CONVECTION IN ROTATING STARS

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S U M M A R Y

It is shown that many features of convection in rotating spheres and spherical shells can be understood on the basis of plane layer models. The phenomenon of differential rotation generated by convection is emphasized. The potential applications and limitations of analytical and numerical models for problems of astrophysical interest are briefly discussed.

I I N T R O D U C T I O N

In thinking about the effects of rotation in stars a variety of thoughts comes to mind. Some are more negative: Rotation is a breaker of symmetry. It spoils our notion of a star as an ideal spherically symmetric body which is in static equilibrium except for the convection zones. Even the latter can be regarded as spherically symmetric with respect to their gross properties in the absence of rotation. Although the deviations from spherical symmetry are small in most rotating stars, the effects of rotation are only partially known and continue to irritate the theoretician involved in computations of stellar evolution.

On the other hand stellar rotation is an exciting subject because of the variety of interesting phenomena associated with it. The generation of magnetic fields is in general connected with rotation. The shape of surfaces of equal potential in a rotating star may become unstable in phases of contraction. Rotation can cause meridional circulations and mixing processes and, in addition, there is a variety of phenomena connected with differential rotation.

For the theoretical fluid dynamicist rotation brings to mind still other thoughts. Whenever the Coriolis force becomes dominant the dynamics of fluids are profoundly altered. The intuition developed from experience with hydrodynamics in non-rotating systems is no longer valid. Intuitive concepts like mixing length theory appear to be even less applicable in the case of low Rossby number convection, i.e. when the vorticity of motion relative to the rotating system is small compared to the rotation rate. On the other hand, theories developed for small amplitude convection appear to hav a much larger range of validity than in a non-rotating system. The two-dimensionality enforced by a dominating Coriolis force tends to suppress instabilities and restricts the degree of freedom for turbulent motion.

In the following theoretical and experimental results for the small Rossby number

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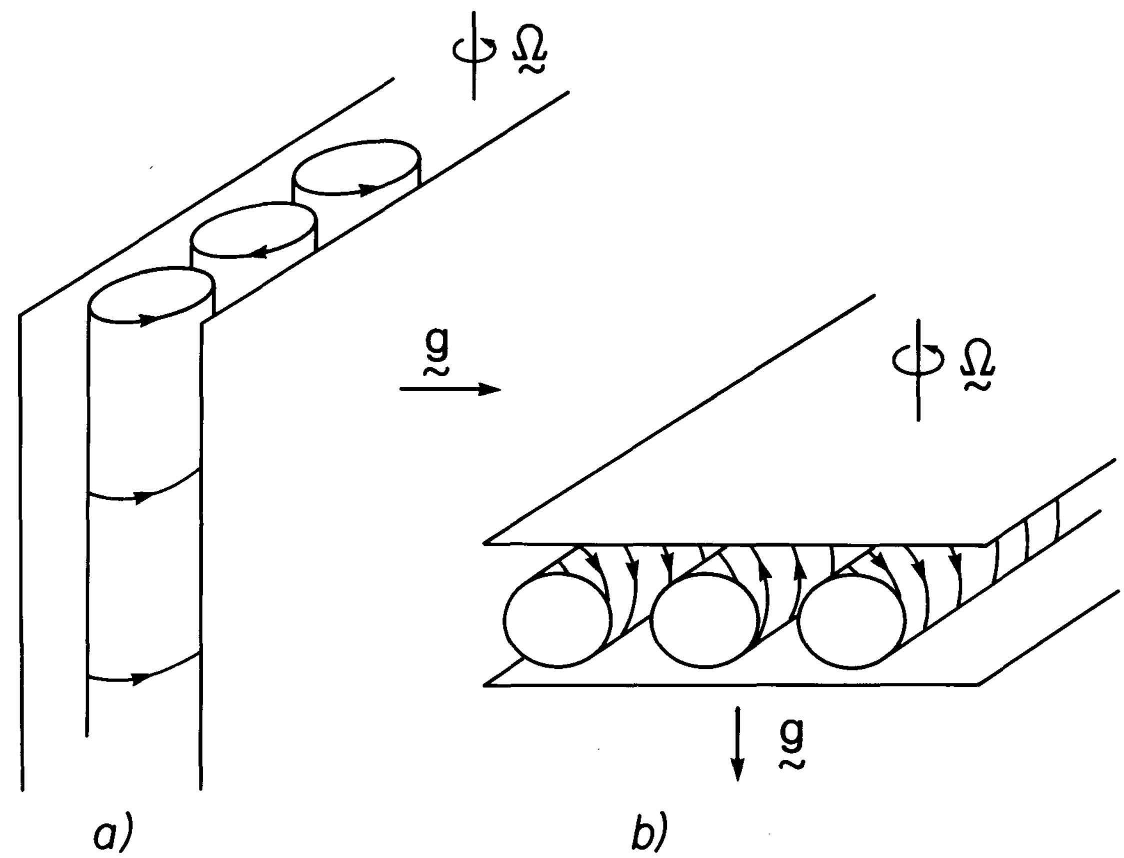

a) Convection columns aligned with the axis of rotation.

b) Convection rolls in a layer rotating about a vertical axis.

F I G U R E 1

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case will be presented. Among the nonlinear phenomena caused by convection in rotating systems we shall emphasize the generation of differential rotation. In discussing the application to rotating stars we shall restrict our attention to the Sun and Jupiter. The detailed surface observations available in both cases offer the best hope for eventual quantitative tests of theoretical concepts.

2 BASIC EFFECTS OF ROTATION ON CONVECTION

The dynamics of nearly stationary motions in a rotating system are governed by the Proudman-Taylor theorem which states that a small amplitude stationary velocity field of an inviscid incompressible fluid must be independent of the coordinate in the direction of the axis of rotation. It is of interest for astrophysical applications that the theorem holds for barotropic fluids as well if the velocity vector $\mathbf{v}$ is replaced by the momentum vector $\rho\mathbf{v}$: By taking the curl of the equation of motion

$$2\,\boldsymbol{\Omega} \times \rho\mathbf{v} = -\nabla p - \rho\nabla\Phi$$

and using the equation of continuity

$$\nabla \cdot \rho\mathbf{v} = 0$$

the relationship

$$2\boldsymbol{\Omega} \cdot \nabla\rho\mathbf{v} = 0 \qquad (1)$$

is obtained. In the following we shall restrict our attention, however, to the case of incompressible fluids, or, more exactly, Boussinesq fluids for which the temperature dependence of the density is taken into account in the gravity term only.

