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A Further Decay Estimate for the Dziubański–Hernández Wavelets

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Abstract. We give a further decay estimate for the Dziubański–Hernández wavelets that are band-limited and have subexponential decay. This is done by constructing an appropriate bell function and using the Paley–Wiener theorem for ultradifferentiable functions.

1 Introduction

Our aim in this note is to give a further decay estimate for the Dziubański–Hernández wavelets. Their wavelet ψ satisfies the estimate of the form $|\psi(x)| \leq C_\delta \exp(-|x|^{1/\delta})$, $\delta > 1$. (See [DH].) By constructing an appropriate bell function, we can show, for example, that some of their wavelets satisfy the estimate of the form $|\psi(x)| \leq C_\delta \exp\{-|x|/(\log|x|)^\delta\}$, $\delta > 1$, for large values of $|x|$. It is to be noted that none of them can have exponential decay.

Their wavelet ψ is constructed by putting its Fourier transform $\widehat{\psi}(\xi) = e^{i\xi/2} b_a(\xi)$, where $b_a(\xi)$, $0 < a \leq \pi/3$, is an even function on $\mathbb{R}$ whose restriction to $[0, \infty)$ is a bell function associated with the interval $[\pi, 2\pi]$. (See also [AWW, BSW, HW].) The function b_a is defined by using a cutoff function φ_a belonging to every Gevrey class Γ^δ, $\delta > 1$.

The outline of this note is as follows. In Section 2, we begin with the definition of the space of all ultradifferentiable functions f of class (M_n) and of class $\{M_n\}$. (See [Ko] as well as [Bj] and [Ro].) After giving a special sequence L_n, $n = 0, 1, 2, \dots$ of positive numbers, we define the p-logarithmic Gevrey classes $\gamma^{p,\delta}$ and $\Gamma^{p,\delta}$, $p = 1, 2, 3, \dots, \delta > 1$. The Paley–Wiener theorem for ultradifferentiable functions is also recalled. Our theorem, giving the further decay estimate mentioned above, is stated in Section 3. The proof is carried out in Section 4, where we give a natural extension of Proposition 2.6 in [DH] and calculate the function associated with the sequence L_n by using the technique in [Ma].

2 Notation, Definitions, and Results

This section is divided into three subsections as described above.

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2.1 Ultradifferentiable Functions

Let $\mathbb{R}$ be a one-dimensional real Euclidean space. Let M_n, $n = 0, 1, 2, \dots$, be a sequence of positive numbers. We assume that M_n satisfies the following conditions:

$$M_n^2 \leq M_{n-1}M_{n+1}, \quad n = 1, 2, 3, \dots, \tag{2.1}$$

$$M_{n+1} \leq AH^n M_n, \quad n = 0, 1, 2, \dots, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty, \tag{2.3}$$

where A and H are constants independent of n. Note that the Gevrey sequences $n^{n\delta}$, $\delta > 1$, satisfy all the conditions.

The function $M(\rho)$ defined by

$$M(\rho) = \sup_{n \geq 0} \log \frac{\rho^n M_0}{M_n}, \quad \rho > 0,$$

is called the *function associated with* the sequence M_n. If $M_n = n^{n\delta}$, then $M(\rho)$ is equivalent to $\rho^{1/\delta}$; $\lim_{\rho\to\infty} M(\rho)/\rho^{1/\delta} = \delta/e$.

An infinitely differentiable function f on an open set Ω in $\mathbb{R}$ is said to be an *ultradifferentiable function* of class (M_n) (resp. of class $\{M_n\}$) if for each compact set K in Ω and each $h > 0$ there is a constant C (resp. there are constants h and C) such that

$$\sup_{x\in K} \left| \left(\frac{d}{dx}\right)^n f(x) \right| \leq Ch^n M_n, \quad n = 0, 1, 2, \dots.$$

We denote by $\mathcal{E}^{(M_n)}(\Omega)$ (resp. $\mathcal{E}^{\{M_n\}}(\Omega)$) the space of all functions of class (M_n) (resp. of class $\{M_n\}$) on Ω. Let K be a compact set in $\mathbb{R}$. Then we denote by $\mathcal{D}_K^{(M_n)}$ (resp. $\mathcal{D}_K^{\{M_n\}}$) the space of all functions of class (M_n) (resp. of class $\{M_n\}$) with compact support contained in K. We can introduce natural locally convex topologies in these spaces.

