) and Lemma 2.8 combine with dominated convergence to give

$$\bar{H}_\lambda \sim \left[\sum_0^\infty \lambda^n\left(\frac{(2d-m)^n - m^n}{2(d-m)}\right)\right]\bar{F} = \frac{\lambda}{(1-m\lambda)(1-(2d-m)\lambda)}\bar{F}. \tag{2.5}$$

(This is valid even when $d = m$.) On the other hand, by Fatou's Lemma, $\lambda \geqslant 1/(2d-m)$ implies

$$\liminf_{t\to\infty} \frac{\bar{H}_\lambda(t)}{\bar{F}(t)} \geqslant \sum_0^\infty \lambda^n\left(\frac{(2d-m)^n - m^n}{2(d-m)}\right) = \infty. \tag{2.6}$$

Let

$$\lambda_0 = \sup\{\lambda : \bar{H}_\lambda \sim k_\lambda \bar{F} \text{ for some } k_\lambda < \infty\}.$$

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

Since k_λ must increase with λ, it is clear from (2.5) and (2.6) that $\lambda_0 = 1/(2d - m)$ and that for all $\lambda < \lambda_0$, $\overline{H}_\lambda \sim k_\lambda \overline{F}$ for some finite k_λ.

Let

$$\lambda_1 = \sup\left\{\lambda: \overline{H_\lambda * H_\lambda} \sim 2d_\lambda \overline{H}_\lambda \text{ for some } d_\lambda < \infty\right\}.$$

By Corollary 2.5, $d_\lambda = m_\lambda + k_\lambda(d - m)$ for all $\lambda < \lambda_0$. Thus $\lambda_1 \geqslant \lambda_0$. Suppose $\lambda_1 > \lambda_0$. (This requires $d > m$.) Let $\mu < \lambda_0 \leqslant \lambda < \lambda_1$. Then $\overline{H}_\mu \sim k_\mu \overline{F} = o(\overline{H}_\lambda)$. Thus,

$$\begin{aligned}\overline{H_\mu * H_\lambda} &= \sum_0^\infty \sum_0^\infty \lambda^j \mu^{n-j} \overline{F}^{*n} = \sum_0^\infty \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu} \overline{F}^{*n} \\ &= \left(\frac{\lambda}{\lambda - \mu}\right)\overline{H}_\lambda - \left(\frac{\mu}{\lambda - \mu}\right)\overline{H}_\mu \sim \left(\frac{\lambda}{\lambda - \mu}\right)\overline{H}_\lambda.\end{aligned}$$

But by Corollary 2.7, $\overline{H_\mu * H_\lambda} \sim m_\mu \overline{H}_\lambda$, a contradiction since $m_\mu < \lambda/(\lambda - \mu)$. Hence $\lambda_1 = \lambda_0$.

Continue to assume $d > m$ and let $\lambda_0 < \nu < \mu < \lambda < 1/m$. Since $\overline{H}_\mu = o(\overline{H_\mu * H_\mu}) < \overline{H_\mu * H_\lambda} < (\lambda/(\lambda - \mu))\overline{H}_\lambda$, it is clear that $\overline{H}_\mu = o(\overline{H}_\lambda)$ and $\overline{H_\mu * H_\lambda} \sim (\lambda/(\lambda - \mu))\overline{H}_\lambda$. Similarly, $\overline{H_\nu * H_\lambda} \sim (\lambda/(\lambda - \nu))\overline{H}_\lambda$ and $\overline{H_\nu * H_\mu} \sim (\mu/(\mu - \nu))\overline{H}_\mu$. Applying Lemma 2.4(ii),

$$\begin{aligned}\left(\frac{\mu}{\mu - \nu}\right)&\left(\frac{\lambda}{\mu - \nu}\right)\overline{H}_\lambda \sim \left(\frac{\mu}{\mu - \nu}\right)\overline{H_\mu * H_\lambda} \\ &= \overline{H_\nu * H_\mu * H_\lambda} + \left(\frac{\nu}{\mu - \nu}\right)\overline{H_\nu * H_\lambda} \\ &\sim \left(\frac{\mu}{\mu - \nu}\right)\overline{H_\mu * H_\lambda} + \left(m_\mu m_\nu - \frac{\mu}{\mu - \nu} m_\mu\right)\overline{H}_\lambda + \left(\frac{\nu}{\mu - \nu}\right)\overline{H_\nu * H_\lambda} \\ &\sim \left[\left(\frac{\mu}{\mu - \nu}\right)\left(\frac{\lambda}{\mu - \nu}\right) + \left(m_\mu m_\nu - \frac{\mu}{\mu - \nu} m_\mu\right) + \left(\frac{\nu}{\mu - \nu}\right)\left(\frac{\lambda}{\mu - \nu}\right)\right]\overline{H}_\lambda.\end{aligned}$$

And this requires $m_\mu m_\nu - (\mu/(\mu - \nu))m_\mu + (\nu/(\mu - \nu))(\lambda/(\lambda - \nu)) = 0$.