We start the discussion of convection in rotating systems by considering two simple cases as shown in Figure 1. In case (a) the vectors of gravity and rotation are at a right angle and convection solutions satisfying the Proudman-Taylor theorem are possible. The Coriolis force is entirely balanced by the pressure gradient in that case and the critical value of the Rayleigh number for the onset of convection becomes the same as in a nonrotating system. Since the Coriolis force always increases the critical Rayleigh number unless it is balanced by the pressure, the solution corresponding to convection rolls aligned with the axis of rotation is physically preferred. It can be easily realized in the laboratory by heating a cylindrical rotating annulus from the outside and cooling it from the inside and using the centrifugal force as gravity (Busse and Carrigan, 1974).

While the stabilizing effect of the Coriolis force vanishes in case (a) it reaches its maximum in case (b) when the vectors of gravity and rotation are parallel. This is realized when a fluid layer heated from below is rotating about a vertical axis. Release of potential energy by convection requires a vertical component of

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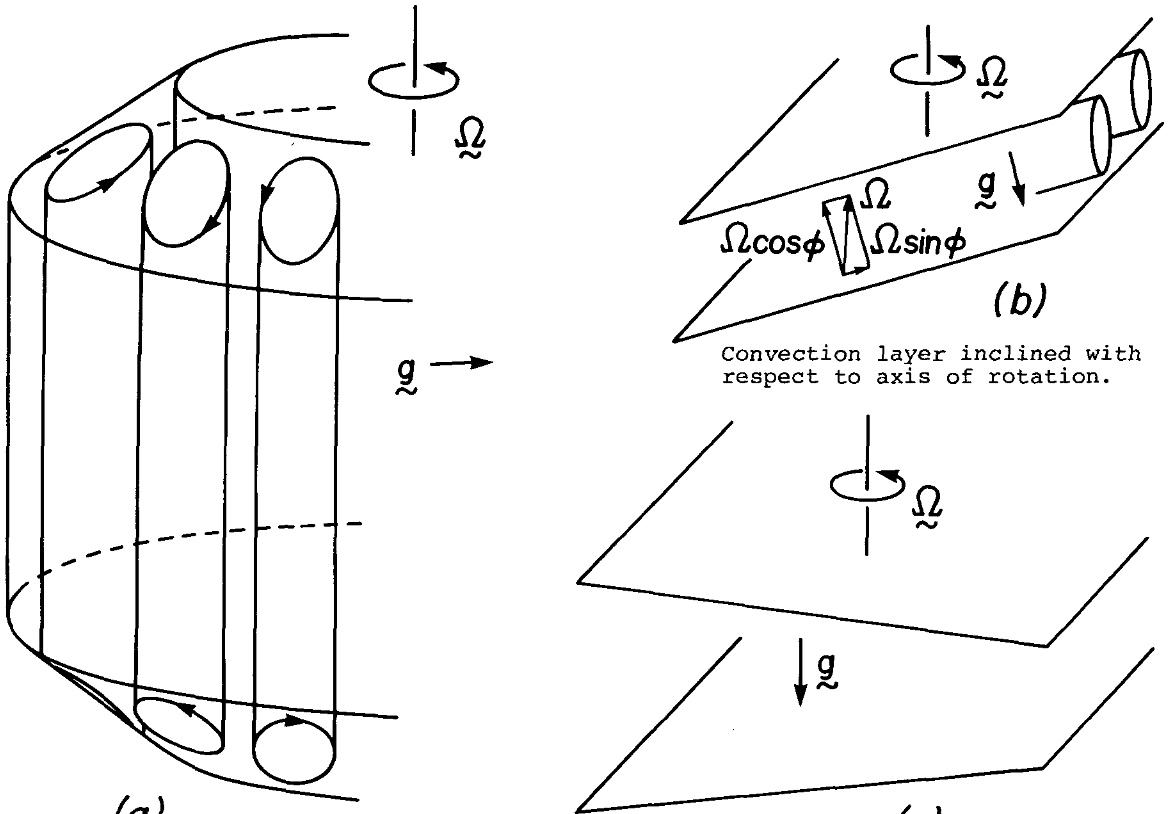

(a) Convection columns in an annulus with inclined top and bottom boundaries.

(b) Convection layer inclined with respect to axis of rotation.

(c) Convection layer with changing light.

Figure 2:

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motion which cannot occur without violating the Proudman-Taylor theorem. In order to overcome the constraint of the Proudman-Taylor theorem viscous friction must become sufficiently strong, thus playing a destabilizing role in this case. The non-dimensional number describing the ratio between viscous friction and Coriolis force is the Ekman number

$$E = \nu / \Omega d^2 \tag{2}$$

where d is the thickness of the layer and ν is the kinematic viscosity. Since E is very small in most applications, the horizontal scale of convection must become much smaller than the vertical in order to increase friction. More detailed analysis (we refer to Chandrasekhar's (1961) book) shows that the horizontal scale decreases like $E^{1/3}$ and the Rayleigh number for onset of convection increases like $E^{-4/3}$ for small E. Besides the Ekman number and the Rayleigh number, which is a measure of the buoyancy, the Prandtl number is the third dimensionless parameter of the problem. It describes the ratio between thermal and viscous time scales of convection. For Prandtl numbers less than a value of about 1, oscillatory convection offers an alternate way to overcome the constraint of the Proudman-Taylor theorem wihtout changing, however, the power laws in the dependences on E.

