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IMAGES OF CLASS-$\mathfrak{C}$ SPACES

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ABSTRACT. In this paper the author characterizes images of class-$\mathfrak{C}$ spaces (as defined by Ishii, Tsuda and Kunugi, Proc. Japan Acad. **44**, (1968), 897–903) under almost-open maps, bi-quotient maps, pseudo-open maps and quotient maps.

1. **Introduction**. In this paper all spaces are T_2, all maps are continuous and onto. Ishii, Tsuda and Kunugi [3] have recently defined a new class of spaces, called $\mathfrak{C}$-spaces, somewhat smaller than the class of M-spaces introduced in [6] by Morita. $\mathfrak{C}$-spaces are countably productive, and the product of a class-$\mathfrak{C}$ space with an M-space is an M-space.

DEFINITION 1.1. Y is of class-$\mathfrak{C}$ iff Y has a sequence of open covers $\mathscr{U}_1, \mathscr{U}_2, \ldots$ such that

(i) $\mathscr{U}_1 > \mathscr{U}_2^* > \mathscr{U}_2 > \mathscr{U}_3^* > \cdots$;

(ii) any point-sequence $\{y_i\}$, where $y_i \in St(y, \mathscr{U}_i)$ for all $i=1, 2, \ldots$ and for some fixed $y \in Y$, has a subsequence whose closure is compact.

The next result is from [1] and [12].

THEOREM 1.2. *The following are equivalent:*

(a) *X is of class-$\mathfrak{C}$;*

(b) *X is M and weakly-k (given $F \subseteq X$, F is closed if $F \cap C$ is finite for every C compact in X);*

(c) *X is M and k_0 (every sequence which clusters has a subsequence whose closure is compact).*

(d) *X is M and weakly-para-k (F is closed in X if $F \cap P$ is finite for every closed paracompact P in X).*

Characterizations of continuous images of various spaces have already been carried out. Some of these results have been summarized in [5] and [11]. For such results on M-spaces, see [1], [7], [9] and [10]. In this paper images of class-$\mathfrak{C}$ spaces under some continuous maps will be characterized.

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2. Characterizations.

DEFINITION 2.1. A map $f:X\to Y$ is said to be almost-open iff for any $y\in Y$ there exists an $x\in f^{-1}(y)$ having a basis of open sets such that the image of each member of the basis is open.

DEFINITION 2.2. A sequence $\{U_1, U_2, \ldots\}$ of sets in a topological space is said to form an r_0-sequence iff any point-sequence of the form $\{y_i:y_i\in U_i\}$ has a subsequence $\{y_{i(n)}\}$ whose closure is compact.

DEFINITION 2.3. A topological space Y is said to be an r_0-space iff for every $y\in Y$ there exists an r_0-sequence $\{U_1, U_2, \ldots\}$ of neighbourhoods of y.

THEOREM 2.4. *A regular space Y is an r_0-space iff it is the almost-open image of a space $X\in\mathfrak{C}$.*

Proof. Let Y be a regular r_0-space, and let y be any point of Y. Then there exists a decreasing r_0-sequence $\{U_1, U_2, \ldots\}$ about y. Put $K=\bigcap\{\mathrm{Cl}U_i:i=1, 2,\ldots\}$. Then each point sequence in K has a subsequence whose closure is compact, so $K\in\mathfrak{C}$. Also, $\{U_i\}$ forms a countable base for K. To show this, let $V\supseteq K$ be an open set in Y and assume that $V\nsupseteq U_i$ for $i=1, 2,\ldots$. Then choose a sequence of points $\{y_i\}$ such that $y_i\in U_i-V$ for all $i=1, 2,\ldots$. Because $\{U_i\}$ is an r_0-sequence, $\{y_i\}$ has a subsequence $\{y_{i(n)}\}$ which has a cluster point p. Then obviously $p\in K$. Since $p\notin V$, $K\nsubseteq V$. Contradiction. Then $\{U_i\}$ is a countable base for K. Consequently there exists a cover $\{K_\alpha:\alpha\in\Omega\}$ of Y by sets having the property that every point sequence in K_α has a subsequence with compact closure, and each K_α has a countable neighbourhood base which forms an r_0-sequence.