2.2 Logarithmic Gevrey Classes

Let us give a sequence L_n, $n = 0, 1, 2, \dots$, satisfying conditions (2.1), (2.2), and (2.3). Putting $\log_0 \sigma = \sigma$ and $\log_p \sigma = \log(\log_{p-1} \sigma)$ for $p = 1, 2, 3, \dots$ (if $\log_{p-1} \sigma > 0$), we define the p-logarithmic function $l_{p,\delta}(x)$ for large values of x as follows:

$$l_{p,\delta}(x) = (\log x)(\log_2 x) \cdots (\log_{p-1} x)(\log_p x)^\delta, \quad p = 1, 2, 3, \dots, \ \delta > 1.$$

It is easy to see that the sequence $L_n = (nl_{p,\delta}(n))^n$ satisfies conditions (2.1), (2.2), and (2.3) for large values of n. We only note that the function $x \log_k x$, $k \geq 1$, is convex for large values of x. For large values of ρ, the function associated with the sequence L_n can be written as

$$L(\rho) = \sup_{n \geq n_0} [n \log \rho - n(\log n + \log_2 n + \cdots + \log_p n + \delta \log_{p+1} n)], \tag{2.4}$$

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where the integer n_0 is sufficiently large. We now define the p-logarithmic Gevrey classes as follows.

Definition Let $p = 1, 2, 3, \dots, \delta > 1$, and $L_n = (n\, l_{p,\delta}(n))^n$. Then

$$\gamma^{p,\delta} = \mathcal{E}^{(L_n)}(\mathbb{R}) \quad \text{and} \quad \Gamma^{p,\delta} = \mathcal{E}^{\{L_n\}}(\mathbb{R}).$$

We remark that the spaces $\gamma^{p,\delta}$ and $\Gamma^{p,\delta}$ are continuously included in $\gamma^{q,\varepsilon}$ and $\Gamma^{q,\varepsilon}$ respectively if and only if $p > q$ or $p = q$, $\delta \leq \varepsilon$. Furthermore, the space $\Gamma^{p,\delta}$ is continuously included in $\gamma^{q,\varepsilon}$ if and only if $p > q$ or $p = q$, $\delta < \varepsilon$. (See [Ko, pp. 52–53].) This is a generalization of [DH, Lemma 2.4].

2.3 The Paley–Wiener Theorem

The most fundamental result for ultradifferentiable functions is the following Paley–Wiener theorem:

Suppose that M_n satisfies conditions (2.1), (2.2), and (2.3) and that K is a compact set in $\mathbb{R}$.

An entire function $u(\zeta)$ on $\mathbb{C}$ is the Fourier–Laplace transform of an ultradifferentiable function $\varphi(x) \in \mathcal{D}_K^{(M_n)}$ (resp. $\mathcal{D}_K^{\{M_n\}}$), $u(\zeta) = \int_{\mathbb{R}} \varphi(x) e^{-ix\cdot\zeta}\, dx$, if and only if for any $h > 0$ there is a constant C (resp. there are constants h and C) such that

$$|u(\zeta)| \leq C \exp\{-M(|\zeta|/h) + H_K(\zeta)\}, \quad \zeta \in \mathbb{C},$$

where $H_K(\zeta) = \sup_{x\in K} \operatorname{Im}(x \cdot \zeta)$ is the support function of K.

3 Statement of the Theorem

Our theorem is now stated as follows.