But this is certainly not true for arbitrary λ. Therefore we must conclude that $d = m$.

The proof by Chover, Wainger and Ney uses Banach algebra elements and relations defined by asymptotic tails. It is similar to our proof in that it also shows certain relationships must hold in a range defined by m. However, it uses complex valued transforms and Cauchy's theorem. Additionally, they prove part of our Theorem 2.13 before proving $d = m$. Embrechts (1983) gives another theorem similar to Theorem 2.9, except that it refers to the probabilities $f_n = F\{n\}$ of a distribution with mass on the integers. His proof relies on a real analytic theorem by Rudin (1973).

(Because $\overline{F * F} \sim 2d\overline{F}$ implies $d = m_F(\alpha)$ when $F \in L_\alpha$, one might also suspect that $\overline{F}^{*n} \sim d_n \overline{F}$ is equally sufficient. The answer to this ultimately depends on whether the bounds in Corollary 2.6(ii) are in fact sharp. For example, $\bar{F}^{*n} \sim d_n \bar{F}$ implies $\overline{F * F} \lesssim 2dF$ for some d, and this alone is sufficient to prove, for Theorem 2.9,

$$\frac{1}{2d-m} \leqslant \lambda_0 = \lambda_1 = \frac{1}{m}.$$

The remaining inequality, however, can only be resolved if one knows that

$$\lim_{t\to\infty} \sup \frac{\overline{F * F}(t)}{\bar{F}(t)} = 2d \quad \text{implies} \quad \lim_{t\to\infty} \sup \frac{\bar{F}^{*n}(t)}{\bar{F}(t)} = \frac{(2d-m)^n - m^n}{2(d-m)}.\Bigg)$$

Knowing that $d = m$, we may revise several of our earlier results, crediting those who have previously stated them in this form.

Corollary 2.10 (Embrechts and Goldie, 1982). *If $F_1 \in S_\alpha$ and $\bar{F}_2 \sim k\bar{F}_1$ for some $k > 0$, then $F_2 \in S_\alpha$ and $F_1 * F_2 \in S_\alpha$. If $\bar{F}_2 = o(\bar{F}_1)$, then $F_1 * F_2 \in S_\alpha$.*

Proof. Corollaries 2.5 and 2.7 and Theorem 2.9.

Corollary 2.11 (Chover, Wainger and Ney, 1973a, b; Embrechts and Goldie, 1982). *The following are equivalent.*

(i) $F \in S_\alpha$.

(ii) $F \in L_\alpha$ *and* $\bar{F}^{*n} \sim nm^{n-1}\bar{F}$ *for some (hence all)* n.

(iii) $F \in L_\alpha$ *and* $F^{*n} \in S_\alpha$ *for some (hence all)* n.

Proof. If $F \in S_\alpha$, then $F \in L_\alpha$ and $\overline{F * F} \sim 2m\bar{F}$ from Theorem 2.9. By Corollary 2.6, $\bar{F}^{*n} \sim nm^{n-1}\bar{F}$. The remainder of the proof is virtually the same as that for Theorem 2.13 and Corollary 2.14, which we provide later.

The remarks following Theorem 2.13 apply equally well to Corollary 2.11, which is a special case.

Lemma 2.12 (Chover, Wainger and Ney, 1973b). *If $F \in S_\alpha$, then for each $\varepsilon > 0$ and some K_ε, $\bar{F}^{*n}(t) \leqslant K_\varepsilon(m+\varepsilon)^{n-1}\bar{F}(t)$ all n, all t.*

Proof. Lemma 2.8 and Theorem 2.9.

The next theorem both strengthens and generalizes the known results for distributions of random sums.

Theorem 2.13. *Assume $F \in L_\alpha$, $m = \int_0^\infty e^{\alpha u} F(du)$. Let $\{\lambda_n\}$ be a sequence of nonnegative coefficients such that $\lambda_j > 0$ for some $j > 1$ and $\sum_0^\infty \lambda_n (m+\varepsilon)^n < \infty$ for some $\varepsilon > 0$. Denote $H = \sum_0^\infty \lambda_n F^{*n}$. The following are equivalent.*

(i) $F \in S_\alpha$.