Now for each $\alpha\in\Omega$, let Y_α be the set Y with the topology in which the open sets are the sets of the form $U\cup V$ where U is open in Y and V is any subset of $Y-K_\alpha$. Y is regular so Y_α is clearly regular. So let $\{V_\alpha^i:i=1, 2,\ldots\}$ be a countable base for K_α in Y_α such that $\mathrm{Cl}V_\alpha^{i+1}\subseteq V_\alpha^i$ for each i, and $\{V_\alpha^i:i=1, 2,\ldots\}$ is an r_0-sequence. For all $i=1, 2,\ldots$, put $\mathscr{W}_\alpha^i=[\{V_\alpha^i\}, \{y:y\in Y_\alpha-V_\alpha^i\}]$. Then $\{\mathscr{W}_\alpha^i: i=1, 2,\ldots\}$ is a normal sequence of open covers in Y_α satisfying the point sequence condition of class-$\mathfrak{C}$ spaces. So Y_α is regular and belongs to class $\mathfrak{C}$. Thus the discrete sum X of the Y_α's also has these properties.

Define $f:X\to Y$ by taking $f|\,Y_\alpha: Y_\alpha\to Y$ as the identity map; then f is clearly onto and continuous. Now let $y\in Y$. Then it is necessary to show that there is an $x\in f^{-1}(y)$ having an open neighbourhood basis $\mathscr{U}$ such that $f(U)$ is open in Y for each $U\in\mathscr{U}$. Let $y\in K_\alpha$ and let $\mathscr{U}$ be an open neighbourhood basis of y; then $\mathscr{B}=\{f^{-1}(U)\cap Y_\alpha:U\in\mathscr{U}\}$ is an open neighbourhood basis of $x=f^{-1}(y)\cap Y_\alpha$ in X and $f(V)$ is open for each $V\in\mathscr{B}$.

To prove necessity, let X be an r_0-space and f an almost-open map of X onto Y. It suffices to show that the almost-open image of an r_0-space is an r_0-space. Let $y \in Y$; then there is an $x \in f^{-1}(y)$ having a neighbourhood base $\mathscr{U}$ of open sets such that $f(U)$ is open in Y for each $U \in \mathscr{U}$. Since X is r_0, $\mathscr{U}$ contains an r_0-sequence $\{U_i\}$. We claim the neighbourhoods $f(U_i)$ of y form an r_0-sequence about y. To see this, let $y_i \in f(U_i)$ and choose $x_i \in U_i$ such that $f(x_i)=y_i$ for $i=1, 2, \ldots$. Then $\{x_i\}$ has a subsequence $\{x_{i(n)}\}$ whose closure is compact, since $\{U_i\}$ is an r_0-sequence. But then $\{y_{i(n)}\} \subset f[\mathrm{Cl}\{x_{i(n)}\}] \subset \mathrm{Cl}\{y_{i(n)}\}$ and $f[\mathrm{Cl}\{x_{i(n)}\}]$ is closed, so $\mathrm{Cl}\{y_{i(n)}\}=f[\mathrm{Cl}\{x_{i(n)}\}]$ is compact. Hence $\{f(U_i)\}$ is an r_0-sequence about y as asserted.

DEFINITION 2.5. Call a set C proto-compact iff every point sequence in C which accumulates has a subsequence whose closure is compact.

DEFINITION 2.6. Y is said to be a proto-k space iff the following condition holds: $V \subseteq Y$ is open iff $V \cap C$ is relatively open in C for every proto-compact $C \subseteq Y$.

THEOREM 2.7. *Y is proto-k iff there exists a regular class-$\mathfrak{C}$ space X and a quotient map $f: X \to Y$.*

Proof. Let $f: X \to Y$ be a quotient map from a regular class-$\mathfrak{C}$ space X. Let V be nonopen in Y. Thus there exists $x \in f^{-1}(V) - \mathrm{Int} f^{-1}(V)$. Since X is $\mathfrak{C}$, there is an r_0-sequence $\{U_1, U_2, \ldots\}$ of neighbourhoods of x. By regularity, we may assume that $U_{n+1} \subset U_n$ for all $n=1, 2, \ldots$. Now $U_n \cap (X-f^{-1}(V)) \neq \varnothing$ for all $n=1, 2, \ldots$. If we put $C(x)=\bigcap\{U_n : n=1, 2, \ldots\}=\bigcap\{\mathrm{Cl} U_n : n=1, 2, \ldots\}$, then $C(x)$ is clearly proto-compact.

Consider two cases.

