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# HILBERT-KUNZ MULTIPLICITY OF TWO-DIMENSIONAL LOCAL RINGS

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**Abstract.** We study the behavior of Hilbert-Kunz multiplicity for powers of an ideal, especially the case of stable ideals and ideals in local rings of dimension 2. We can characterize regular local rings by certain equality between Hilbert-Kunz multiplicity and usual multiplicity.

We show that rings with "minimal" Hilbert-Kunz multiplicity relative to usual multiplicity are "Veronese subrings" in dimension 2.

## Introduction

Throughout this paper, let $(A, \mathfrak{m}, k)$ be a commutative Noetherian local ring of characteristic $p > 0$ with dimension $d := \dim A \geq 1$. For an $A$-module $M$, the minimal number of generators, the multiplicity and the length of $M$ is denoted by $\mu_A(M)$, $e_A(M)$ and $l_A(M)$, respectively.

Now let $I$ be an $\mathfrak{m}$-primary ideal of $A$ and $M$ a finite $A$-module. Then there exists a positive real constant $c$ such that

$$l_A(M/I^{[q]}M) = cq^d + o(q^d) \quad \text{for all large } q = p^e,$$

where $I^{[q]} = (a^q \mid a \in I)A$. We define

$$e_{\mathrm{HK}}(I, M) := \lim_{e \to \infty} \frac{l_A(M/I^{[q]}M)}{q^d}, \quad \text{where } \; q = p^e.$$

and we call $e_{\mathrm{HK}}(I, M)$ the *Hilbert-Kunz multiplicity* of $M$ with respect to $I$. In particular, we write as $e_{\mathrm{HK}}(I) := e_{\mathrm{HK}}(I, A)$ and $e_{\mathrm{HK}}(A) := e_{\mathrm{HK}}(\mathfrak{m})$; see [Mo] for details. Moreover, since Hilbert-Kunz multiplicity does not change under base field extension, we may assume that $A/\mathfrak{m}$ is infinite unless specified.

The notion of Hilbert-Kunz multiplicity has been introduced by [Ku1] in 1969, and has been studied in detail by Monsky [Mo]. Moreover, in recent

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years, Hochster and Huneke [HH] have pointed out that the tight closure $I^*$ of $I$ is the unique largest ideal containing $I$ having the same Hilbert-Kunz multiplicity as $I$; see also Lemma 1.6.

On the other hand, it is well-known that the integral closure $\overline{I}$ of $I$ is the unique largest ideal containing $I$ having the same usual multiplicity as $I$; see [Re1]. Since $I \subseteq I^* \subseteq \overline{I}$ for any ideal $I$, the Hilbert-Kunz multiplicity gives us more information than the usual one. For example, for a Cohen-Macaulay local ring $A$ with $e(A) = 2$, it is weakly F-regular (resp. not F-regular) if and only if $e_{\mathrm{HK}}(A) < 2$ (resp. $e_{\mathrm{HK}}(A) = 2$). Therefore the Hilbert-Kunz multiplicity seems to be important to study the singularities of local rings of positive characteristic.

The study of Hilbert-Kunz multiplicity attracted recently the attention of many researchers, see e.g. [BC], [BCP], [Co], [HM], [Se1], [Se2] and [WY]. The goal of this paper is the investigation of the structure of rings (or singularities) via the study of the behavior of Hilbert-Kunz multiplicity of powers of an ideal.

Our first result is the following theorem, which is a generalization of the fundamental Lemma 1.2.

Theorem 1.1. *For any $\mathfrak{m}$-primary ideal $I$ of $A$ and for any positive integer $n$, we have the following inequalities*:

$$\frac{e(I^n)}{d!} \leq e_{\mathrm{HK}}(I^n) \leq \frac{\binom{n+d-1}{d}}{n^d} e(I^n),$$

*where $e(I)$ denotes the usual multiplicity of $I$. In particular,*

$$\lim_{n\to\infty} \frac{e_{\mathrm{HK}}(I^n)}{e(I^n)} = \frac{1}{d!}.$$

In [WY], the authors noticed that the following conjecture is fundamental for the study of Hilbert-Kunz multiplicity.