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

(ii) $\overline{H} \sim c\overline{F}$ *for* $c = \sum_1^\infty n\lambda_n m^{n-1}$.

(iii) $H \in S_\alpha$ *and* $\overline{F} \neq o(\overline{H})$.

Proof. That (i) implies (ii) follows by dominated convergence using Corollary 2.11 and Lemma 2.12. By Corollary 2.5, it immediately follows that (i) implies (iii).

To show that (ii) implies (i) we use Corollary 2.6(i). Assume $\lambda_j > 0$ for some $j \geq 2$. Then $\overline{F}^{*j} \gtrsim jm^{j-1}\overline{F}$ and

$$\lambda_j \overline{F}^{*j} = \overline{H} - \sum_{n \neq j} \lambda_n \overline{F}^{*n}$$

$$\lesssim c\overline{F} - \sum_{n \neq j} n\lambda_n m^{n-1}\overline{F}$$

$$= j\lambda_j m^{j-1}\overline{F}.$$

Thus $\overline{F}^{*j} \sim jm^{j-1}\overline{F}$. Also from Corollary 2.6(ii) $\overline{F}^{*j} \gtrsim m^{j-2}\overline{F * F} + (j-2)m^{j-1}\overline{F}$, so that by Lemma 2.1(i, ii), $\overline{F * F} \lesssim 2m\overline{F} + m^{2-j}\overline{F}^{*j} - jm\overline{F} \sim 2m\overline{F}$. This shows $F \in S_\alpha$, which is (i).

To show that (iii) implies (i), we first note that according to Corollary 2.6, $\overline{F}^{*n} \gtrsim nm^{n-1}\overline{F}$ for all n. Hence $H \gtrsim (\sum_0^\infty n\lambda_n m^{n-1})\overline{F} = c\overline{F}$. However, $\overline{F} \neq o(\overline{H})$, so by Corollary 3.2(i) in the next section, $F \in S_\alpha$.

In case $\alpha = 0$, the assumption $F \in L_0$ may be replaced by $m = \overline{F}(0)$ (and $F \in S_0$ by $\overline{F * F} \sim 2m\overline{F}$, similarly for H). Indeed, as remarked following Lemma 2.2, $F \in L_0$ is a consequence of $\overline{F * F} \sim 2m\overline{F}$ (and $H \in L_0$ of $\overline{H * H} \sim 2\overline{H}(0)\overline{H}$). The same remark allows us to use Corollary 2.6(i) to show (ii) implies (i). As Embrechts and Goldie (1982) point out, the requirement $F \in L_\alpha$ when $\alpha > 0$ seems to be related to the unsolved question of whether $\overline{F * F} \sim 2d\overline{F}$ actually implies $F \in L_\alpha$ for some α.

Chover, Wainger and Ney (1973) have already shown that (i) implies (ii) in Theorem 2.13, while Embrechts and Goldie (1982) and Embrechts, Goldie and Veraverbeke (1979) proved the equivalency of all three statements for the special cases where the λ_n are Poisson and geometric probabilities. We now show that for these cases and for the case $H = F^{*n}$ the assumption $\overline{F} \neq o(\overline{H})$ in (iii) is unnecessary.

Corollary 2.14. *Suppose* $F \in L_\alpha$, $H = \sum_0^\infty \lambda_n F^{*n}$ *and* $H \in S_\alpha$. *Then each of the following implies* $F \in S_\alpha$.

(i) $\lim_{n \to \infty} \sup(\lambda_{n+1})/\lambda_n < 1/m$.

(ii) *For* $q = m_H(\overline{F}(0)/\overline{H}(0))$, $\sum_0^\infty \lambda_n (q + \varepsilon)^n < \infty$ *for some* $\varepsilon > 0$.

Proof. For both conditions, the proof relies on Theorem 2.13 and the demonstration that $\bar{F} \neq o(\bar{H})$.