Theorem *There exists a Dziubański–Hernández wavelet ψ that satisfies the following decay estimate:*

For any $p = 1, 2, 3, \dots$ and any $\delta > 1$, there is a constant $C_{p,\delta}$ such that

$$|\psi(x)| \leq C_{p,\delta}\ \exp\{-|x|/l_{p,\delta}(|x|)\}$$

for large values of $|x|$.

4 Proof of the Theorem

The Denjoy–Carleman–Mandelbrojt theorem states that if M_n satisfies conditions (2.1), (2.2), and (2.3), then for any compact set K with interior points there is a function $\varphi \in \mathcal{D}_K^{(M_n)}$ ($\subset \mathcal{D}_K^{\{M_n\}}$) such that $\varphi(x) \geq 0$ and $\int_{\mathbb{R}} \varphi(x)\, dx = 1$. A natural extension of Proposition 2.6 in [DH] can be stated as follows.

Proposition 1 *There exists a cutoff function which belongs to every $\Gamma^{p,\delta}$ with $p = 1, 2, 3, \dots$ and $\delta > 1$.*

Proof We employ the regularization procedure used in the proof of Proposition 2.6 in [DH], taking one additional parameter p $(= 1, 2, 3, \dots)$ into consideration. Let h be an infinitely differentiable nonnegative function with compact support contained in the interval $[-1, 1]$ such that $h(-x) = h(x)$, $x \in \mathbb{R}$, and $\int_{\mathbb{R}} h(x)\,dx = 1$. Let N_m be an increasing sequence of large positive integers such that

$$\sum_{n \ge N_m} \big(n\, l_{m,\delta_m}(n)\big)^{-1} < 2^{-m},$$

where $\delta_m = 1 + 1/m$. By choosing $a_n = \big(n\, l_{m,\delta_m}(n)\big)^{-1}$ for $N_m \le n < N_{m+1}$, we have that $\sum_{n \ge N_1} a_n \le \sum_{m=1}^{\infty} 2^{-m} = 1$.

Define the function $\varphi_{(n)}(x)$, $n \ge N_1$, as $\varphi_{(n)}(x) = h_{a_{N_1}} * h_{a_{N_1+1}} * \cdots * h_{a_n}(x)$, where $h_a(x) = (1/a)\, h(x/a)$, so that $\int_{\mathbb{R}} h_a(x)\,dx = 1$. We note that $\varphi_{(n)}$ is a function with compact support contained in the interval $[-1, 1]$.

We shall show that for any $p = 1, 2, 3, \dots$ and any $\delta > 1$, there is a constant $C_{p,\delta}$ such that for every $N = 0, 1, 2, \dots$ and every $x \in \mathbb{R}$, it holds that

$$\Big| \Big(\frac{d}{dx}\Big)^N \varphi_{(n)}(x) \Big| \le (C_{p,\delta})^{N+1} \big(N\, l_{p,\delta}(N)\big)^N \tag{4.1}$$

for all $n \ge n(p, \delta, N)$. Take m and n so large that $m > p$, $\delta_m < \delta$, and $n > N_m + N$. Then

$$\Big(\frac{d}{dx}\Big)^N \varphi_{(n)}(x) = h_{a_{N_1}} * h_{a_{N_1+1}} * \cdots * h_{a_{N_m}} * \Big(\frac{d}{dx}\Big) h_{a_{N_m+1}} * \cdots * \Big(\frac{d}{dx}\Big) h_{a_{N_m+N}} * \cdots * h_{a_n}(x).$$

We have, for $n \ge N_m$,

$$\int_{\mathbb{R}} \Big| \Big(\frac{d}{dx}\Big) h_{a_n}(x) \Big|\,dx = \frac{1}{a_n} \int_{\mathbb{R}} \frac{1}{a_n} \Big| \Big(\frac{d}{dx} h\Big) \Big(\frac{x}{a_n}\Big) \Big|\,dx \le \frac{C}{a_n} \le C\, n\, l_{m,\delta_m}(n).$$