Conjecture 1. *Let $I$ be an $\mathfrak{m}$-primary ideal of $A$.*

(1) *$e_{\mathrm{HK}}(I) \geq l_A(A/I^*)$, where $I^*$ denotes the tight closure of $I$.*

(2) *If $A$ is Cohen-Macaulay, then $e_{\mathrm{HK}}(I) \geq l_A(A/I)$.*

For example, if $A$ is a regular local ring then $e_{\mathrm{HK}}(I) = l(A/I)$ for any $\mathfrak{m}$-primary ideal $I$ of $A$. Moreover, it is well-known that (2) in Conjecture

1 is valid if $A$ is a complete intersection; see [Du]. See also [WY] in detail. However, they seem to be open even if $A$ is a two-dimensional Cohen-Macaulay local ring.

On the other hand, the authors [WY] have seen that an unmixed local ring $A$ with $e_{\mathrm{HK}}(A) = 1$ is regular. So as special case of Conjecture 1, we consider the following one.

Conjecture 2. *Suppose that $A$ is Cohen-Macaulay. Then*

(1) $e_{\mathrm{HK}}(\mathfrak{m}^n) \geq l_A(A/\mathfrak{m}^n)$ *for all* $n \geq 1$.

(2) *If* $e_{\mathrm{HK}}(\mathfrak{m}^n) = l_A(A/\mathfrak{m}^n)$ *for some* $n \geq 1$, *then $A$ is regular.*

The first aim of this paper is to prove that Conjecture 2 is true in case of two-dimensional local rings. In Theorem 2.5, we will show

$$e_{\mathrm{HK}}(\mathfrak{m}^n) \geq \frac{e(A)}{2}n^2 + \frac{n}{2} \geq l_A(A/\mathfrak{m}^n)$$

for all $n \geq 1$. This yields that Conjecture 2 is true in case of two-dimensional local rings.

In general, since the values of $e_{\mathrm{HK}}(A)$ have more variety compared to the usual multiplicity, sometimes $e_{\mathrm{HK}}(A)$ "determines" the ring almost uniquely. For instance, as described before, $e_{\mathrm{HK}}(A) = 1$ characterizes that $A$ is regular. Therefore the following question has naturally occurred in the study of lower bounds for Hilbert-Kunz multiplicity.

Question 3. *Let $A$ be a two-dimensional Cohen-Macaulay local ring. When does the equality* $e_{\mathrm{HK}}(\mathfrak{m}^n) = \frac{e(A)}{2}n^2 + \frac{n}{2}$ *hold for each* $n \geq 1$ *?*

The second purpose of this paper is to give a complete answer to this question. It turns out that the ring $A$ which satisfies this condition is almost uniquely determined for each multiplicity. Namely, in Section 3, we show the following.

Theorem 3.1. (See also Theorem 2.5.) *Let $A$ be a Cohen-Macaulay local ring of characteristic $p > 0$ and with* $\dim A = 2$. *Put $e = e(A)$. Assume that $k = A/\mathfrak{m}$ is algebraically closed. Then the following conditions are equivalent.*

(1) $e_{\mathrm{HK}}(A) = \frac{e+1}{2}$.

(2) $e_{\mathrm{HK}}(\mathfrak{m}^n) = \frac{e}{2}n^2 + \frac{n}{2}$ *for all (some)* $n \geq 1$.

(3) $G := \mathrm{gr}_{\mathfrak{m}}(A) \cong (k[X, Y])^{(e)}$, *where $G$ is the associated graded ring with respect to the maximal ideal and $(k[X, Y])^{(e)}$ is the subring of $k[X, Y]$ generated by all forms of degree $e$.*

Now let us explain the organization of this paper. In Section 1, we will give an upper bound for $e_{\mathrm{HK}}(I^n)$ in terms of $e_{\mathrm{HK}}(I)$ for any $\mathfrak{m}$-primary ideal $I$ of a local ring $A$ with arbitrary dimension. Namely, we prove the following

$$e_{\mathrm{HK}}(I^n) \le e(I)\binom{n+d-2}{d} + e_{\mathrm{HK}}(I)\binom{n+d-2}{d-1}.$$

We also show that if $I$ is stable then equality holds in the above equation. As a consequence, one can also see that the Conjecture 2 is true in case of Cohen-Macaulay local rings with minimal multiplicity (in any dimension); see Corollary 1.10.

