ON THE LINE GRAPH OF A FINITE AFFINE PLANE

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1. Introduction. Let Π be a finite affine plane with n points on a line. We denote by $G(\Pi)$ the graph whose vertices are all points and lines of Π, with two vertices adjacent if and only if one is a point, the other is a line, and the point and line are incident. Let $L(\Pi)$ denote the line graph of $G(\Pi)$, i.e., the vertices of $L(\Pi)$ are the edges of $G(\Pi)$, and two vertices of $L(\Pi)$ are adjacent if the corresponding edges of $G(\Pi)$ are adjacent. It is clear that $L(\Pi)$ is a regular, connected graph with $n^2(n+1)$ vertices and valence $2n-1$.

The adjacency matrix $A(G)$ of a graph G is a square (0, 1) matrix whose rows and columns correspond to the vertices of G, and $a_{ij} = 1$ if and only if vertices i and j are adjacent. The problem we pose here is whether $L(\Pi)$ is characterized by the distinct eigenvalues of its adjacency matrix, a problem of a type that has been studied recently for several classes of graphs, especially some line graphs; cf. **(2)** and the bibliography in **(1)**. As will be noted below, since the eigenvalues depend only on n, they cannot possibly distinguish between two planes Π which are not isomorphic (and hence have non-isomorphic $L(\Pi)$) of the same order n. But it is still reasonable to ask whether any regular connected graph H of $n^2(n+1)$ vertices, such that $A(H)$ has the same distinct eigenvalues as $A(L(\Pi))$, where Π is an affine plane of order n, is such that $H \cong L(\Pi')$, where Π' is an affine plane of order n. We shall prove that the answer is "yes." The corresponding question for projective planes was answered affirmatively in **(4)**.

2. Determination of the eigenvalues of $A(L(\Pi))$. Let B be a (0, 1) matrix with n^2 rows corresponding to the points of Π and $n^2 + n$ columns corresponding to the lines of Π, with $b_{ij} = 1$ if and only if point i is on line j. Let K be a (0, 1) matrix with $n^2 + (n^2 + n)$ rows, and $n^2(n+1)$ columns, where the first n^2 rows correspond to the points of Π, the remaining $n^2 + n$ rows correspond to the lines of Π, and the columns correspond to incidences of points and lines (namely, the vertices of $L(\Pi)$). Each column of K contains exactly two 1's, one in the top set of rows, the other in the bottom set, and the selected rows are an incident point-line pair. Then

$$KK^T = \left(\begin{array}{c|c} (n+1)I & B \\ \hline B^T & nI \end{array}\right) \tag{2.1}$$

and

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$$K^T K = 2I + A, \tag{2.2}$$

where $A = A(L(\Pi))$.

Let us first calculate the eigenvalues of KK^T. If α is an eigenvalue and z the corresponding eigenvector, let x be the vector formed by the first n^2 co-ordinates of z and y the vector formed by the last $n^2 + n$ co-ordinates of z. Then

$$(n + 1)\, x + By = \alpha x, \tag{2.3}$$

$$B^T x + ny = \alpha y. \tag{2.4}$$

Case 1. $x = 0$. Then, as long as $By = 0$ (and such y exist, since B has more columns than rows), $\alpha = n$ is an eigenvalue.

Case 2. $x \neq 0$. Multiply (2.4) on the left by B, and invoke (2.3) to obtain

$$BB^T x = (\alpha - n)(\alpha - (n + 1))x. \tag{2.5}$$

Since $x \neq 0$, $(\alpha - n)(\alpha - (n + 1))$ is an eigenvalue of BB^T. So the distinct eigenvalues of KK^T other than n are included among the roots of

$$(\alpha - n)(\alpha - (n + 1)) = \beta, \tag{2.6}$$

where β is any eigenvalue of BB^T.

But $BB^T = nI + J$, where I and J are square matrices of order n^2, and J is the matrix every entry of which is unity. Hence the values of β are $n^2 + n$ and n.