3 EFFECTS OF INCLINED BOUNDARIES

The two extreme cases (a) and (b) of Figure 1 obviously correspond to equatorial and polar regions, respectively, of rotating spherical fluid shells heated from within and subjected to spherically symmetric gravity. There are, however, some important deviations because of the finite dimensions of the spherical shells. To discuss these effects let us consider the influence of inclined boundaries in (a) and (b). If top and bottom boundaries are added in case (a) convective motions satisfying the Proudman-Taylor theorem are still possible as long as the boundaries are parallel and viscous friction is negligible. Boundaries inclined with respect to each other, however, require a dependence of the velocity field on the coordinate in the direction of the axis of rotation, which we shall cal z-coordinate. A typical example is shown in Figure 2 (a). The deviation from the Proudman-Taylor condition is accomplished by a combination of time dependence and viscous friction in this case: Convection still has the form of columns aligned with the z-axis, but the columns are travelling like Rossby waves in the prograde or retrograde azimuthal direction depending on whether the distance between top and bottom boundaries decreases or increases with distance from the axis. In addition the azimuthal wave number α becomes large in order to increase frictional effects. In the limit of small values of E we find

$$R \propto \left(\frac{\eta}{E}\right)^{4/3}, \quad \alpha \propto \left(\frac{\eta}{E}\right)^{1/3}, \quad \omega \propto \alpha^{-1} \tag{3}$$

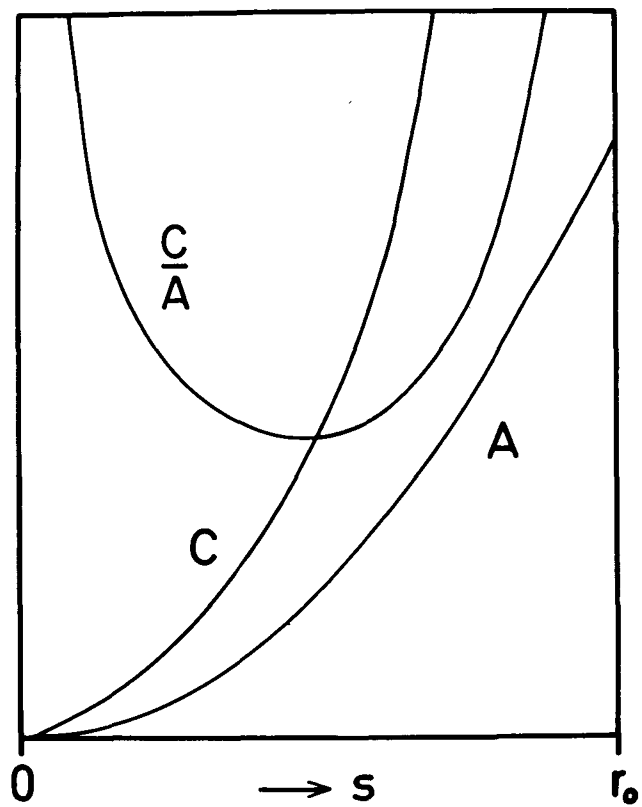

Figure 3: Buoyancy force A and inhibition C caused by inclined boundaries in a sphere as a function of distance S from the axis.

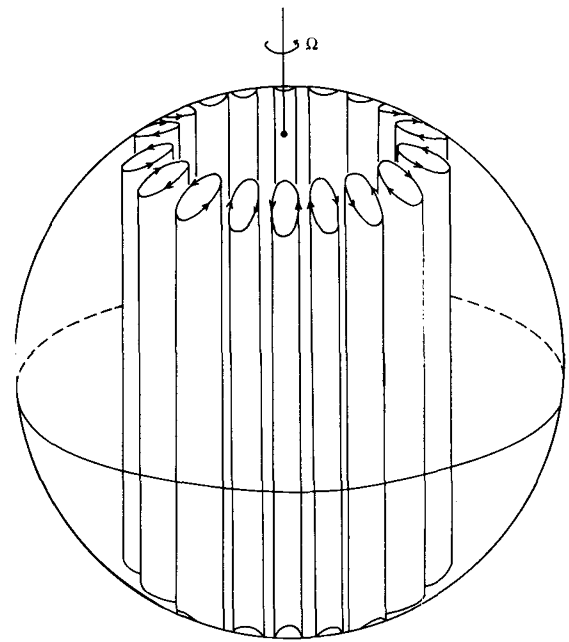

Figure 4: Sketch of motion at the onset of convection in a rotating sphere.

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rotation rate, and η is the tangent of half of the angle between the inclined boundaries. A detailed theory and an experimental study of the stabilizing effect of inclined boundaries can be found in the papers by Busse (1970b) and Busse and Carrigan (1974). In section 4 we show how the theory can be applied more or less directly to the case of convection in a sphere.

Since convection at the mid-latitudes of a rotating spherical shell corresponds to intermediate angles between gravity and the rotation vector, it may be anticipated that it shows properties intermediate to those of the extreme cases (a) and (b) of Figure 1. Indeed, it is easily shown (Chandrasekhar, 1961) that both the Rayleigh number and the wave number at the onset of convection are governed by the expressions derived for case (a) if the rotation rate Ω is replaced by its vertical component $\Omega \cos \phi$. Convection occurs in the form of rolls aligned with the horizontal component of $\underset{\sim}{\Omega}$, as indicated in Figure 2(b). Accordingly, the component of the Coriolis force proportional to $\Omega \sin \phi$ is balanced by the pressure and drops out of the dynamical considerations.

When applying the theory of plane parallel convection layers to spherical shells the strong dynamical coherence of the fluid along any line parallel to the axis of rotation must be kept in mind. For this reason convection in a spherical shell exhibits the effects of non-parallel boundaries even though the distance between the boundaries is constant. Since the tangential surfaces to the spherical boundaries of the shell are not parallel at the points intersected by the same line parallel to the z-axis, the dynamics of convection exhibit the same effects as in the case of the convective layer shown in Figure 2(c). The variation of "height" with distance from the axis of rotation induces a wave propagation property of the convective motions similar to that of the convection columns in Figure 2(a). Because of the particular phase relationship between buoyancy force and motion the phase propagation velocity is opposite that of Rossby waves, at least for Prandtl numbers of the order 1/3 and larger (Busse and Cuong, 1976).