(i) Assume that $x \in \mathrm{Cl}[C(x) \cap (X-f^{-1}(V))]$. Then $f(C(x))$ is a proto-compact set such that $f(C(x)) \cap V)$ is nonopen in $f(C(x))$. To show this last, let W be a given neighbourhood of $f(x) \in Y$. Then $f^{-1}(W)$ is a neighbourhood of x in X, and

$$f^{-1}(W) \cap C(x) \cap (X-f^{-1}(V)) \neq \varnothing.$$

So $W \cap f(C(x)) \cap (Y-V) \neq \varnothing$ in Y, and

$$f(x) \in f(C(x)) \cap V \cap \mathrm{Cl}[f(C(x)) \cap (Y-V)].$$

This last says that $f(C(x)) \cap V)$ is nonopen in $f(C(x))$, and Y is then proto-k.

(ii) Now assume that $x \notin \mathrm{Cl}[C(x) \cap (X-f^{-1}(V))]$. (Thus $C(x) \cap [X-f^{-1}(V)]$ may be empty). Since X is regular, there exists an open neighbourhood U of x such that

$$\mathrm{Cl} U \cap C(x) \cap (X-f^{-1}(V)) \neq \varnothing.$$

Since $x \in Cl[X-f^{-1}(V)]$, we can choose $x_n \in (U_n \cap U) \cap (X-f^{-1}(V))$, for each $n=1, 2, \ldots$. The point sequence $\{x_n : n=1, 2, \ldots\}$ then has a subsequence $\{x_{n(j)} : j=1, 2, \ldots\}$ whose closure K is compact. If x_0 is any accumulation point of $\{x_{n(j)}\}$, then $x_0 \in ClU \cap C(x)$, so that $x_0 \in f^{-1}(V)$. Since $x_0 \in K$, this implies $f(x_0) \in f(K) \cap V$. Now take any neighbourhood W of $f(x_0)$; $f^{-1}(W)$ is a neighbourhood of x_0 in X. Thus, for any $n=1, 2, \ldots$, there exists an $n(j) \geq n$ such that $x_{n(j)} \in f^{-1}(W)$. This last says that $W \cap f(K) \cap (Y-V) \neq \varnothing$ since

$$f^{-1}(W) \cap K \cap (X-f^{-1}(V)) \neq \varnothing.$$

Then $f(x_0) \in Cl[f(K) \cap (Y-V)]$, and $f(K) \cap V$ is nonopen in $f(K)$. This last says that Y is proto-k.

Conversely, let Y be a given regular proto-k space. Then let $\{K_\alpha : \alpha \in A\}$ be the family of all proto-compact sets of Y. Take X the discrete sum of the K_α's. X is clearly of class-$\mathfrak{C}$, and the map formed from the direct sum of natural injections $f_\alpha : K_\alpha \to Y$ is a quotient map.

COROLLARY 2.8. *For a regular space Y, the following are equivalent:*

(a) *Y is proto-k;*

(b) *Y is the quotient of a regular r_0-space;*

(c) *Y is the quotient of a regular class-$\mathfrak{C}$ space;*

(d) *Y is the quotient of a regular locally proto-compact space (i.e., a space in which every point has a proto-compact neighbourhood).*

The next map, originally defined by O. Hajek, has been studied in some detail by Michael [4].

DEFINITION 2.9. A map $f: X \to Y$ is said to be bi-quotient iff: given $\mathscr{B}$ a filter-base in Y, if $y \in ClB$ for all $B \in \mathscr{B}$, then there exists an $x \in f^{-1}(y)$ such that $x \in Clf^{-1}(B)$ for all $B \in \mathscr{B}$.

DEFINITION 2.10. Y is bi-proto-k iff any maximal filter $\mathscr{F}$ which converges to $y \in Y$ contains an r_0-sequence $\{F_1, F_2, F_3, \ldots\}$ of members of $\mathscr{F}$ such that $y \in \bigcap\{F_i : i=1, 2, \ldots\}$.

THEOREM 2.11. *Among regular spaces, Y is bi-proto-k iff there exists a class-$\mathfrak{C}$ space X and a bi-quotient map $f: X \to Y$.*

Proof. Let $X \in \mathfrak{C}$ and let $\mathscr{G}$ be a maximal filter converging to $y \in Y$. Then there exists an $x \in f^{-1}(y)$ which is a cluster point of $f^{-1}(\mathscr{G})$. Now $X \in \mathfrak{C}$, so an r_0-sequence of neighbourhoods $\{U_1, U_2, \ldots\}$ of x exists such that $U_1 \supseteq ClU_2 \supseteq U_2 \supseteq ClU_3 \supseteq U_3 \supseteq \cdots$. Since $f(U_i) \cap G \neq \varnothing$ for all $i=1, 2, \ldots$ and all $G \in \mathscr{G}$, this says that $f(U_i) \in \mathscr{G}$.

