Canad. Math. Bull. Vol. **49** (3), 2006 pp. 407–413

Intermediate Model Structures for Simplicial Presheaves

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Abstract. This note shows that any set of cofibrations containing the standard set of generating projective cofibrations determines a cofibrantly generated proper closed model structure on the category of simplicial presheaves on a small Grothendieck site, for which the weak equivalences are the local weak equivalences in the usual sense.

Suppose that $\mathcal{C}$ is a small Grothendieck site, and let $s\mathrm{Pre}(\mathcal{C})$ denote the category of simplicial presheaves on the site $\mathcal{C}$. It has been known for some time [9] that the category of simplicial presheaves carries a proper closed simplicial model structure for which the cofibrations are inclusions of simplicial presheaves, the weak equivalences are the local weak equivalences, and the fibrations are the global fibrations.

In the presence of an adequate supply of stalks (or if the category of sheaves on the site has enough points), a local weak equivalence is a map $f\colon X \to Y$ of simplicial presheaves which induces a weak equivalence $f_*\colon X_x \to Y_x$ of simplicial sets in all stalks. Alternatively, f is a local weak equivalence if

(1) the induced map $f_*\colon \tilde{\pi}_0 X \to \tilde{\pi}_0 Y$ of sheaves of path components is an isomorphism, and

(2) the comparison diagram

$$\begin{array}{ccc} \tilde{\pi}_n X & \xrightarrow{f_*} & \tilde{\pi}_n Y \\ \downarrow & & \downarrow \\ \tilde{X}_0 & \xrightarrow[f_*]{} & \tilde{Y}_0 \end{array}$$

of sheaves of group objects is a pullback, for $n \geq 1$.

Here, for example, $\tilde{X}_0$ is the sheaf associated to the presheaf X_0 of vertices of X. A global fibration is a map which has the right lifting property with respect to all morphisms which are simultaneously cofibrations and local weak equivalences.

The indicated structure is a standard tool for calculational applications of the homotopy theory of simplicial presheaves.

Recently, another useful model structure for $s\mathrm{Pre}(\mathcal{C})$ has appeared which has the same weak equivalences and hence describes the same homotopy category, but which has different cofibrations and fibrations. This is the local projective model structure, which was introduced by Blander [2] as a tool for investigating some of the models for the motivic stable category.

Received by the editors August 10, 2004; revised November 5, 2004.
This research was supported by the Natural Sciences and Engineering Research Council of Canada.
AMS subject classification: Primary: 18G30; secondary: 18F20, 55U35.

Specifically, a cofibration $i\colon A \to B$ is said to be projective if it has the left lifting property with respect to all maps $p\colon X \to Y$ which are trivial fibrations of simplicial sets $f\colon X(U) \to Y(U)$, $U \in \mathcal{C}$, in all sections. A map $p\colon X \to Y$ is then said to be a local projective fibration if it has the right lifting property with respect to all maps which are simultaneously projective cofibrations and local weak equivalences. Blander [2] proves the following:

Theorem 1 *The category of simplicial presheaves* $\mathrm{sPre}(\mathcal{C})$ *on a small Grothendieck site* $\mathcal{C}$*, with the classes of projective cofibrations, local weak equivalences and local projective fibrations, satisfies the axioms for a proper closed simplicial model category.*

The closed model structure given by this result is the local projective model structure for the category $\mathrm{sPre}(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$.

Say that a map $f\colon X \to Y$ of simplicial presheaves is a pointwise weak equivalence (respectively, pointwise fibration) if all of its component maps $f\colon X(U) \to Y(U)$, $U \in \mathcal{C}$, are weak equivalences (respectively, fibrations) of simplicial sets.

The special case of Theorem 1 corresponding to the trivial (or chaotic) topology on $\mathcal{C}$, for which the local projective fibrations are the pointwise fibrations, was proved long ago by Bousfield and Kan [5]. Explicitly, the chaotic topology on a site $\mathcal{C}$ is the topology for which every covering family of an object U contains the identity map $1\colon U \to U$. Every presheaf is a sheaf for the chaotic topology.