4 CONVECTION IN ROTATING SPHERES AND SPHERICAL SHELLS

The problem of convection in a self-gravitating rotating fluid sphere has been traditionally considered for the case of homogeneous internal heating. Both gravity vector and temperature gradient vary linearly with distance r from the center in this case. Roberts (1968) gave a detailed mathematical analysis of the problem. The physically realized mode was determined by Busse (1970b).

An approximate solution of the problem can be obtained without any numerical analysis by applying the concept of convection in a rotating annulus, as shown in Figure 2(a). Because of the coherence in the z-direction enforced by rotation and the small length scale of the convection columns in the perpendicular direction, convection in any cylindrical section of the sphere behaves as in the corresponding

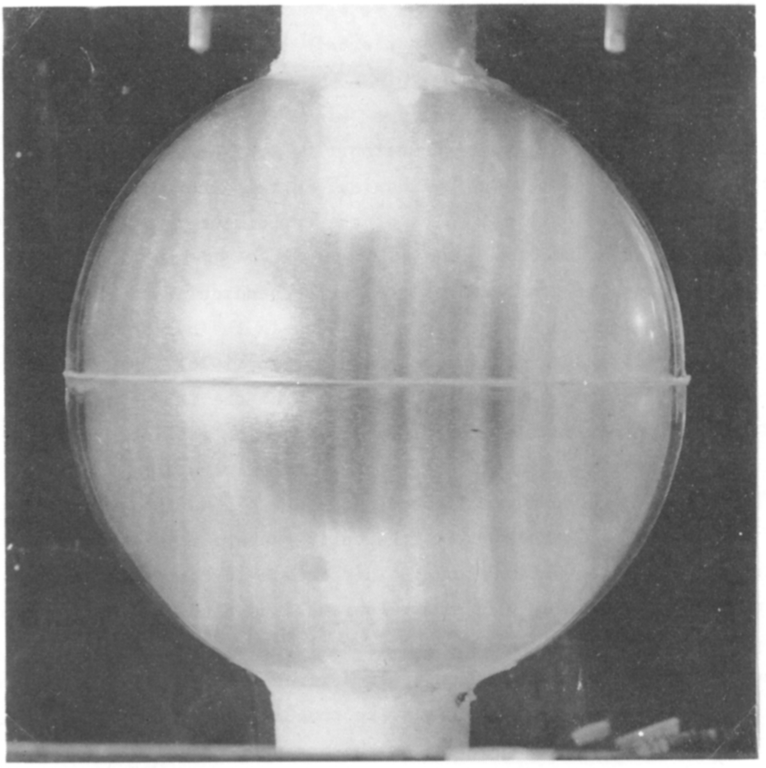

Figure 5: Laboratory simulation of convection in a rapidly rotating sphere. The motions are made visible by small flaky particles which align with the shear.

annulus problem. In Figure 3 the stabilizing effect C of the Coriolis force owing to the inclination of the boundary has been plotted as a function of the distance s from the axis together with the buoyancy force A, which is given by the product of the s-components of gravity and temperature gradient, since the z-component of the buoyancy force has little effect on the convection motion. The minimum of C/A at a distance s of about half the radius indicates the cylindrical surface where the onset of convection will occur as the critical value of the temperature gradient is reached. Figure 4 gives a qualitative sketch of the solution of the problem.

The fact that only the component of gravity perpendicular to the axis of rotation enters the dynamics in a first approximation is the basis for the laboratory simulation of the convection process (Busse and Carrigan, 1976). By using centrifugal force in place of gravity and by cooling the sphere from the inside and heating it from the outside the convection flow described above can be realized in a laboratory experiment. The onset of convection occurs in the form of regularly spaced columns, as shown in Figure 4. When the buoyancy force increases beyond the critical value, the region of convection is extended until the entire sphere is filled by convection columns. While amplitude fluctuations and the difference in the speed of propagation cause deviations from the regular picture at low amplitudes the perfect alignment of the columns persists, as shown in Figure 5.

The analysis of the spherical case applies directly to the equatorial region of spherical shells outside the cylindrical surface touching the inner boundary at the equator. In all cases the Rayleigh number for the onset of convection is lower in that region than in the other parts of the fluid shell. Inside the cylindrical surface the onset of convection can be described approximately by applying locally the theory of an inclined convection layer if the effects discussed in connection with Figures 2(b) and 2(c) are taken into account. Of particular interest is the prograde propagation of convection modes everywhere except at the poles. An asymptotic analysis for different radius ratios and for varying Prandtl number P is given by Busse and Cuong (1976). Figure 6 shows the local Rayleigh number for onset of convection as a function of the distance from the axis in a typical case. The corresponding wave number and frequency of convection are also shown. The asymptotic results agree reasonably well with the earlier numerical results obtained by Gilman (1975) at finite values of E in the case $P = 1$ and for a radius ratio $r_i/r_o = 0.8$, which is appropriate for the solar convection zone.

Figure 7 illustrates the most important feature of convection in a rapidly rotating spherical shell: The change in the character of convection across the cylindrical interface $s = r_i$. While the vorticity of the motions is nearly z- independent for $s > r_i$ the z- component of vorticity changes sign between lower and upper parts of the convection cell for $s < r_i$. This change in the symmetry of convection has important effects on the nature of the differential rotation generated by convection and on the heat transport.