(i) Clearly, $\sum_0^\infty \lambda_n(m+\varepsilon)^n < \infty$ for some $\varepsilon > 0$, as required. Assume $\bar{F} = o(\bar{H})$. Then if $\bar{F} \lesssim \varepsilon\bar{H}$ and $\bar{F}^{*n} \lesssim \varepsilon k_n \bar{H}$, Lemma 2.4(i) yields,

$$\begin{aligned}\bar{F}^{*n+1} &\lesssim \bar{F}^{*n+1} + 2\varepsilon^2 k_n m_H \bar{H} \\ &\lesssim \varepsilon^2 k_n \overline{H*H} + m^n\bar{F} + m\bar{F}^{*n} \\ &\lesssim \varepsilon\left[m^n + mk_n + 2\varepsilon m_H k_n\right]\bar{H},\end{aligned}$$

since $\overline{H*H} \sim 2m_H\bar{H}$. Hence, $\bar{F} = o(\bar{H})$ implies $\bar{F}^{*n} = o(\bar{H})$ for all n. Modifying a finite number of the λ_n, therefore, will not change the assumption $H \in S_\alpha$. We thus assume that $\lambda_0 = 0$ and $(m+\varepsilon)\lambda_{n+1} \leqslant \lambda_n$ for all $n \geqslant 1$. From Corollary 2.7, $\bar{F} = o(\bar{H})$ implies $\overline{H*F} \sim m\bar{H}$. Thus

$$\begin{aligned}\bar{H} &= \sum_1^\infty \lambda_n \bar{F}^{*n} = \lambda_1\bar{F} + \sum_1^\infty \lambda_{n+1}\bar{F}^{*n+1} \\ &\leqslant \lambda_1\bar{F} + \frac{1}{m+\varepsilon}\overline{H*F} \\ &\sim \frac{m}{m+\varepsilon}\bar{H}.\end{aligned}$$

This contradiction demonstrates that $\bar{F} \neq o(\bar{H})$.

(ii) We assume without any loss that $\bar{F}(0) = 1$, $\lambda_0 = 0$ and $\sum_1^\infty \lambda_n = 1$ (so that $\bar{H}(0) = 1$, also). Since $\bar{F}(t) \leqslant \bar{F}^{*n}(t)$, then for all t, $\bar{F}(t) \leqslant \sum_1^\infty \lambda_n \bar{F}^{*n}(t) = \bar{H}(t)$ and $\bar{F}^{*n}(t) \leqslant \bar{H}^{*n}(t)$. By Lemma 2.12, applied to H, $\bar{F}^{*n}(t) \leqslant \bar{H}^{*n}(t) \leqslant K_\varepsilon(m_H+\varepsilon)^{n-1}\bar{H}(t)$. If we further assume $\bar{F} = o(\bar{H})$, then, as above, $\bar{F}^{*n} = o(\bar{H})$ and dominated convergence yields

$$1 = \lim_{t\to\infty}\frac{\bar{H}(t)}{\bar{H}(t)} = \sum_1^\infty \lambda_n \lim_{t\to\infty}\frac{\bar{F}^{*n}(t)}{\bar{H}(t)} = 0,$$

a contradiction. Thus $\bar{F} \neq o(\bar{H})$.

3. Closure and factorization

This section primarily improves results in Embrechts and Goldie (1980) for subexponential distributions and generalizes them to include exponential tail distributions. We investigate two properties for distributions $F_j \in S_\alpha$,

$$\text{closure (under } *\text{)}: F_1 * F_2 \in S_\alpha,$$

$$\text{factorization}: \overline{F_1 * F_2} \sim m_2\bar{F}_1 + m_1\bar{F}_2.$$

We show in Theorem 3.4, as did Embrechts and Goldie for S_0, that these properties are equivalent when both F_1, $F_2 \in S_\alpha$. Although we still cannot verify that S_α is in fact closed under convolutions, we can provide fairly general conditions on F_1 and F_2 for which the property holds.

We start with a useful lemma.

LEMMA 3.1. *Suppose F_1, $F_2 \in L_\alpha$, $F_3 \in S_\alpha$. If $\bar{F}_1 \lesssim \bar{F}_3$, $\bar{F}_2 \lesssim \bar{F}_3$ and $F_3 \lesssim k\overline{F_1 * F_2}$ for some $k < \infty$, then each of the following holds.*

(i) $\overline{F_1 * F_2} \in S_\alpha$.

(ii) $\overline{F_1 * F_2} \sim m_2\bar{F}_1 + m_1\bar{F}_2$.