Now let Y be a given bi-proto-k space. Let $\alpha=\{A_1, A_2, \ldots\}$ be an arbitrary r_0-sequence of neighbourhoods in Y with $\bigcap\{A_i:i=1, 2, \ldots\}\neq\varnothing$. Then define $C(\alpha)=\bigcap\{A_i:i=1, 2, \ldots\}$, and define a new topological space Y_α by the following:

$N(x)\in\mathcal{N}(x)$, a neighbourhood basis of $x\in X$ if

(a) $N(x)=A_i\cap U(x)$ for $i=1, 2, \ldots$ and $U(x)$ a Y-neighbourhood of $x\in C(\alpha)$;
(b) $N(x)=\{x\}$ if $x\in Y_\alpha-C(\alpha)$.

Let X be the discrete sum of all Y_α, $\alpha\in A$, where A is the collection of all r_0-sequences of Y with nonempty intersection. Define an open cover $\mathcal{U}_i$ of X by

$$\mathcal{U}_i=\bigcup\{\mathcal{U}_{i,\alpha}:\alpha\in\Omega\}$$

where

$$\mathcal{U}_{i,\alpha}=[\{A_i\}, \{z:z\in Y_\alpha-A_i\}].$$

If $\{x_i\}$ is a point-sequence in X such that $x_i\in St(x_0, \mathcal{U}_i)$, then an α exists such that $x_0\in Y_\alpha$. If $x_0\notin C(\alpha)$, x_0 is a cluster point of $\{x_i\}$, so the subsequence consisting of the singleton $\{x_0\}$ obviously has compact closure. On the other hand, if $x_0\in C(\alpha)$, then $x_i\in A_i$ for all $i=1, 2, \ldots$. Thus $\{x_i\}$ has a subsequence which has compact closure in Y_α, and hence in X. Now let f be a map from X to Y defined by taking $f\mid Y_\alpha=f_\alpha$ as the identity map. Then f is clearly continuous and onto since the topology of each Y_α is stronger than that of Y.

So let $\mathcal{G}$ be a filterbase in Y and let y be a cluster point of $\mathcal{G}$. Then a maximal filter $\mathcal{G}'$ exists containing $\mathcal{G}$ and converging to y. Now an r_0-sequence of neighbourhoods exists, call it $A_1\supseteq A_2\supseteq\cdots$, such that $A_i\in\mathcal{G}'$, $y\in\bigcap\{A_i:i=1, 2, \ldots\}$. Call $\alpha=\{A_1, A_2, \ldots\}$; then $f^{-1}(y)\cap Y_\alpha$ is a single point x_α. Let $A_i\cap U(x_\alpha)$ be a basic neighbourhood of $x_\alpha\in Y_\alpha$. Now A_i and $U(x_\alpha)\in\mathcal{G}'$, so $A_i\cap U(x_\alpha)\in\mathcal{G}'$. Then $A_i\cap U(x_\alpha)\cap G\neq\varnothing$ for every $G\in\mathcal{G}$. This last says that $A_i\cap U(x)\cap f^{-1}(G)\neq\varnothing$ in X, in which case x is a cluster point of $f^{-1}(\mathcal{G})$ in X. Thus f is bi-quotient, which finishes the proof.

Definition 2.12. A map $f:X\to Y$ is pseudo-open iff for every $y\in Y$ and for every neighbourhood U of $f^{-1}(y)$, $y\in \operatorname{Int} f(U)$.

Definition 2.13. Y is said to be singly bi-proto-k iff the following condition holds: $y\in \operatorname{Cl}B$ for some $B\subseteq Y$ iff there exists an r_0-sequence $\{U_1, U_2, \ldots\}$ of subsets of Y such that

(1) $y\in U_i$ for every $i=1, 2, \ldots$;
(2) $y\in \operatorname{Cl}(U_iB)$ for every $i=1, 2, \ldots$.

The next theorem has a proof essentially the same as that of the theorem in [10]. The proof will be omitted.

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THEOREM 2.14. *Y is singly bi-proto-k iff Y is the image of a class-$\mathfrak{C}$ space X by means of a pseudo-open map.*

We have an analog to Corollary 2.1.

COROLLARY 2.15. *A regular space Y is singly bi-proto-k iff Y is the pseudo-open image of a regular r_0-space X.*

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