Observe that the definition of projective cofibration makes no reference to an underlying topology, and thus coincides with the notion arising in the Bousfield–Kan result. That result is easy to prove: if L_U denotes the left adjoint to the U-sections functor $\mathrm{sPre}(\mathcal{C}) \to \mathbf{S}$ taking values in simplicial sets, then the cofibrations $L_U(\partial\Delta^n) \to L_U(\Delta^n)$ generate the projective cofibrations in the usual sense, and the maps $L_U(\Lambda^n_k) \to L_U(\Delta^n)$ generate the acyclic projective cofibrations for the Bousfield–Kan theory. The function spaces $\mathbf{hom}(X, Y)$ have the form that we expect, and there is a minor amount of fussing (in the proof of the simplicial model axiom) to show that if $i\colon A \to B$ is a projective cofibration and $j\colon K \to L$ is an inclusion of simplicial sets then the induced map

$$(i, j)_*\colon (B \times K) \cup_{(A\times K)} (A \times L) \to B \times L$$

is a projective cofibration. One resolves this, however, by showing that the collection of all inclusions i for which the cofibration $(i, j)_*$ is projective is saturated, and includes all $L_U(\partial\Delta^n) \to L_U(\Delta^n)$.

The purpose of this note is to show (in Theorem 2 below) that there are other model structures on the simplicial presheaf category $\mathrm{sPre}(\mathcal{C})$ which are intermediate between the local projective and standard theories. A little more work (Proposition 5) shows that these structures are cofibrantly generated. Questions of what the fibrant objects look like, or equivalently what descent should mean for these theories, are much more delicate, and will not be discussed here, see [8, 11, 12].

To start the discussion, let $\mathbf{C}_P$ denote the class of projective cofibrations, and let $\mathbf{C}$ denote the full class of cofibrations in $\mathrm{sPre}(\mathcal{C})$. Then there is clearly an inclusion relation $\mathbf{C}_P \subset \mathbf{C}$. Recall that $\mathbf{C}_P$ has a generating family of cofibrations $S_0 = \{L_U(\partial\Delta^n) \to L_U(\Delta^n)\}$.

Suppose that $S = \{A_j \to B_j\}$ is some other set of cofibrations which contains S_0, and let $\mathbf{C}_S$ denote the saturation of the set $\overline{S}$ of cofibrations

$$(B_j \times \partial\Delta^n) \cup_{(A_j \times \partial\Delta^n)} (A_j \times \Delta^n) \subset B_j \times \Delta^n \tag{1}$$

where $n \geq 0$, and the cofibrations $A_j \to B_j$ belong to the set S. Note that the case $n = 0$ reduces to maps $A_j \to B_j$ in the generating set for $\mathbf{C}_S$. The class $\mathbf{C}_S$ will also be called the class of S-cofibrations. If S is a generating set for the full class of cofibrations, then $\mathbf{C}_S = \mathbf{C}$ and in that case the set of S-cofibrations is the full set of cofibrations.

Say that a map $p \colon X \to Y$ is a local S-fibration if p has the right lifting property with respect to all maps which are S-cofibrations and local weak equivalences.

Theorem 2 *With these definitions, the category* sPre($\mathcal{C}$) *and the classes of S-cofibrations, local weak equivalences and local S-fibrations together satisfy the axioms for a proper closed simplicial model category.*

Proof Among the usual closed model axioms **CM1–CM5** (see, for example, [7, p. 66]), only the factorization axiom **CM5** requires proof. This axiom is proved with a small object argument, which is displayed here in a highly compressed form.

Every map $f \colon X \to Y$ of simplicial presheaves has a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

where the map j is a member of $\mathbf{C}_S$ and p has the right lifting property with respect to all morphisms of $\mathbf{C}_S$. It follows that p has the right lifting property with respect to all projective cofibrations, so that p is a projective fibration and a pointwise (hence local) weak equivalence. The map p is also a local S-fibration.