(iii) *At least one of F_1, F_2 is in S_α. The other, if not in S_α, has an asymptotically negligible tail.*

PROOF. We obtain (ii) using first Lemma 2.4(i), $\overline{F_1 * F_2} + 2m_3\bar{F}_3 \lesssim \overline{F_3 * F_3} + (m_2\bar{F}_1 + m_1\bar{F}_2)$. Thus, by Lemma 2.1(i), $\overline{F_1 * F_2} \lesssim (m_2\bar{F}_1 + m_1\bar{F}_2)$. And by Lemma 2.2(ii), this is sufficient for $\overline{F_1 * F_2} \sim m_2\bar{F}_1 + m_1\bar{F}_2$. We can also show $\overline{F_1 * F_2} \sim m_2\bar{F}_1 + m_1\bar{F}_2 \lesssim (m_2 + m_1)\bar{F}_3$ and $\bar{F}_3 \lesssim k_1\overline{(F_1 * F_2)^{*2}}$ for $k_1 = k/\bar{F}_1(0)\bar{F}_2(0)$. Therefore, a repeat argument gives $\overline{(F_1 * F_2)^{*2}} \sim 2m_1m_2\overline{F_1 * F_2}$. That is, $F_1 * F_2 \in S_\alpha$.

Because of the equivalency, $m_2\bar{F}_1 + m_1\bar{F}_2 \in S_\alpha$ also. Therefore,

$$\begin{aligned} 4m_1m_2\left(m_2\bar{F}_1 + m_1\bar{F}_2\right) &\sim m_2^2\overline{F_1 * F_1} + 2m_1m_2\overline{F_1 * F_2} + m_1^2\overline{F_2 * F_2} \\ &\sim 2m_1m_2\left(m_2\bar{F}_1 + m_1\bar{F}_2\right) + m_2^2\overline{F_1 * F_1} + m_1^2\overline{F_2 * F_2}. \end{aligned}$$

This implies

$$2m_1m_2^2\bar{F}_1 + 2m_2m_1^2\bar{F}_2 \sim m_2^2\overline{F_1 * F_1} + m_1^2\overline{F_2 * F_2}.$$

But each term on the left is dominated by the corresponding term on the right, asymptotically. Thus by Lemma 2.1(iv), either $2m_1\bar{F}_1 \sim \overline{F_1 * F_1}$ or $2m_2\bar{F}_2 \sim \overline{F_2 * F_2}$, and if not both, then one set of terms must be negligible with respect to the other. In other words, we must have one of

$$\begin{aligned} &F_1 \in S_\alpha, \quad F_2 \in S_\alpha, \\ &F_1 \in S_\alpha, \quad \overline{F_2 * F_2} = o\left(\bar{F}_1\right), \quad \text{or} \\ &F_2 \in S_\alpha, \quad \overline{F_1 * F_1} = o\left(\bar{F}_2\right). \end{aligned}$$

This immediately gives the following corollary which provides one way to determine if a distribution is in S_α.

COROLLARY 3.2. *Suppose $F_1 \in S_\alpha$ and $F_2 \in L_\alpha$.*

(i) *If $\bar{F}_2 \lesssim \bar{F}_1$, then $F_1 * F_2 \in S_\alpha$. If $\bar{F}_2 \neq o(\bar{F}_1)$ also, then $F_2 \in S_\alpha$.*

(ii) *If $\bar{F}_1 \lesssim \bar{F}_2$, then $F_1 * F_2 \in S_\alpha$ if and only if $F_2 \in S_\alpha$.*

PROOF. (i) Follows from Lemma 3.1 with $F_3 = F_1$.

(ii) Suppose $F_2 \in S_\alpha$. Then $F_1 * F_2 \in S_\alpha$ by (i).

If instead we assume $F_1 * F_2 \in S_\alpha$, then using Lemma 3.1 with $F_3 = F_1 * F_2$, we must have $F_2 \in S_\alpha$ since $\overline{F}_2 \neq o(\overline{F}_1)$.

We have already used Corollary 3.2 in the proof of Theorem 2.13.

The next lemma is interesting for its simplicity.

LEMMA 3.3. *Suppose* F_1, $F_2 \in S_\alpha$. *Let* $G = F_1 * F_2$, $H = m_2 F_1 + m_1 F_2$. *Then*

$$\overline{H * H} \sim 2m_1 m_2 \overline{G} + 2m_1 m_2 \overline{H},$$

$$\overline{H * G} \sim 4m_1 m_2 \overline{G} - m_1 m_2 \overline{H}$$

and

$$\overline{G * G} \sim 4m_1 m_2 \overline{G} - 2m_1 m_2 \overline{H}.$$

PROOF. Straightforward application of $\overline{F_j * F_j} \sim 2m_j \overline{F}_j$ and Lemma 2.4.