Thus, using the equality $\int u * v = (\int u)(\int v)$ and Young's inequality $\sup |u * v| \le (\int |u|)(\sup |v|)$, we conclude that

$$\begin{aligned} \Big| \Big(\frac{d}{dx}\Big)^N \varphi_{(n)}(x) \Big| &\le \frac{1}{a_{N_1}} \Big(\sup_{x \in \mathbb{R}} h(x)\Big) \cdot C^N \big((N_m + N)\, l_{m,\delta_m}(N_m + N)\big)^N \\ &\le (C_{p,\delta})^{N+1} \big(N\, l_{p,\delta}(N)\big)^N. \end{aligned}$$

Observe that N_m depends on both p and δ. It is now easy to see that $\varphi_{(n)}$ converges to a function φ which satisfies (4.1) with $\varphi_{(n)}$ replaced by φ. The proof is complete. ■

In order to use the Paley–Wiener theorem stated in Subsection 2.3, it is necessary to have an asymptotic estimate for $L(\rho)$ of (2.4) in Subsection 2.2. Let $\sigma = \log \rho$. Then we can write

$$L(\rho) = B(\sigma) = \sup_{n \ge n_0} [n\sigma - n(\log n + \log_2 n + \cdots + \log_p n + \delta \log_{p+1} n)].$$

Let

$$B_1(\sigma) = \frac{e^{\sigma - 1}}{\sigma(\log \sigma) \cdots (\log_{p-2} \sigma)(\log_{p-1} \sigma)^{\delta}}.$$

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Proposition 2 *We have that*

$$B(\sigma) \sim B_1(\sigma) \text{ as } \sigma \to \infty, \quad \text{that is,} \quad \lim_{\sigma\to\infty} \frac{B(\sigma)}{B_1(\sigma)} = 1.$$

Proof We use the same technique as in [Ma, pp. 122–123], which treats the case where $\delta = 1$. We note again that the number n_0 appearing in the definition of $B(\sigma)$ is sufficiently large when $\sigma = \log \rho$ is very large.

Put $\alpha_n = \log L_n - \log L_{n-1}$, $n = 1, 2, 3, \dots$. Then we have

$$\alpha_n = (1 + \log n) + \log_2 n + \cdots + \log_p n + \delta \log_{p+1} n + o(1), \quad n \to \infty$$

and that α_n is increasing for $n \geq n_0$.

Let $N(\alpha)$ be the distribution function of the sequence $\{\alpha_n\}$, that is, let $N(\alpha)$ be equal to the cardinality of the set $\{n : \alpha_n < \alpha\}$. If $m = N(\alpha)$, then we have

$$\begin{aligned}(1 + \log m) + \log_2 m + \cdots + \log_p m + \delta \log_{p+1} m + o(1) < \alpha \leq \\ (1 + \log(m+1)) + \log_2(m+1) + \cdots + \log_p(m+1) + \delta \log_{p+1}(m+1) + o(1).\end{aligned}$$

Because $\log_k \alpha = \log_{k+1} m + o(1)$ if $k \geq 1$, it holds that

$$\log N(\alpha) = \log m = (\alpha - 1) - \log \alpha - \cdots - \log_{p-1} \alpha - \delta \log_p \alpha + o(1).$$

We now need to use [Ma, Lemma 1.8 III], which reads as follows.