The map $f \colon X \to Y$ can be factored

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & W \arrow[ur, "q"'] & \end{tikzcd}$$

where q is a global fibration (hence a local S-fibration) and i is a cofibration and a local weak equivalence. The map i has a factorization

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "j"'] & & W \\ & Z \arrow[ur, "p"'] & \end{tikzcd}$$

https://doi.org/10.4153/CMB-2006-040-8 Published online by Cambridge University Press

where p is a local S-fibration and a local weak equivalence and j is an S-cofibration. Then j must also be a local weak equivalence, and the composite qp is a local S-fibration.

Suppose that the map $p\colon X \to Y$ is a local S-fibration and a local weak equivalence. Then p has a factorization

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "j"'] & & Y \\ & Z \arrow[ur, "q"'] & \end{tikzcd}$$

where j is an S-cofibration and q has the right lifting property with respect to all S-cofibrations. As before, q is therefore a local weak equivalence as well as a local S-fibration. The S-cofibration j is thus a local weak equivalence, and so the indicated lifting exists in the diagram

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ Z \arrow[r, "q"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

The map p is therefore a retract of q, and thus has the right lifting property with respect to all S-cofibrations.

The function complex is the standard one, and the model structure satisfies Quillen's axiom **SM7**, because the class C_S was cooked up so that it would do so: it includes all maps

$$(B_i \times \partial\Delta^n) \cup_{(A_i \times \partial\Delta^n)} (A_i \times \Delta^n) \subset B_i \times \Delta^n.$$

To establish properness, note first that all local S-fibrations are pointwise fibrations and pullback along pointwise fibrations preserves local weak equivalences, by a Boolean localization argument [10]. "Dually", all S-cofibrations are cofibrations and the standard structure is proper, so that pushout along S-cofibrations preserves local weak equivalences. ■

The case $S = S_0$ for Theorem 2 is the Blander result Theorem 1, and the proof of Theorem 2 is an abstraction of Blander's proof.

The meaning of the term "local" in the statement of Theorem 2 can also vary wildly. In particular, if the ambient topology is the chaotic topology, then local weak equivalences are pointwise weak equivalences. In this case the corresponding local S-fibrations will just be called S-fibrations; thus an S-fibration is a map which has the right lifting property with respect to all maps which are S-cofibrations and pointwise weak equivalences.

Corollary 3 *With these definitions, the category* sPre($\mathcal{C}$) *and the classes of S-cofibrations, pointwise weak equivalences and S-fibrations together satisfy the axioms for a proper closed simplicial model category.*

It is a rather silly way to prove the result, but the Bousfield–Kan theorem is the case $S = S_0$ of Corollary 3.

Remark 4 F. Lárusson [11] has given an example, coming from analytic geometry, of an S-structure that coincides with neither the projective nor the standard structure.

The question of whether or not the closed model structure given by Theorem 2 is cofibrantly generated is somewhat more subtle, but has an affirmative answer which is displayed in Proposition 5 below.

The proof of Proposition 5 can be achieved either by general techniques such as those displayed by Beke [1], or by the more ad hoc argument given here. The overall method, as in [1], is to verify a solution set condition. The solution set technique was originally introduced by Bousfield in [4]. This proof also involves an application of the bounded cofibration condition [6], which again was first introduced by Bousfield [3].

The bounded cofibration condition says that for the diagram of cofibrations

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

with $A \to X$ a local weak equivalence, then for a sufficiently large choice of cardinal λ there is a λ-bounded object C with $B \subset C \subset X$ and such that the induced cofibration $C \cap A \to C$ is a local weak equivalence.

For the statement of the bounded cofibration condition to hold, the cardinal λ must be chosen larger than the cardinality of the power set of the set of morphisms of the underlying site. In this case, the set of all λ-bounded trivial cofibrations generates the class of trivial cofibrations in the standard model structure.