We now provide the primary theorem for this section. Under the assumption that F_1, $F_2 \in S_0$, Embrechts and Goldie (1980) proved a similar theorem, namely that (i), (ii) and (iv) are equivalent. In addition to proving the theorem for all classes S_α, our contribution has been to weaken the assumptions and to weaken (ii) to (v).

THEOREM 3.4. *Suppose* F_1, $F_2 \in L_\alpha$, $G = F_1 * F_2$, $H = m_2 F_1 + m_1 F_2$. *Then the following are equivalent.*

(i) $G \in S_\alpha$.

(ii) $\overline{G} \sim \overline{H}$, *with* (vi) *below.*

(iii) $H \in S_\alpha$.

(iv) $a_1 F_1 + a_2 F_2 \in S_\alpha$ *for some (hence all)* a_1, $a_2 > 0$.

(v) $\overline{G} \sim a_1 \overline{F}_1 + a_2 \overline{F}_2$ *for some* a_1, a_2, *with* (vi) *below.*

Statements (ii) *and* (v) *refer to*

(vi) *At least one of* F_1 *and* F_2 *is in* S_α. *The other, if not in* S_α, *has an asymptotically negligible tail.*

PROOF. Note that (ii) implies (v) because (ii) is a special case of (v).

That (i) implies (ii) (and (v)) follows immediately from Lemma 3.1.

On the other hand (ii) implies (i), because if $F_1 \in S_\alpha$ and $\overline{F}_2 = o(\overline{F}_1)$, then $F_1 * F_2 \in S_\alpha$ by Corollary 2.10. And if F_1, $F_2 \in S_\alpha$ then by Lemma 3.3, $\overline{G * G} \sim 4m_1 m_2 \overline{G} - 2m_1 m_2 \overline{H} \sim 2m_1 m_2 \overline{G}$.

Furthermore, (i) implies (iv) (with "all") when $F_1, F_2 \in S_\alpha$, since

$$\left(a_1\bar{F}_1 + a_2\bar{F}_2\right)^{*2} = a_1^2\overline{F_1 * F_1} + 2a_1a_2\bar{G} + a_2^2\overline{F_2 * F_2}$$
$$\sim 2(a_1m_1 + a_2m_2)\left(a_1\bar{F}_1 + a_2\bar{F}_2\right).$$

When $F_1 \in S_\alpha$, $\bar{F}_2 = o(\bar{F}_1)$, (iv) holds because $a_1\bar{F}_1 + a_2\bar{F}_2 \sim a_1\bar{F}_1$. To show that (iv) (with "some") implies (i), let $c = \max(a_1/m_2, a_2/m_1)$ and $K = a_1F_1 + a_2F_2$. Then $a_1\bar{F}_1 \lesssim \bar{K}$, $a_2\bar{F}_2 \lesssim \bar{K}$ and $\bar{K} \lesssim c\bar{H} \lesssim c\bar{G}$. Applying Lemma 3.1, it follows that $K \in S_\alpha$ implies $G \in S_\alpha$. Statement (iii) is equivalent to (i) because (iv) is.

It remains only to show that (v) implies (i). Assume without loss that $F_1 \in S_\alpha$. Note first that if $\bar{F}_2 \sim k\bar{F}_1$, then $G \in S_\alpha$ by Corollary 2.10. So we assume $\bar{F}_2 \nsim k\bar{F}_1$ for any k (finite, zero or infinite) and we must also assume $F_2 \in S_\alpha$, since (vi) holds. Note that $m_2\bar{F}_1 + m_1\bar{F}_2 \lesssim a_1\bar{F}_1 + a_2\bar{F}_2$. Since (ii) implies (i), the result is obvious if $a_1 = m_2$, $a_2 = m_1$. If $a_2 < m_1$, then by Lemma 2.1(i), $m_2\bar{F}_1 + (m_1 - a_2)\bar{F}_2 \lesssim a_1\bar{F}_1$, which requires $a_1 \geqslant m_2$. Thus

$$\bar{F}_2 \lesssim (a_1 - m_2)/(m_1 - a_2)\bar{F}_1.$$

By Corollary 3.2(i), $G \in S_\alpha$. (Note that in fact $a_1\bar{F}_1 + a_2\bar{F}_2 \sim \bar{G} \sim m_2\bar{F}_1 + m_1\bar{F}_2$. This implies $a_1 = m_2$, $a_2 = m_1$ since $\bar{F}_1$ and $\bar{F}_2$ are not asymptotically equivalent (Lemma 2.1(iii)).) The case $a_1 < m_2$ is handled similarly. On the other hand, we intend to show that $a_1a_2 = m_1m_2$ and thus no other cases are possible.