Now let us see whether every solution α of (2.6) is an eigenvalue of KK^T. Clearly $\alpha \neq n$, for $\alpha = n$ implies $\beta = 0$. Let x and y satisfy

$$BB^T x = \beta x, \tag{2.7}$$

$$y = B^T x/(\alpha - n). \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.3) and (2.4) we verify that, when α satisfies (2.6), α is an eigenvalue of KK^T. Thus the distinct eigenvalues of KK^T are

$$2n + 1,\quad 0,\quad \tfrac{1}{2}[2n + 1 + \sqrt{(4n + 1)}],\quad \tfrac{1}{2}[2n + 1 - \sqrt{(4n + 1)}],\quad n. \tag{2.9}$$

To determine the distinct eigenvalues of A, observe first that KK^T and $K^T K$ have the same non-zero eigenvalues, but 0 is an eigenvalue of $K^T K$ (since K has more columns than rows) and also an eigenvalue of KK^T (by (2.9)). So KK^T and $K^T K$ have the same set of distinct eigenvalues; hence (from (2.2) and (2.9)), the distinct eigenvalues of A are

$$\begin{aligned} \alpha_1 &= 2n - 1, \\ \alpha_2 &= -2, \\ \alpha_3 &= \tfrac{1}{2}[2n - 3 + \sqrt{(4n + 1)}], \\ \alpha_4 &= \tfrac{1}{2}[2n - 3 - \sqrt{(4n + 1)}], \\ \alpha_5 &= n - 2. \end{aligned} \tag{2.10}$$

We now determine the multiplicities m_i^0 of the α_i ($i = 1, \ldots, 5$). Clearly,

$$m_1^0 = 1, \tag{2.11}$$

since α_1 is the valence of $L(\Pi)$ and $L(\Pi)$ is connected; see **(3)** for a simple proof of this fact, a special case of the well-known Peron–Frobenius theorem on non-negative matrices.

To calculate the remaining multiplicities, recall that the non-zero eigenvalues of KK^T and K^TK are identical. Then, as we saw in Case 1 of the discussion of (2.3) and (2.4), m_5^0 must be the dimension of the null-space of B, i.e.,

$$m_5^0 = n. \tag{2.12}$$

Also,

$$m_3^0 = m_4^0 = n^2 - 1, \tag{2.13}$$

by (2.6), since the eigenvalue n of BB^T has multiplicity $n^2 - 1$, and (from (2.7) and (2.8)) the $2(n^2 - 1)$ roots α of (2.6) can be seen to correspond to orthogonal eigenvectors of KK^T. Since $\sum m_i^0 = n^2(n + 1)$, it follows from (2.11)–(2.13) that

$$m_2^0 = n^3 - n^2 - n + 1. \tag{2.14}$$

3. Determination of the spectrum of H. Throughout this section, we assume that H is a regular connected graph on $n^2(n + 1)$ vertices, and $A = A(H)$ has distinct eigenvalues $\alpha_1, \ldots, \alpha_5$ given by (2.10). We shall prove that, for each i, the multiplicity m_i of α_i is given by m_i^0 in (2.11)–(2.14).

LEMMA 1. *If $n \neq k(k + 1)$, $k > 1$, then $m_i = m_i^0$.*

Proof. Clearly, α_1 is the dominant eigenvalue of A, so that

$$m_1 = 1. \tag{3.1}$$

Also,

$$\sum m_i = n^2(n + 1). \tag{3.2}$$

Next,

$$\sum \alpha_i m_i = 0, \tag{3.3}$$

for A has only zeros on the main diagonal. Therefore,

$$0 = \sum a_{ii} = (\text{sum of the eigenvalues of } A) = \sum \alpha_i m_i = 0.$$

Further,

$$\sum \alpha_i^2 m_i = (2n - 1)(n + 1)n^2, \tag{3.4}$$

for A^2 has $(2n - 1)$ everywhere on the main diagonal. Hence

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$$(2n - 1)(n + 1)n^2 = \sum (A^2)_{ii} = (\text{sum of the eigenvalues of } A^2) = \sum \alpha_i^2 m_i.$$

Now (3.1)–(3.4) are four linear relations connecting the five m_i. In general, these do not suffice to determine the m_i uniquely. In the special case $n = 2$, however, the only non-negative integral m_i satisfying (3.1)–(3.4) are the values given by (2.11)–(2.14). Since a multiplicity must be a non-negative integer, we are finished in the case in which $n = 2$.

To treat $n > 2$, we invoke the assumption $n \neq k(k + 1)$. Then $4n + 1$ is not a square; so α_3 and α_4 are algebraic conjugates. Since A is rational, we must have

$$m_3 - m_4 = 0. \tag{3.5}$$

Now we have five linear relations for the m_i. It can be easily verified that the matrix of this system is non-singular, so that the solution is unique, and that (2.11)–(2.14) satisfy (3.1)–(3.5).