Proposition 5 *The class consisting of all maps which are simultaneously S-cofibrations and local weak equivalences has a set J of generators.*

Proof Suppose given a commutative diagram

(2)
$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ B \arrow[r, "\beta"'] & Y \end{tikzcd}$$

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where the cofibration j is a member of $\overline{S}$ (recall that this set is defined in (1)) and f is a local weak equivalence. Find a factorization $f = q \cdot i$ where i is a cofibration and a local weak equivalence and q is a global fibration. Then q is a local weak equivalence, so the lifting θ exists in the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dd, "j"'] & X \arrow[d, "i"] \\ & Z \arrow[d, "q"] \\ B \arrow[r, "\beta"'] \arrow[ur, dotted, "\theta"] & Y \end{tikzcd}$$

Pick a cardinal λ which is sufficiently large that the set of all λ-bounded local trivial cofibrations generates the class of trivial cofibrations, and such that $\lambda > |B|$. Let C be the image of B in Z. Then $|C| < \lambda$, so there is a subobject $D \subset Z$ containing C such that $|D| < \lambda$ and the induced map $D \cap X \to D$ is a local weak equivalence. Then it follows that the diagram (2) has a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & D \cap X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

Take a factorization

$$\begin{tikzcd} D \cap X \arrow[r, "i'"] \arrow[dr] & E \arrow[d, "p"] \\ & D \end{tikzcd}$$

where i' is an S-cofibration and a local weak equivalence and p is an S-fibration and a local weak equivalence. Then p has the right lifting property with respect to j, so it follows that each diagram (2) has a factorization

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & D \cap X \arrow[r] \arrow[d, "i'"] & X \arrow[d, "f"] \\ B \arrow[r] & E \arrow[r] & Y \end{tikzcd}$$

where the maps i' belong to some set T_j of trivial S-cofibrations which is determined by the map j.

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Let $K = \bigcup_{j \in \bar{S}} T_j$ and let $C(K)$ denote the saturation of the set K in the class of cofibrations. If $i\colon U \to V$ is an S-cofibration and a local weak equivalence, then i has a factorization

$$\begin{tikzcd} U \arrow[r, "\tilde{j}"] \arrow[dr, "i"'] & W \arrow[d, "q"] \\ & V \end{tikzcd}$$

where $\tilde{j}$ is a member of $C(K)$ and q has the right lifting property with respect to all members of $C(K)$. But then q is a local weak equivalence, and so it follows from the constructions of the first paragraph that q has the right lifting property with respect to all S-cofibrations. The dotted arrow lifting therefore exists in the diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{j}"] \arrow[d, "i"'] & W \arrow[d, "q"] \\ V \arrow[r, "1"'] \arrow[ur, dotted] & V \end{tikzcd}$$

so that i is a retract of $\tilde{j}$, and is therefore in $C(K)$. ∎

References

[1] T. Beke, *Sheafifiable homotopy model categories.* Math. Proc. Cambridge Philos. Soc. **129**(2000), no. 3, 447–475.

[2] B. Blander, *Local projective model structures on simplicial presheaves.* K-Theory **24**(2001), no. 3, 283–301.

[3] A. K. Bousfield, *The localization of spaces with respect to homology.* Topology **14**(1975), 133–150.

[4] ———, *Constructions of factorization systems in categories.* J. Pure Appl. Algebra **9**(1976/77), 207–220.

[5] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations.* Lecture Notes in Math. (Second corrected printing), Springer-Verlag, Berlin, 1987.

[6] P. G. Goerss and J. F. Jardine, *Localization theories for simplicial presheaves.* Can. J. Math. **50**(1998), no. 5, 1048–1089.

[7] ———, *Simplicial Homotopy Theory.* Progress in Mathematics 174, Birkhäuser, Basel, 1999.

[8] D. C. Isaksen, *Flasque model structures for presheaves.* Preprint (2004), `Math.ArXivAT/0401132`, K-Theory Preprint Archive **679**.

[9] J. F. Jardine, *Simplicial presheaves.* J. Pure Appl. Algebra **47**(1987), no. 1, 35–87.

[10] ———, *Boolean localization, in practice.* Doc. Math. **1**(1996), no. 13, 245–275.

[11] F. Lárusson, *Model structures and the Oka principle.* J. Pure Appl. Algebra **192**(2004), 203–223.

[12] F. Lárusson, *Mapping cylinders and the Oka principle.* Preprint (2004), `Math.ArXivCV/0408373`.

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