Since $\bar{G} \sim a_1\bar{F}_1 + a_2\bar{F}_2$ and $\overline{F_j * F_j} \sim 2m_j\bar{F}_j$, then from Lemma 2.4(ii),

(3.1)
$$\overline{G * G} \sim a_1^2\overline{F_1 * F_1} + 2a_1a_2\bar{G} + a_2^2\overline{F_2 * F_2} + 2(m_1m_2 - a_1m_1 - a_2m_2)\bar{G}$$
$$\sim 2a_1(a_1a_2 + m_1m_2 - a_2m_2)\bar{F}_1 + 2a_2(a_1a_2 + m_1m_2 - a_1m_1)\bar{F}_2.$$

By Lemma 3.3, another asymptotic expression for $\overline{G * G}$ is

$$\overline{G * G} \sim 4m_1m_2\bar{G} - 2m_1m_2\bar{H} \tag{3.2}$$
$$\sim 2m_1m_2(2a_1 - m_2)\bar{F}_1 + 2m_1m_2(2a_2 - m_1)\bar{F}_2.$$

Since $\bar{F}_1$ and $\bar{F}_2$ are not asymptotically equivalent, by Lemma 2.1(iii) the respective coefficients in (3.1) and (3.2) equal. That is,

$$a_1(a_1a_2 + m_1m_2 - a_2m_2) = m_1m_2(2a_1 - m_2)$$

and

$$a_2(a_1a_2 + m_1m_2 - a_1m_1) = m_1m_2(2a_2 - m_1).$$

These reduce to $(a_1 - m_2)(a_1a_2 - m_1m_2) = 0$ and $(a_2 - m_1)(a_1a_2 - m_1m_2) = 0$, requiring at least $a_1a_2 = m_1m_2$ and justifying our argument above.

For $F_1, F_2 \in L_\alpha$, Theorem 3.4 characterizes the case $G \in S_\alpha$ and the case $a_1F_1 + a_2F_2 \in S_\alpha$. It does not however characterize the case $\bar{G} \sim \bar{H}$ without additional assumptions. This suggests there may be examples for which $F_1 * F_2$ is not in S_α, yet we still have $\overline{F_1 * F_2} \sim m_2\bar{F}_1 + m_1\bar{F}_2$ and $F_1, F_2 \in L_\alpha$.

https://doi.org/10.1017/S1446788700029633 Published online by Cambridge University Press

In Corollary 3.2 we have given a condition for $F_1 * F_2$ to be in S_α which is easily checked. We will now give a more general condition.

THEOREM 3.5. *Suppose $F_1, F_2 \in S_\alpha$. Define $b(t) = \bar{F}_1(t)/\bar{F}_2(t)$. If for some $x_1 > 1 > x_0 > 0$, $\sup_t \sup_{x_0 < x < x_1} b(xt)/b(t) < \infty$, then $F_1 * F_2 \in S_\alpha$. In particular, if $b(t) \in RV_\rho$ for any ρ, then $F_1 * F_2 \in S_\alpha$.*

PROOF. The condition on b is called the R-O variation property (Seneta (1976), Appendix). Since $x_0 < 1 < x_1$, the values of x_0 and x_1 may be extended arbitrarily. We set $K = \sup_t \sup_{1/2 \leqslant x \leqslant 2} b(xt)/b(t)$.

Thus, for $0 \leqslant u \leqslant t/2$,

$$\frac{\bar{F}_1(t-u)}{\bar{F}_1(t)} \leqslant K \frac{\bar{F}_2(t-u)}{\bar{F}_2(t)}.$$

Since $F_2 \in S_\alpha$, then $\overline{F_2 * F_2} \sim 2m_2 \bar{F}_2$ so that

$$\lim_{t\to\infty} \int_0^{t/2} \frac{\bar{F}_2(t-u)}{\bar{F}_2(t)} F_2(du) = m_2 = \int_0^\infty \lim_{t\to\infty} \left[\frac{\bar{F}_2(t-u)}{\bar{F}_2(t)} 1_{[0,t/2]}(u) \right] F_2(du)$$

and $\lim_{t\to\infty} \bar{F}_2^2(t/2)/\bar{F}_2(t) = 0$. Therefore, by dominated convergence

$$\lim_{t\to\infty} \int_0^{t/2} \frac{\bar{F}_1(t-u)}{\bar{F}_1(t)} F_2(du) = m_2.$$