LEMMA 2. *Let $C = A^2 - (n - 2)A - (2n - 1)I$. Then the sum of the entries in C is $n^3(n + 1)(2n - 1)$. If, further, $m_i = m_i^0$ $(i = 1, \ldots, 5)$, then* $\operatorname{Tr} C^2 = n^3(n + 1)(2n - 1)$; *i.e. every entry in C is 0 or 1.*

Proof. Let u be the vector of all 1's. Since $Au = (2n - 1)u$, $A^2u = (2n-1)^2u$. Hence

$$Cu = [(2n - 1)^2 - (n - 2)(2n - 1) - (2n - 1)]u = n(2n - 1)u.$$

Therefore, the sum of the entries in C is $u'Cu = n^3(n + 1)(2n - 1)$.

Because C is an integral matrix, the lemma will be proved if we can show that $\operatorname{Tr} C^2 = n^3(n + 1)(2n - 1)$. But $\operatorname{Tr} C^2$ can be evaluated, since the trace of any power of A is the sum of that power of the eigenvalues. Hence one may calculate that $\operatorname{Tr} C^2 = n^3(n + 1)(2n - 1)$.

It is perhaps worth remarking that the calculations are not really necessary, however. If $H = L(\Pi)$, C is obviously a (0, 1) matrix; hence (in that case)

$$\operatorname{Tr} C^2 - n^3(n + 1)(2n - 1) = 0.$$

But the left-hand side of this equation is a polynomial in the variable $\sqrt{(4n + 1)}$. Because there is an infinite number of values of n for which there exists an affine plane of order n, there exists an infinite number of values of the variable $\sqrt{(4n + 1)}$ annulling the left side of the equation. Hence the polynomial is identically zero.

Before stating the next lemma, we need the concept of the polynomial of a graph (**3**). Let J be the matrix all of whose entries are unity. Then, for any graph G with adjacency matrix $A = A(G)$, there exists a polynomial $P(x)$ such that $P(A) = J$ if and only if G is regular and connected. The unique polynomial of least degree satisfying this equation is called the polynomial of G, and may be calculated as follows: if G has v vertices, valence d, and the other distinct eigenvalues are $\alpha_1, \ldots, \alpha_t$, then

$$P(x) = v\frac{\prod (x - \alpha_i)}{\prod (d - \alpha_i)}. \tag{3.6}$$

LEMMA 3.

$$P(x) = \frac{1}{2n+1}(x^4 + (7-3n)x^3 + (3n^2 - 17n + 18)x^2 - (n^3 - 12n^2 + 32n - 20)x + (n^2 - 4n + 2)(4 - 2n)).$$

Proof. Apply (2.10) to (3.6).

LEMMA 4. *If* $n \geqslant 4$, *then*

$$\operatorname{Tr} A^3 \geqslant \sum_{i=1}^{5} \alpha_i^3 m_i^0. \tag{3.7}$$

Proof. Define C as in Lemma 2. Then

$$C^2 = A^4 + (7-3n)A^3 + (n-3)A^3 + (n^2 - 8n + 6)A^2 + (4n-2)(n-2)A + (2n-1)^2 I. \tag{3.8}$$

Notice that $\operatorname{Tr} I$, $\operatorname{Tr} A$, $\operatorname{Tr} A^2$ do not depend on m_i. It follows from Lemma 3, since $P(A) = J$, also that $\operatorname{Tr}(A^4 + (7-3n)A^3)$ does not depend on m_i. Therefore, from (3.8), Lemma 2, and the fact that the sum of the squares of integers is not less than their sum, we conclude that

$$(n-3)\operatorname{Tr} A^3 \geqslant (n-3)\sum_{i=1}^{5} \alpha_i^3 m_i^0.$$

Since we are assuming $n \geqslant 4$, the conclusion of the lemma follows.

LEMMA 5. *Assume that* $n = k(k+1)$, *and define* $n_i = m_i - m_i^0$ $(i = 1, \ldots, 5)$. *Then*

$$n_1 = 0, \qquad n_3 = \frac{-n_2 k^3}{2k+1}, \qquad n_4 = \frac{-n_2(k+1)^3}{2k+1}, \qquad n_5 = n_2 k(k+1).$$