Lemma **([Ma, 1.8 III])** *Let $\{\nu_n\}$ and $\{\lambda_n\}$ be two increasing sequences of positive numbers tending to infinity.*

Put

$$\begin{cases} N(x) = 0, & 0 < x \leq \nu_1, \\ N(x) = \lambda_n, & \nu_n < x \leq \nu_{n+1}, \quad n \geq 1, \end{cases}$$

and

$$\lambda_0 = 0, \quad N_n = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})\nu_i.$$

Then we have for $\nu_m \leq x \leq \nu_{m+1}$, $m \geq 1$, that

$$\int_0^x N(t)\,dt = \max_{n\geq 1}(\lambda_n x - N_n) = \lambda_m x - N_m.$$

By putting $\nu_n = \alpha_n$ and $\lambda_n = n$ in this lemma, we obtain

$$B(\sigma) = \int_{\alpha_1}^{\sigma} N(\alpha)\,d\alpha = \int_{\alpha_1}^{\sigma} [1 + o(1)] \frac{e^{\alpha-1}\,d\alpha}{\alpha(\log \alpha)\cdots(\log_{p-2}\alpha)(\log_{p-1}\alpha)^{\delta}}.$$

It is easy to see that $B_1(\sigma) \sim B_1'(\sigma)$, so in [Ha, Theorem 19] tells us that $B(\sigma) \sim B_1(\sigma)$. The proof is complete. ■

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If we put $\sigma = \log \rho$ in Proposition 2, then we have

$$L(\rho) \sim \frac{\rho}{e(\log \rho) \cdots (\log_{p-1} \rho)(\log_p \rho)^{\delta}}. \tag{4.2}$$

We finally return to the argument given in [DH, Section 3]. By Proposition 1, we can choose, as in [DH, p. 401], a cutoff function φ_a belonging to every $\Gamma^{p,\delta}$, $p = 1, 2, 3, \dots, \delta > 1$. The function φ_a is compactly supported in the interval $[-a, a]$, $0 < a \leq \pi/3$. Set $\theta_a(x) = \int_{-\infty}^{x} \varphi_a(t)\, dt$, which belongs to every $\Gamma^{p,\delta}$, $p = 1, 2, 3, \dots, \delta > 1$. Then, [Ru, Theorem A] tells us that $S_a(x) = \sin(\theta_a(x))$ and $C_a(x) = \cos(\theta_a(x))$ belongs to every $\Gamma^{p,\delta}$, $p = 1, 2, 3, \dots, \delta > 1$, since the sequence A_n in [Ru] is equal to $(L_n/n!)^{1/n}$ and is increasing. We remark that the compositions of two ultradifferentiable functions are well studied. (See also [DH, Theorem 3.3].) Since $\Gamma^{p,\delta}$ is a topological algebra under the pointwise multiplication by [Ko, Theorem 2.8], it follows that the function $b_a(x) = S_a(x - \pi)C_{2a}(x - 2\pi)$ belongs to every $\Gamma^{p,\delta}$, $p = 1, 2, 3, \dots, \delta > 1$. By the remark in Subsection 2.2 about the inclusions among the classes of ultradifferentiable functions, the function b_a also belongs to every $\gamma^{p,\delta}$, $p = 1, 2, 3, \dots, \delta > 1$. By using the Paley–Wiener theorem in Subsection 2.3 (with the choice of $h = 1/e$) and the asymptotic estimate (4.2), the Dziubański–Hernández wavelet ψ, whose construction based on a bell function is described in Section 1, satisfies the estimate

$$\begin{aligned} |\psi(x)| = C|\widehat{b}_a(x + 1/2)| &\leq C_{p,\delta} \exp\{-L(e|x + 1/2|)\} \\ &\leq C_{p,\delta} \exp\{-|x|/l_{p,\delta}(|x|)\} \end{aligned}$$

for large values of $|x|$. The proof of the theorem is now complete. ■

Remark 1. We had tried to include the case where $p = 0$ in the theorem, but in vain. This is because we cannot have Proposition 2 in that case. Mandelbrojt treats the problem for all $p \geq 0$, and this is possible because his problem treats the case where $\delta = 1$.

Remark 2. Our attempt to use as many logarithms as possible in search of better estimates seems to be quite universal in analysis. See, for example, the "calculus of infinities" of du Bois-Reymond in [Ha].

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