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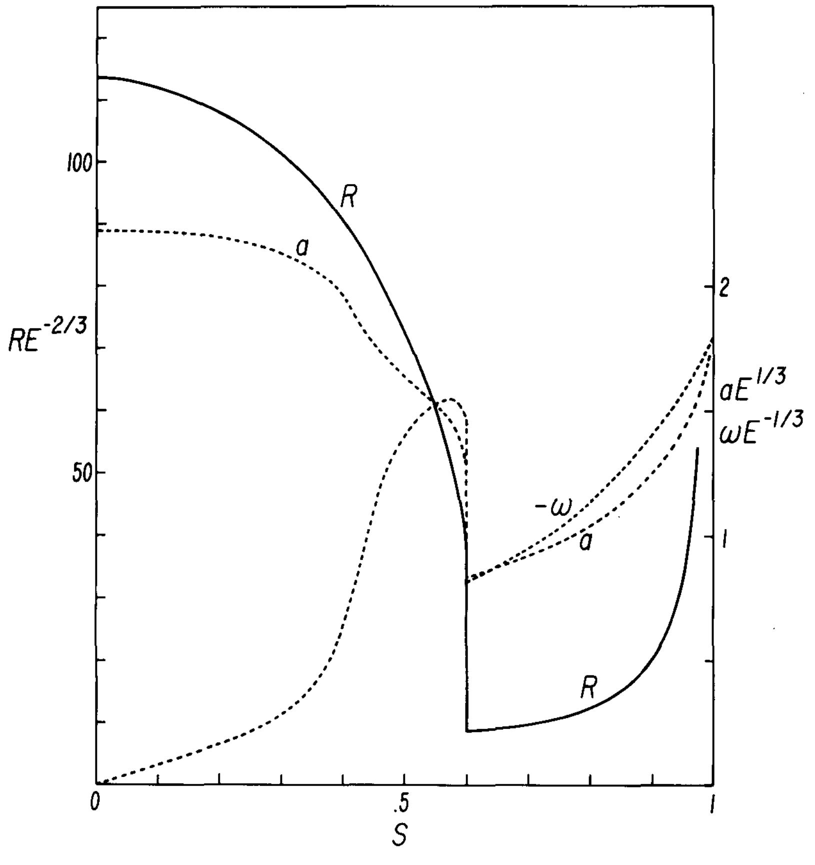

Figure 6: Rayleigh number R for the onset of convection in a spherical shell with radius ratio $r_i/r_o = 0.6$ as a function of distance S from the axis. Wave number a and frequency ω of convection columns are shown by dashed lines.

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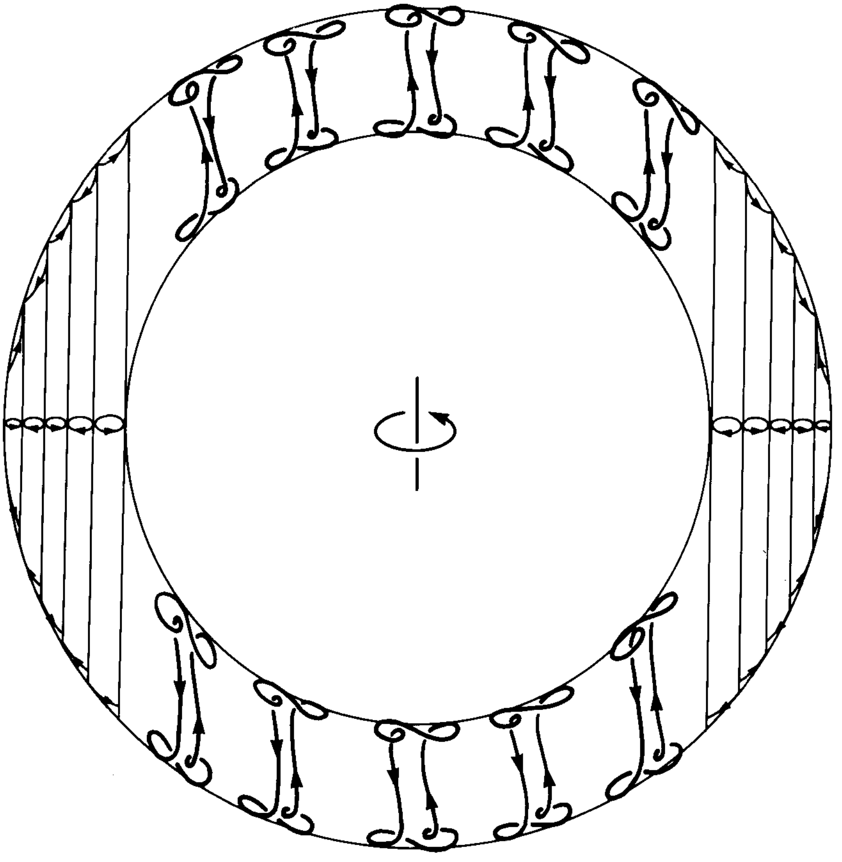

Figure 7: Sketch of convection modes in a spherical shell.