Proof. Obviously $n_1 = 0$. Since

$$\operatorname{Tr} A^t = \sum_{i=1}^{5} \alpha_i^t m_i,$$

$t = 0, 1, 2$, we have the following equations:

$$n_3 + n_4 + n_5 = -n_2,$$

$$n_3 \alpha_3 + n_4 \alpha_4 + n_5 \alpha_5 = -\alpha_2 n_2,$$

$$n_3 \alpha_3^2 + n_4 \alpha_4^2 + n_5 \alpha_5^2 = -\alpha_2^2 n_2.$$

Substituting the expressions for the roots, α_3, α_4, α_5, and α_2, we obtain

$$n_3 + n_4 + n_5 = -n_2,$$

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$$
\begin{aligned}
&n_3(k^2 + 2k - 1) + n_4(k^2 - 2) + n_5(k^2 + k - 2) = +2n_2, \\
&n_3(k^2 + 2k - 1)^2 + n_4(k^2 - 2)^2 + n_5(k^2 + k - 2)^2 = -4n_2.
\end{aligned}
\tag{3.9}
$$

The equations (3.9) express n_3, n_4, and n_5 uniquely in terms of n_2. It can be checked that

$$
n_3 = \frac{-n_2 k^3}{2k+1}, \qquad n_4 = -n_2 \frac{(k+1)^3}{2k+1}, \qquad n_5 = \frac{n_2 k(k+1)(2k+1)}{2k+1}
$$

is a solution of the equations (3.9).

Lemma 6. *If $n = k(k+1)$ and $n \geqslant 4$, then the multiplicities of the roots of A are given by $m_i = m_i^0$, $i = 1, 2, 3, 4, 5$.*

Proof. We have

$$
\operatorname{Tr} A^3 = \sum_{i=1}^{5} m_i \alpha_i^3.
$$

Therefore, using Lemma 5,

$$
\begin{aligned}
\operatorname{Tr} A^3 - \sum_{i=1}^{5} \alpha_i^3 m_i^0 &= n_2 \left(\alpha_2^3 - \frac{\alpha_3^3 k^3}{2k+1} - \frac{\alpha_4^3 (k+1)^3}{2k+1} + \alpha_5^3 k(k+1) \right) \\
&= -\frac{n_2}{2k+1} (2k^7 + 7k^6 + 9k^5 + 5k^4 + k^3).
\end{aligned}
$$

Since $n \geqslant 4$, by Lemma 4, $n_2 \leqslant 0$. Also

$$
m_5 = m_5^0 + n_5 = k(k+1) + n_2 k(k+1) = k(k+1)(1+n_2).
$$

Since $m_5 \geqslant 1$, and n_2 is an integer, $n_2 \geqslant 0$. It follows that $n_2 = 0$ and $m_i = m_i^0$, $i = 1, 2, 3, 4, 5$.

Thus, Lemma 6 and Lemma 1 together establish that, for all n, $m_i = m_i^0$. Consequently, Lemma 2 holds for all n.

4. Theorem. *If H is a regular connected graph on $n^2(n+1)$ vertices such that $A = A(H)$ has distinct eigenvalues α_i given by* (2.10), *then there exists an affine plane Π of order n such that $H \cong L(\Pi)$.*

We shall say that two vertices of H are point-mates if they are adjacent, and the number of verticies adjacent to both is $n - 1$. If two vertices are adjacent, and the number of vertices adjacent to both is $n - 2$, they will be called line-mates. Let P_i be the set of point-mates of i, p_i the number of vertices in P_i, $\bar{p}$ the average of the p_i. Define L_i, l_i, and $\bar{l}$ similarly.

Lemma 7. *For each i,*

$$
p_i + l_i = 2n - 1, \tag{4.1}
$$

and

$$
\bar{p} = n, \qquad \bar{l} = n - 1. \tag{4.2}
$$

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Proof. Equation (4.1) follows at once from Lemma 2. Summing (4.1) over all i, we obtain

$$\bar{p} + \bar{l} = 2n - 1. \tag{4.3}$$

From Lemma 2, we also obtain

$$(n-1)\bar{p} + (n-2)\bar{l} = \frac{\operatorname{Tr} A^3}{n^2(n+1)}. \tag{4.4}$$

The reason for this is that $(A^3)_{ii}$ is clearly equal to $(n-1)p_i + (n-2)l_i$. Summing over i we obtain (4.4). Now the matrix of coefficients in (4.3) and (4.4) is non-singular, and $\operatorname{Tr} A^3$ is the sum of the cubes of the eigenvalues of A, which may be calculated under our hypotheses. The unique solution to (4.3) and (4.4) satisfies (4.2).

We now define a (three-fingered) claw as a graph consisting of four vertices $\{1, 2, 3, 4\}$, with 4 adjacent to each of $\{1, 2, 3\}$, but no two vertices in $\{1, 2, 3\}$ adjacent. The word claw was first used in **(1)**, but the concept has been important in almost all the studies of line graphs cited in the references.