We also have

$$\lim_{t\to\infty} \frac{\bar{F}_1(t/2)\bar{F}_2(t/2)}{\bar{F}_1(t)} \leqslant K \lim_{t\to\infty} \frac{\bar{F}_2^2(t/2)}{\bar{F}_2(t)} = 0.$$

And by a similar argument

$$\lim_{t\to\infty} \int_0^{t/2} \frac{\bar{F}_2(t-u)}{\bar{F}_2(t)} F_1(du) = m_1.$$

Thus $\overline{F_1 * F_2}(t) \sim m_2 \bar{F}_1(t) + m_1 \bar{F}_2(t)$ and $F_1 * F_2 \in S_\alpha$.

With similar conditions, one may easily discuss tails of finitely many convolutions. Results for infinite convolutions are forthcoming in another paper.

4. Conclusion

Except for Theorem 3.5, all of our results relied on the Lemmas 2.1–2.4 and 2.8. These were the only results which required analytic proofs. This has made it possible to generalize results and, in particular to obtain a real analytic proof of the basic result, Theorem 2.9. Our method, however, relies heavily on the monotonicity of the distribution tails. Embrechts (1983) used the work of Rudin (1973) to get a real analytic proof when f is the atomic density of a lattice distribution F. Letting $f_n = F\{n\}$ and $f_n * f_n = F*F\{n\}$, then Embrechts' result is

$$\text{If } f_n \sim Rf_{n+1} \text{ and } f_n * f_n \sim 2df_n, \text{ then } d = \sum_0^\infty R^n f_n.$$

However, it is also apparent that $f_n = \bar{F}(n) - \bar{F}(n+1) \sim (1-R)\bar{F}(n)$. Thus, for $R < 1$, Embrechts' result is a special case of ours.

Chover, Wainger and Ney (1973a) proved the lattice result using a Banach algebra technique. Their technique also gave a result for the density f of absolutely continuous F, namely:

$$\text{If } f(t) \sim e^{\alpha u} f(t+u) \text{ and } f * f(t) \sim 2df(t), \text{ then } d = \int_0^\infty e^{\alpha u} f(u)\,du.$$

This is different from Theorem 2.9 for the density case, because $F \in L_\alpha$ is necessary but not sufficient for $f(t) \sim e^{\alpha u} f(t+u)$. We have not yet obtained a real analytic proof for the density version.

Acknowledgement

The referee's careful critique is greatly appreciated.

References

V. P. Chistyakov (1964), 'A theorem on sums of independent positive random variables and its applications to branching random processes,' *Theory Probab. Appl.* **9**, 640–648.

J. Chover, P. Ney and S. Wainger (1973a), 'Functions of probability measures,' *J. Analyse Math.* **26**, 255–302.

J. Chover, P. Ney and S. Wainger (1973b), 'Degeneracy properties of subcritical branching process,' *Ann Probab.* **1**, 663–673.

D. B. H. Cline (1986), 'Convolution tails, product tails and domains of attraction', *Probab. Theor. Rel. Fields* **72**, 525–557.

R. A. Davis and S. I. Resnick (1985a), 'Limit theory for moving averages of random variables with regularly varying tail probabilities,' *Ann. Probab.* **13**, 179–195.

R. A. Davis and S. I. Resnick (1985b), 'More limit theory for the sample correlation function of moving averages', *Stochastic Process. Appl.* **20**, 257–279.

P. Embrechts (1983), 'The asympotic behaviour of series and power series with positive coefficients', *Verh. Konink. Acad. Wetensch., Belgie.*

P. Embrechts and C. M. Goldie (1980), 'On closure and factorization properties of subexponential distributions', *J. Aust. Math. Soc. (Ser. A)* **29**, 243–256.

P. Embrechts and C. M. Goldie (1982), 'On convolution tails,' *Stochastic Process. Appl.* **13**, 263–278.
P. Embrechts, C. M. Goldie and N. Veraverbeke (1979), 'Subexponentially and infinite divisibility', *Z. Wahrsch. Verw. Gebiete* **49**, 335–347.
W. Feller (1971), *An introduction to probability theory and its applications*, Vol. II (Wiley).
W. Rudin (1973), 'Limits of ratios of tails of measures', *Ann. Probab.* **1**, 982–994.
E. Seneta (1976), *Regularly varying functions* (Lecture Notes in Mathematics 508, Springer-Verlag).

Department of Statistics
Texas A & M University
College Station, Texas 77843
U.S.A.