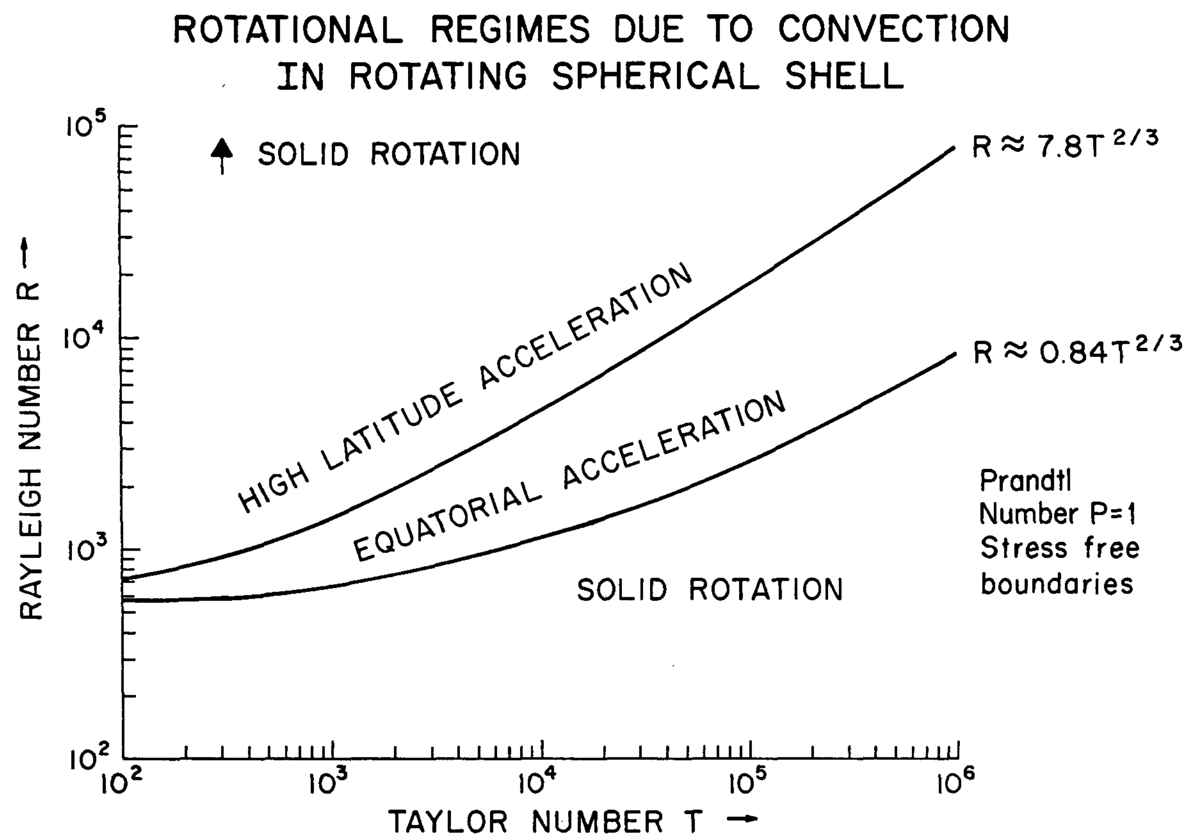

Figure 8: Regimes of differential rotation from Gilman (1976a).

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5 NONLINEAR ASPECTS

The phenomenon of solar differential rotation has stimulated much of the recent effort to understand convection in rotating spherical shells. It was first shown by Busse (1970a) that convection in a spherical shell can generate a differential rotation of the same form as that observed on the Sun. While Busse used an analytical perturbation method in the thin shell limit, Durney (1970) independently developed a mean field approach for the solution of the problem from which he obtained--after using the wave propagation property demonstrated by the analytical theory--essentially the same results. The exciting aspect of the observed solar phenomenon as well as of the theoretical results is that a prograde differential rotation occurs at the equator. This contradicts the earlier notion of angular mixing by convection which would have led to a deceleration of the equatorial region.

That the hypothesis of angular momentum mixing by convection is incorrect can easily be demonstrated in the case of convection in a cylindrical annulus discussed earlier. Since the Coriolis force can be entirely balanced by the pressure in this case, the influence of rotation disappears from the full nonlinear equation for two-dimentional convection rolls. Differential rotation cannot be a part of the solution since the basic equations are identical to those in a nonrotating region in this case and since a preferred azimuthal direction cannot be distinguished. Generation of differential rotation obviously depends on secondary features such as the curvature of the boundaries, and cannot be predicted by simple physical arguments.

How complicated the phenomenon of differential rotation in a convecting spherical shell can become at higher Rayleigh and Taylor numbers is evident from the numerical computations of Gilman (1972, 1976a,b). Because both the Reynolds stresses of the fluctuating convection velocity field and the meridional circulations caused by the inhomogeneity of convection contribute to the generation of differential rotation, small changes in the parameters of the problem may change the form of differential rotation dramatically. Figure 8 from Gilman (1976a) shows how the equatorial maximum of angular velocity changes into a relative minimum as the Rayleigh number is increased. The influence of boundary conditions also appears to be important. The almost exclusively used stress-free boundaries actually represent a singular case in the thin shell limit (Busse, 1973) since an equilibration between Reynolds stresses and viscous stresses can take place only in the latitudinal direction.

In order to investigate the generation of differential rotation in a conceptually simple case, the problem of convection in a rotating cylindrical annulus has recently been studied both experimentally and theoretically. Since the measurements are still in progress we restrict our attention to the qualitative picture, as shown in Figure 9. No differential rotation is generated in the case of straight top and bottom boundaries

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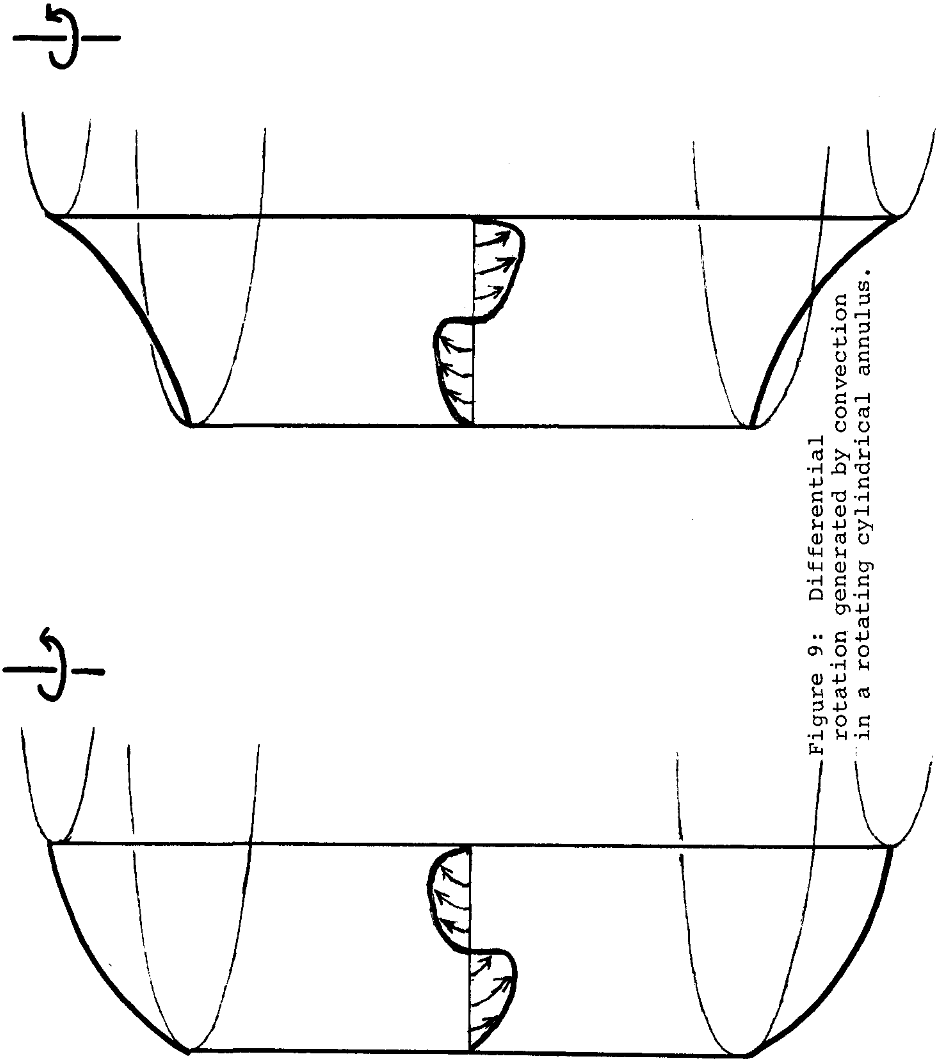

