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On a problem in the theory of ordered groups

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The group G presented on two generators a, c with the single defining relation $a^{-1}c^2a = c^2a^2c^2$ [proposed by B.H. Neumann in 1949 (unpublished), discussed by Gilbert Baumslag in *Proc. Cambridge Philos. Soc.* 55 (1959)] has been considered as a possible example of an orderable group which can not be embedded in a divisible orderable group, contrary to the conjecture that no such examples exist. It is known from Baumslag's discussion that G can not be embedded in any divisible orderable group. However, it is shown in this note that G is not orderable, and thus is not a counter-example to the conjecture.

DEFINITIONS. A group, G, is an orderable group (O-group) if G admits a linear order, $\leq$, which has the property that if $x \leq y$ then $axb \leq ayb$ for a, b, x, y in G.

G is an R-group if it has the property that $x^n = y^n$ implies $x = y$ for x, y in G.

G is a divisible group if for each g in G and integer, n, there exists a (not necessarily unique) x in G such that $x^n = g$.

It is convenient to ignore the presentation of the group G given in the abstract, and instead to construct G as a generalized free product, as follows:

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Take groups $A = \text{gp}(a, b : a^{-1}ba = b^2)$ (whose elements may be written uniquely in the form $a^n b^\beta$ where n is an integer and $\beta = m2^{-k}$, m an integer and k a non-negative integer - see Fuchs [2], p. 60) and C, the infinite cyclic group with generator, c. Let G be the generalized free product of A and C with amalgamated subgroup

$$H = \text{gp}(c^2 = ba^{-2}) = \text{gp}(c^2 = a^{-2}b^{1/4}) .$$

Baumslag [1] has shown that G is an R-group which cannot be embedded in a divisible R-group. Thus G can not be embedded in a divisible O-group, because every O-group is an R-group (Fuchs [2], p. 61). We show that G cannot be linearly ordered.

LEMMA 1. *$ac^2a \neq ca^2c$ in G.*

Proof. We use a normal form argument. (See Neumann [3] for the theory of normal form in a free product with amalgamation.)

Let S be a system of left coset representatives of A with respect to H such that both $b^{1/4}$ and $b^{1/2}$ belong to S. $T = \{1, c\}$ is a system of left coset representatives of C with respect to H. (Observe that $b^{1/4}$ and $b^{1/2}$ lie in different cosets of H because every non-identity element of H has a non-trivial power of a in its unique representation in A.)

Now

$$ac^2a = ac^2a^2a^{-1} = aba^{-1} = b^{1/2} .$$

Since $b^{1/2} \in S$, $b^{1/2}$ is the normal form of ac^2a in G (with respect to S, T and H).

But

$$\begin{aligned} ca^2c &= ca^2ba^{-2}a^2b^{-1}c \\ &= cb^{1/4}a^2b^{-1}c \\ &= cb^{1/4}a^2(c^2a^2)^{-1}c \\ &= cb^{1/4}c^{-1} \\ &= cb^{1/4}cc^{-2} . \end{aligned}$$

Since $b^{1/4} \in S$, $c \in T$ and $c^{-2} \in H$, $cb^{1/4}cc^{-2}$ is the normal form of ca^2c in G .

The next lemma shows that G is not an O-group.

LEMMA 2. *Let K be an O-group with elements x and y which satisfy*

(1) $$x^{-1}y^2x = y^2x^2y^2 .$$

Then $xy^2x = yx^2y$.

Proof. We show that neither

(2) $$xy^2x < yx^2y$$

nor

(3) $$xy^2x > yx^2y$$

hold in K .

If we assume that (2) holds, we have

$$\begin{aligned} xy^2x < yx^2y &\Rightarrow xy^2x < y^{-1}x^{-1}y^2xy^{-1} \quad \text{by (1)} \\ &\Rightarrow xy^2x < y^{-1}x^{-2}xy^2xy^{-1} \\ &\Rightarrow xy^2x < y^{-1}x^{-2}yx^2yy^{-1} \quad \text{by (2)} \\ &\Rightarrow xy^2 < y^{-1}x^{-2}yx \\ &\Rightarrow xy^2 < y^{-1}x^{-2}y^{-1}y^2x \\ &\Rightarrow xy^2 < x^{-1}y^{-2}x^{-1}y^2x \quad \text{by (2)} \\ &\Rightarrow xy^2 < x^{-1}y^{-2}y^2x^2y^2 \quad \text{by (1)} \\ &\Rightarrow xy^2 < xy^2 \quad \text{- impossible.} \end{aligned}$$

So $xy^2x \nless yx^2y$.

Now assume that (3) holds. By substituting $>$ for $<$ and (3) for (2) in the above argument, the validity of this argument is not affected. So $xy^2x \ngtr yx^2y$. Hence $xy^2x = yx^2y$ and Lemma 2 is proven.

Finally, we observe that a and c in G satisfy (1). (Because

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$b = c^2a^2$ and $a^{-1}ba = b^2$ imply $a^{-1}(c^2a^2)a = (c^2a^2)^2$; that is $a^{-1}c^2a = c^2a^2c^2$.) So, if G were an O-group, then, by Lemma 2, $ac^2a = ca^2c$ would hold, contrary to Lemma 1, so G is not an O-group.

References

[1] Gilbert Baumslag, "Wreath products and p-groups", *Proc. Cambridge Philos. Soc.* 55 (1959), 224-231.

[2] László Fuchs, *Teilweise geordnete algebraische Strukturen* (Akadémiai Kiadó, Budapest, 1966).

[3] B.H. Neumann, "An essay on free products of groups with amalgamations", *Philos. Trans. Roy. Soc. London Ser. A* 246 (1954), 503-554.

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