Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 434–438

ON THE LOWER CENTRAL FACTORS OF A FREE ASSOCIATIVE RING

ROBERT TYLER

Let R be a free associative ring with identity freely generated by $r_1, r_2, \ldots, r_k$. In analogy to group theory the lower central series for R is defined inductively by

$$\gamma_0 = R \quad \text{and} \quad \gamma_n = [\gamma_{n-1}, R],$$

where γ_n is the ideal generated by the indicated ring commutators. Using P. Hall's collection process [**2**; **1**, Chapter 11] γ_n/γ_{n+1} will be shown to be free as a Z-module and as an R/R'-module for each non-negative integer n. In each case a basis will be exhibited.

Definition 1. Commutators of order zero are the free generators of R. A commutator, c, of order n (denoted by $o(c) = n$) is of the form $[x, y]$, where x and y are commutators and $o(x) + o(y) = n - 1$.

The commutators of R are ordered in any manner respecting the condition that x preceed y whenever $o(x) < o(y)$.

Definition 2. Basic commutators of order zero are the commutators of order zero. A basic commutator of order n is of the form $[x, y]$; where x and y are basic commutators, $o(x) + o(y) = n - 1$, y precedes x in the ordering on the commutators, and if $x = [r, s]$, where r and s are basic commutators, then either $s = y$ or s precedes y in the ordering.

Definition 3. Basic products of order k in R are defined to be products of the form $b_{i_1}b_{i_2} \ldots b_{i_m}$, where the b_{i_j} are basic commutators ordered by their subscripts, $i_1 \leqq i_2 \leqq \ldots \leqq i_m$, and

$$\sum_{i=1}^{m} o(b_{i_j}) = k.$$

Recall that the identity together with the basic products of R form an additive basis for R [**1**, p. 172, Theorem 11.2.3].

Definition 4. The order, o, of an element r of R is the least of the orders of the basic products which appear when r is expressed in terms of the basis described above.

Received October 17, 1973. The author would like to acknowledge his debt to Dr. Jacques Lewin of Syracuse University whose guidance, patience, and inspiration made possible the dissertation from which this paper was taken.

https://doi.org/10.4153/CJM-1975-051-6 Published online by Cambridge University Press

S. A. Jennings has shown that for non-negative integers p, q, r, and s

$$[\gamma_p, \gamma_q] \subseteq \gamma_{p+q+1} \quad \text{and} \quad \gamma_r\gamma_s \subseteq \gamma_{r+s}$$

[**3**, p. 345, Theorems 3.3 and 3.4]. It follows from the definition of basic commutators and his first result that basic commutators of order n belong to γ_n. Then it follows from his second result that basic products of order n belong to γ_n.

Thus elements of R of order at least n belong to γ_n. The problem is to show that the non-zero elements of γ_n are of order at least n. To this end we will use the fact that a commutator of order n may be expressed as a sum of basic commutators each of order n [**4**, p. 327, Theorem 5.9].

Consider the product (not necessarily basic)

$$b_{p1}b_{p2} \ldots b_{pk},$$

where the b_{pj} are basic commutators. The pseudo-order, $\bar{o}$, of this product is defined to be the sum of the orders of the b_{pj}.

LEMMA 1. $\bar{o}(b_{p1}b_{p2} \ldots b_{pk}) \leqq o(b_{p1}b_{p2} \ldots b_{pk})$.

Proof. The deficiency of a factor in a product of basic commutators is defined to be the number of succeeding factors in the product that have lower subscripts. The deficiency, d, of a product of basic commutators is the sum total of the deficiencies of its factors.

For each non-negative integer i and positive integer j let $A(i, j)$ represent the following statement: If $b = b_{p1}b_{p2} \ldots b_{pj}$ is a product of basic commutators and $d(b) = i$ then $\bar{o}(b) \leqq o(b)$. Note that $A(i, 1)$ is true for each i and $A(0, j)$ is true for each j since the order and pseudo-order of a basic product are the same. We proceed by double induction.

$A(m, n)$ represents the statement that

$$\bar{o}(b_{p1}b_{p2} \ldots b_{pn}) \leqq o(b_{p1}b_{p2} \ldots b_{pn}),$$

where the deficiency of $b_{p1}b_{p2} \ldots b_{pn}$ is m. If $A(m, n)$ is not vacuously true, then there are adjacent b_{pi} and b_{pi+1} with $p_i > p_{i+1}$. Since

$$b_{pi}b_{pi+1} = b_{pi+1}b_{pi} + [b_{pi}, b_{pi+1}]$$

it follows that

$$b_{p1}b_{p2} \ldots b_{pn} = b_{p1}b_{p2} \ldots b_{pi+1}b_{pi} \ldots b_{pn} + b_{p1}b_{p2} \ldots [b_{pi}, b_{pi+1}] \ldots b_{pn}.$$

