THE CHORDS OF THE NON-RULED QUADRIC IN $PG(3, 3)$

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The 8-cage **(3)** may be defined as the simplest cubic graph having no circuit of fewer than eight edges. To construct it we first observe that it must contain a tree T whose vertices are of degrees 1 and 3 and in which each vertex of degree 1 is separated by an arc of just three edges from a central edge AB. These properties fix the structure of T uniquely. Starting with T we join the vertices of degree 1 by new edges so as to form a cubic graph, taking care to introduce no circuit of fewer than eight edges. It is found by trial that this can be done in essentially only one way. The resulting graph is the 8-cage. It has 30 vertices and 45 edges.

The 8-cage is 5-regular, that is, if S and S' are any two oriented arcs of length 5 in it there is a unique automorphism of the 8-cage transforming S into S'. Hence it may be calculated that the automorphism group of the 8-cage is of order 1440. It is shown in **(3)** that if in any s-regular graph the length of the shortest circuit is m, then $s \leqslant \frac{1}{2}m + 1$. Thus if $s = 5$ then $m \geqslant 8$. Accordingly the 8-cage is the simplest 5-regular cubic graph. It is also shown that there is no s-regular cubic graph such that $s > 5$. The 8-cage has therefore been called "the most regular of all graphs" **(1)**.

In **(1)** the 8-cage is exhibited as the Levi graph of the Cremona-Richmond configuration. The object of the present note is to describe another geometrical occurrence of the graph.

Let P denote the finite 3-dimensional projective space $PG(3, 3)$ whose 40 points have homogeneous co-ordinates x, y, z, t over the field of residues mod 3. Let Q denote the quadric $x^2 + y^2 + z^2 - t^2 = 0$ in P. The points of Q are the 4 points for which $t = 0$ while the remaining co-ordinates are non-zero, together with the 6 points for which two of the first three co-ordinates are zero and the other two co-ordinates are non-zero: 10 in all. It is easily verified that each plane of P is on at least one of these 10 points and that each tangent plane of Q is on just one of them. Thus Q has no generators. An account of the geometry of the "ellipsoid" Q is given in **(2)**; here we are concerned only with the relation of this geometry to the 8-cage.

Let V be the set of the 30 points of P not on Q, and let E be the set of the 45 lines of P meeting Q in two distinct points. We can regard V and E as sets of vertices and edges respectively of a graph G, the intersections of the edges in points of Q being regarded as irrelevant to the graph structure. We proceed to show that G is an 8-cage.

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If $X \in V$ the polar plane of X meets Q in the four points of a proper conic. The lines of E through V meet the other six points of Q in pairs. Hence G is cubic. It is now only necessary to show that G has no circuit of fewer than eight edges.

Let $v_0, v_1, \ldots, v_n = v_0$ be the vertices of any circuit of G, taken in their natural cyclic order. We have $n \geqslant 4$, since no six points of Q are coplanar. We write E_i for the edge joining v_i and v_{i+1} and F_i for the edge incident with v_i but not with v_{i-1} or v_{i+1}. (Addition in the suffices is mod n.) Since no five points of Q are coplanar we have the rule that no two of the five edges of G incident with v_i or v_{i+1} have a common point on Q. In other words, each point of Q is on just one of these five edges.

Suppose E_0 is on the points A and B of Q. Then by the rule just stated neither A nor B is on any other edge of G incident with v_1 or v_2, but each is on some edge of G incident with v_2 or v_3. Hence each of A and B is on one of the lines E_3 and F_3. Since neither of these lines is E_0 we may suppose E_3 is on A and F_3 is on B. Applying the same argument, but going round the circuit in the other direction, we find that each of A and B is on one of the lines E_{n-3} and F_{n-2}. (Each of these is incident with v_{n-2}.) These results have the following consequences:

(i) If $n = 4$ then A is on two edges of G incident with v_2 or v_3.
(ii) If $n = 5$ then A is on two edges of G incident with v_3 or v_4.
(iii) If $n = 6$ then B is on two edges of G incident with v_3 or v_4.
(iv) If $n = 7$ then A is on two edges of G incident with v_4 or v_5.

Applying our rule we deduce that $n \geqslant 8$. Thus each circuit of G has 8 or more edges.

Now let the points of Q be denoted by the letters a to j according to the following scheme:

a (1, 0, 0, 1), b (1, 0, 0, 2), c (0, 1, 0, 1), d (0, 1, 0, 2),
e (0, 0, 1, 1), f (0, 0, 1, 2), g (1, 1, 1, 0), h (1, 1, 2, 0),
i (1, 2, 1, 0), j (1, 2, 2, 0).

Postmultiplication of the co-ordinate vectors by the matrices

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

effects the permutations $R = (ajecifdg)(bh)$ and $L = (ajbhecgfdi)$ of the points of Q. These induce automorphisms r and l of G.

There is an oriented arc S of G of length 5 determined by the sequence (0, 0, 1, 0) (0, 0, 0, 1) (0, 1, 0, 0) (1, 0, 1, 0) (1, 0, 2, 2) (1, 1, 2, 1) of vertices of G. Its edges, taken in the corresponding order, are ef, cd, gi, af, dj. The automorphisms r and l convert S into oriented arcs given by the edge-sequences

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(cd, gi, af, dj, eg) and (cd, gi, af, dj, bi), respectively. Using the theory of (**3**, § 3), we deduce that r and l generate the whole group of automorphisms of G. Hence each automorphism of G is induced by some automorphism of Q, that is, some projective transformation of P under which Q is invariant. On the other hand, it is clear that at most one permutation of the letters a to j can correspond to a given automorphism of G. It follows that the automorphism groups of G and Q are isomorphic.

Inspection of a diagram of G shows that when the edges through a specified point of Q are removed, the graph falls into two connected parts, each of these being a Thomsen graph with its edges once subdivided.

References

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