Figure 9: Differential rotation generated by convection in a rotating cylindrical annulus.

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of the annulus. The experimental observations show an increase of the gradient of angular momentum for convex boundaries and a decrease for concave boundaries, in agreement with theoretical predictions.

Meridional circulation and latitudinal variation of the convective heat transport are other important nonlinear properties of convection in spherical shells. Both phenomena are closely linked since the variation of the mean temperature caused by an inhomogeneous heat transport is the most important cause of meridional circulation. The lack of observational evidence for either phenomenon on the solar surface has been a source of controversy in the interpretation of theoretical models. We shall return to this point in the next section.

6 APPLICATIONS TO THE SUN AND JUPITER

It is fortunate for the theory of convection in rotating stars that there exist two quite different celestial bodies for which detailed surface observations are available. In the case of the Sun the influence of rotation is relatively small: The Rossby number is large compared to unity at least for the velocity field in the upper part of the convection zone. Jupiter represents the opposite case of a rapidly rotating system characterized by a small Rossby number. Although about half of the energy emitted from the surface of Jupiter is received from the Sun, the convective heat transport required for the other half is the dominating source of motions in the Jovian interior. In this respect Jupiter does indeed represent a low Rossby number example of a rotating convecting star.

The application of theoretical models which are valid at best for systems of laboratory scales to systems of stellar dimensions faces obvious difficulties. It is common practice to take into account the effects of turbulence owing to motions of smaller scale than those considered in the form of an eddy viscosity ν_e which replaces molecular viscosity in the equations of motion. The main justification for this procedure is that it appears to work well in many cases.

If ν_e is chosen sufficiently large that the Rayleigh number and Taylor number $4E^{-2}$ are not too large the differential rotation observed on the Sun resembles that predicted by the theoretical models fairly well. There is also evidence for the large-scale convection cells, often called giant cells, girdling the equator like a cartridge belt (Howard and Yoshimura, 1976). Figure 10 shows a laboratory simulation. The radius ratio in the laboratory experiment is closer to unity than in the solar case and the number of cells is correspondingly larger. Otherwise the cells show a surprising resemblance to those observed on the Sun by Walter and Gilliam (1976). Because the latter authors show magnetic regions a direct physical interpretation of the phenomenological resemblance is difficult, especially since the simultaneous occurrence of magnetic features which are symmetric or antisymmetric with respect to the solar equator is not well understood.

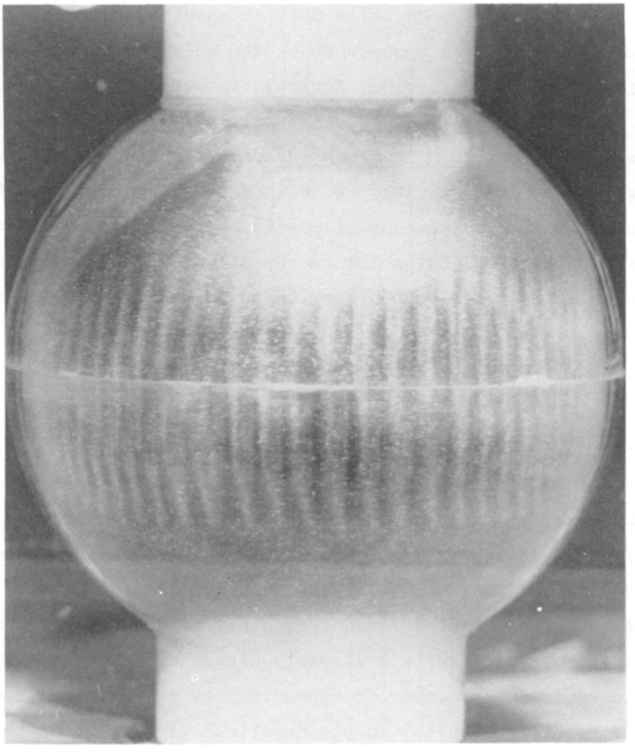

Figure 10: Laboratory simulation of convection in a rotating spherical fluid shell with inner radius $r_i = 4.45$ cm and $r_o = 4.77$ cm.