It follows from the assumption that $A(m - 1, n)$ is true that the first term on the right of this equation is of order greater than or equal to the pseudo-order of $b_{p1}b_{p2} \ldots b_{pn}$. Furthermore, since $[b_{pi}, b_{pi+1}]$ is a commutator of order $o(b_{pi}) + o(b_{pi+1}) + 1$ it may be expressed as a sum of basic commutators each of order $o(b_{pi}) + o(b_{pi+1}) + 1$. Hence by an application of the distributive law and the assumption that $A(i, n - 1)$ is true it follows that we may express the second term on the right as the sum of terms each of order greater

https://doi.org/10.4153/CJM-1975-051-6 Published online by Cambridge University Press

than or equal to $\bar{o}(b_{p_1}b_{p_2}\ldots b_{p_n}) + 1$. Hence

$$\bar{o}(b_{p_1}b_{p_2}\ldots b_{p_n}) \leqq o(b_1 b_{p_2}\ldots b_{p_n}).$$

LEMMA 2. *If x and y are basic products and $[x, y] \neq 0$, then $o([x, y]) > o(x) + o(y)$.*

Proof. Let $x = b_{i_1}b_{i_2}\ldots b_{i_s}$ and $y = b_{j_1}b_{j_2}\ldots b_{j_t}$. It follows from the distributive law and successive applications of the identities

$$[a, b_{p_1}b_{p_2}\ldots b_{p_k}] = [a, b_{p_1}]b_{p_2}\ldots b_{p_k} + b_{p_1}[a, b_{p_2}\ldots b_{p_k}]$$

and

$$[a, b] = -[b, a],$$

where a and b are elements of R, that $[x, y]$ may be written as the sum of terms of the form

$$A[b_i, b_j]B,$$

where b_i and b_j are factors of x and y respectively, and A and B are monomials in the remaining b_k. Then since $[b_i, b_j]$ may be written as a sum of basic commutators each of order $o(b_i) + o(b_j) + 1$ we have from an application of the distributive law that $[x, y]$ may be expressed as a sum in which each term has pseudo-order $o(x) + o(y) + 1$. Hence, from Lemma 1 it follows that

$$o[x, y] > o(x) + o(y).$$

LEMMA 3. *Nonzero elements of γ_n are of order at least n.*

Proof. For $n = 0$ the result follows trivially. Proceeding by induction let x be a nonzero element of γ_n. It follows from the identity

$$a[b, c]d = [b, ac]d - [b, a]cd,$$

where a, b, c and d are elements of R, and from the definition of γ_n, that x may be expressed as the sum of nonzero elements of the form

$$[g_{n-1}, r]s,$$

where g_{n-1} is a nonzero element of γ_{n-1} and r and s are elements of R. Then from the induction hypothesis and the linearity of the bracket operation

$$[g_{n-1}, r]s = \left[\sum_i B_i, r\right]s = \sum_i [B_i, r]s,$$

where the B_i are basic products of order at least $n - 1$. It then follows from the linearity of the bracket operation, the distributive law, and the fact that r and s may be expressed as sums of basic products and constants that

$$\sum_i [B_i, r]s = \sum_{i,j,k} [B_i, C_j]D_k,$$

https://doi.org/10.4153/CJM-1975-051-6 Published online by Cambridge University Press

where the B_i are basic products of order at least $n - 1$, the C_j are basic products, and the D_k are basic products or constants.

Assume, without loss of generality, that none of the terms in $\sum_{i,j,k}[B_i, C_j]D_k$ are zero. Then by Lemmas 1 and 2

$$o([B_i, C_j]D_k) \geqq n$$

for all i, j, and k. Thus

$$o(x) \geqq n$$

and the proof is complete.

Note that R/R' is just the polynomial ring with identity in the k commuting variables $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_k$, where $\bar{r}_i = r_i + \gamma_1$. In other words, γ_0/γ_1, is a free Z-module with the identity and basic products of order zero for a basis. This observation will be generalized in the theorem below.

For notational convenience we identify the basic products of order n with the basic products of order n modulo γ_{n+1}.

THEOREM. *For each positive integer n, γ_n/γ_{n+1} is free as a Z-module and as an R/R'-module with bases given respectively by the basic products of order n and the basic products of order n without factors of order zero.*

Proof. It follows from Lemma 3 that the basic products of order at least n span γ_n/γ_{n+1} as a Z-module. Then since basic products of order greater than n belong to γ_{n+1} it follows that the basic products of order exactly n span γ_n/γ_{n+1}.

To show that this set is linearly independent let

$$\sum_i n_i B_i = 0 \mod \gamma_{n+1},$$

where the n_i are integers and the B_i are distinct basic products of order n. Then by Lemma 3

$$\sum_i n_i B_i = 0.$$

But since the B_i are elements of an additive basis for R this implies that $n_i = 0$ for each i.

Similarly, since the elements of R/R' are linear combinations of the identity of R and basic products of order zero it follows that the basic products of order n without factors of order zero, form an R/R' basis for γ_n/γ_{n+1}. In particular, γ_n/γ_{n+1} has a finite basis.

From Jennings' results and Lemma 3 it follows that for n greater than zero, R/γ_n is a free Z-module with the identity and the basic products of order less than n for a basis. Thus R/γ_n provides a natural prototype for a finitely generated ring of finite class [**3**, p. 343].

References

1. M. Hall, Jr., *The theory of groups* (The Macmillan Company, New York, 1959).
2. P. Hall, *A contribution to the theory of groups of prime power order*, Proc. London Math. Soc. *36* (1933), 29–95.
3. S. Jennings, *Central chains of ideals in an associative ring*, Duke Math. J. *9* (1942), 341–355.
4. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory* (Interscience, New York, 1966).

Susquehanna University,
Selinsgrove, Pennsylvania

https://doi.org/10.4153/CJM-1975-051-6 Published online by Cambridge University Press