LEMMA 8. *H contains no claw as a subgraph.*

Proof. Assume the contrary. Let a_i be the number of vertices adjacent to both 4 and i $(i = 1, 2, 3)$. No vertex can be adjacent to 4, i, and j $(j \neq i;\ i, j = 1, 2, 3)$; for that vertex and 4 would be two vertices adjacent to both i and j. By Lemma 2, since i and j are not adjacent, there can be at most one vertex adjacent to both i and j. Hence, 4 is adjacent to the $\sum a_i$ vertices just described, as well as 1, 2, and 3. Thus the valence of 4 is at least $3 + \sum a_i \geqslant 3 + 3(n-2)$, by Lemma 2. But the valence of 4 is $2n - 1$. Hence,

$$2n - 1 \geqslant 3 + 3(n-1). \tag{4.5}$$

If $n > 2$, (4.5) is impossible. All that remains is to consider the case $n = 2$.

If $n = 2$, and the claw exists, then $l_4 = 3$, $p_4 = 0$ (by 4.1). Since, by (4.2), $\bar{p} = 2$, there must be some vertex k such that $p_k \geqslant 2$. But $p_k \leqslant 3$, so $p_k = 3$. This means that the set of three vertices adjacent to k would have the property that each is adjacent to exactly one of the other two. This is impossible, so the proof of the lemma is complete.

LEMMA 9. *For each i,*

(i) $p_i = n$, $l_i = n - 1$;
(ii) *$\{i, P_i\}$ form a clique, $\{i, L_i\}$ form a clique;*
(iii) *no vertex in P_i is adjacent to a vertex in L_i.*

Proof. We first show that if j and k are in L_i, they are adjacent. Assume otherwise. By Lemma 8, each of the $2n - 3$ remaining vertices adjacent to i is adjacent to at least one of j and k. Let b_j be the number adjacent to j, b_k

the number adjacent to k. We have $b_j + b_k \geqslant 2n - 3$. But since j and k are in L_i, $b_j = b_k = n - 2$. This is a contradiction.

Hence, any vertices in L_i are adjacent, proving the second part of (ii). If, for some i, $l_i > n - 1$, we would have a contradiction of the definition of line-mate. Thus, no l_i can exceed $n - 1$, from which (4.2) is the average l_i. Hence, for all i, $l_i = n - 1$. This statement combined with (4.1), yields (i). Statements (iii) and (ii) follow readily.

We are now ready to prove our theorem. Define a point-clique as any set of $n + 1$ vertices any pair of which are point-mates. Define a line-clique as any set of n vertices any pair of which are line-mates. Since there are $n^2(n+1)$ vertices in all, and each vertex belongs to exactly one point-clique and exactly one line-clique, it follows that there are altogether n^2 point-cliques and $n^2 + n$ line-cliques. The point-cliques are disjoint and their union contains all vertices of H, and similarly for the line-cliques. Each point-clique has a common vertex with $n + 1$ line-cliques; each line-clique has a common vertex with n point-cliques. Let us call any point-clique an "affine point," any line-clique an "affine line," and say that an affine point and affine line are incident if the respective cliques contain a common vertex. We now show that two affine points can be incident with at most one affine line. Assume otherwise. Then H contains four vertices (say 1, 2, 3, 4) such that 1 and 2 are point-mates, 3 and 4 are point-mates, 1 and 3 are line-mates, 2 and 4 are line-mates. Now 1 and 4 cannot be adjacent, for if they were adjacent and point-mates, then 1 and 3 would be point-mates; if they were line-mates, 1 and 2 would be line-mates. Thus 1 and 4 are not adjacent, yet 2 and 3 are adjacent to each. This violates Lemma 2.

Our count of the number of affine points and affine lines, and the incidence of affine points with affine lines, together with the foregoing paragraph, is sufficient to establish that the defined affine points and lines and the defined relation of incidence yield an affine Π of order n. It is then clear that the vertices of H correspond to point-line incidences of Π, and that $H \cong L(\Pi)$.

REFERENCES

1. R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math., *13* (1963), 389–419.

2. R. H. Bruck, *Finite nets* II, *uniqueness and embedding*, Pacific J. Math., *13* (1963), 421–457.

3. A. J. Hoffman, *On the polynomial of a graph*, Amer. Math. Monthly, *70* (1963), 30–36.

4. ——— *On the line graph of a projective plane*, to appear in Proc. Amer. Math. Soc.

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