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The measurement of the Coriolis deflection of the horizontal motion in supergranules by Kubicela (1973) appears to be the only direct determination of the effect of rotation on solar convection. Kubicela interprets the observed deflection of the velocity as the Coriolis acceleration multiplied by the lifetime of a supergranule. Using a lifetime of 20 h he finds reasonable agreement with the measurements. Since the supergranular velocity field is defined as the mean over a field of highly fluctuating granular motions, the eddy viscosity concept can be used as an alternative possibility of interpretation. Using the linear solution for a convection cell in a rotating layer with stress-free boundaries (Chandrasekhar, 1961) we find the expression

$$\mathrm{tg}\gamma = \frac{2\Omega}{\nu_e \pi^2 d^2}$$

for the angle γ of deflection, where d is the depth of the supergranular layer. For simplicity we have assumed that the horizontal wavelength of the cells is large in comparison with d. Using $\approx 10^9$ cm and $\Omega = 2.6 \cdot 10^{-6}$ sec^{-1} we derive from the observed angle $\gamma \approx 10°$ an eddy viscosity of the order $2 \cdot 10^{12} \mathrm{cm}^2 \mathrm{sec}^{-1}$, which is in reasonable agreement with values derived from other more heuristic considerations. For the larger scale of giant cells a slightly higher value of ν_e appears to be appropriate yielding an Ekman number of approximately 10^{-2}, which is of the same order as the value used by Gilman (1976b) in his numerical simulation of the solar convection zone.

It should be mentioned that earlier theories of the solar differential rotation by Kippenhahn (1963) and others used the concept of an anisotropic eddy viscosity proposed by Biermann (1958). This concept often mimics the anisotropic dynamical influence of large-scale eddies. If the deviations from rigid rotation are described in terms of an anisotropic viscosity it would seem reasonable in view of the more detailed theory described above to use a latitude-longitude anisotropy rather than a horizontal-vertical anisotropy as proposed by Biermann.

Rayleigh numbers for stellar convection zones are based on the superadiabatic part of the temperature gradient, which amounts in general to only a small fraction of the total temperature gradient. A small change in surface temperature causes a disproportionately large change in the Rayleigh number and an even larger in the convective heat transport. The convection zone reacts like a high gain amplifier to any change of the temperature at the surface and it is not surprising that no subcritical large-scale variations of the solar surface temperature are observed. Since the temperature determines the energy emission, the convective heat flux must adjust itself to a uniform value. Ingersoll (1976) has emphasized this point in the case of Jupiter, where the convection heat transport adjusts itself in such a way that large-scale variations of the surface temperature vanish.

For this reason the heat flux variations and associated meridional circulations

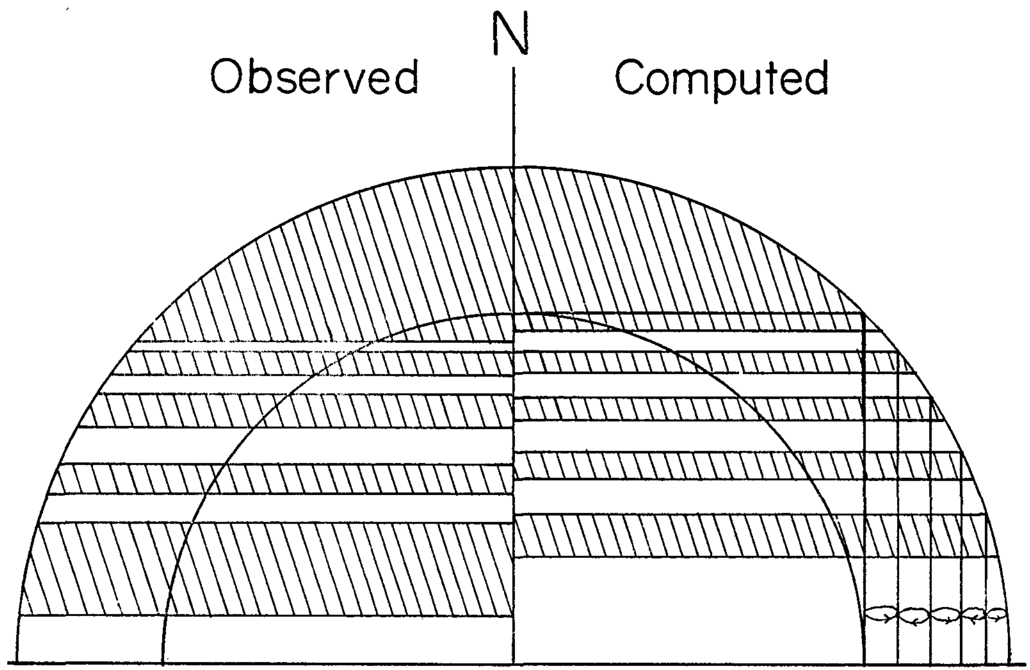

Figure 11: Comparison between theoretical predictions and observations of bands on Jupiter (from Busse, 1976).

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of low Rayleigh number models do not have much meaning for high Rayleigh number stellar convection zones. Even in laboratory experiments it is apparent that the inhibiting influence of rotation on the convective heat transport reverses itself with increasing Rayleigh number. Rossby's (1969) measurements even show a slight increase in heat transport owing to rotation at high Rayleigh numbers. The generation of differential rotation, on the other hand, depends on the alignment effect rather than the inhibition effect of rotation. It seems intuitively reasonable that the former effect, which does not have direct energetic consequences, persists at high Rayleigh numbers, while the latter effect is diminished by nonlinear processes.

Because of its low Rossby number, convection in the planet Jupiter may be more accessible than solar convection to theoretical analysis. A simple model has recently been proposed (Busse, 1976). It is generally believed that a transition from molecular to metallic hydrogen occurs at a radius of about 5/7 of Jupiter's outer radius and that the interface inhibits penetration by convection. Accordingly we are faced with the problem of convection in a rotating shell as sketched in Figure 7, which was actually drawn to apply to Jupiter. The fact that a relatively sharp transition from the low latitude band structure to the polar region of random eddy motion is observed on Jupiter at about 45° latitude appears to be the strongest argument for a dynamical influence of rotation along the lines outlined in this paper. To obtain a more detailed comparison as shown by Figure 11 the concept of an eddy viscosity must be invoked again. The value of ν_e required for a fivefold layer of convection columns is in good agreement, however, with the eddy viscosity deduced from convection models for the heat transport. More elaborate models are clearly possible and Jupiter may well become the testing ground for future theories of convection in rotating stars.

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