From Section 2 to the end of this paper, we direct our attention to two-dimensional Cohen-Macaulay local rings.

In Section 2, we give a lower bound for $e_{\mathrm{HK}}(\mathfrak{m}^n)$ using Lemma 2.1; see Theorem 2.5. Using this, we will show that Conjecture 2 is true in case of two-dimensional local rings.

Section 3 is devoted to proving Theorem 3.1. In our proof we prove a theorem on Cohen-Macaulay homogeneous algebra with minimal multiplicity (degree) over an algebraically closed field, which may be of some interest itself (cf. Proposition 3.2). In fact, we will show that if $\mathrm{gr}_{\mathfrak{m}}(A)$ is not an integral domain then $e_{\mathrm{HK}}(A) > (e(A)+1)/2$.

In Section 4, we present several examples of Hilbert-Kunz multiplicities of stable ideals.

## §1. Asymptotic behavior of Hilbert-Kunz multiplicity

The following theorem shows the relationship between $e_{\mathrm{HK}}(I^n)$ and $e(I^n)$ and their asymptotic behavior.

THEOREM 1.1. *For any $\mathfrak{m}$-primary ideal $I$ of $A$ and for any positive integer $n$, we have the following inequalities*:

$$\frac{e(I^n)}{d!} \le e_{\mathrm{HK}}(I^n) \le \frac{\binom{n+d-1}{d}}{n^d} e(I^n), \tag{1.1}$$

*where $e(I)$ denotes the usual multiplicity of $I$. In particular,*

$$\lim_{n\to\infty} \frac{e_{\mathrm{HK}}(I^n)}{e(I^n)} = \frac{1}{d!}. \tag{1.2}$$

*Proof.* The inequality of the left-hand side follows from the following lemma, which gives a relationship between the Hilbert-Kunz multiplicity and the usual one.

LEMMA 1.2. (cf. [Hu2, Lemma 6.1]) *For any $\mathfrak{m}$-primary ideal $I$ of $A$, we have*

$$\frac{e(I)}{d!} \leq e_{\mathrm{HK}}(I) \leq e(I).$$

We notice that the second statement of the theorem follows from the first one and from $\lim_{n\to\infty} \binom{n+d-1}{d}/n^d = 1/d!$. Thus in order to complete the proof, it suffices to prove the inequality of the right-hand side in Eq.(1.1).

Let $J$ be a minimal reduction of $I$, that is, $J$ is a parameter ideal of $A$ which is contained in $I$ such that $I^{m+1} = JI^m$ for some non-negative integer $m$. Then since $I^n$ is integral over $J^n$, we have $e(I^n) = e(J^n) = n^d \cdot e(J)$. Moreover, since $J^n \subseteq I^n$ for all $n$, we have $e_{\mathrm{HK}}(J^n) \geq e_{\mathrm{HK}}(I^n)$. Thus the proof of the inequality Eq.(1.1) can be reduced to the following lemma.

LEMMA 1.3. *For any parameter ideal $J$ of $A$ and for any positive integer $n$, we have*

$$e_{\mathrm{HK}}(J^n) = \binom{n+d-1}{d} e(J).$$

*In particular, if $A$ is Cohen-Macaulay, then $e_{\mathrm{HK}}(J^n) = l_A(A/J^n)$ for all $n \geq 1$.*

*Proof.* We may assume that $A$ is complete. Take a system of parameters $\underline{a} = a_1, \ldots, a_d$ which is a minimal basis of $J$. Further, let $K$ be a coefficient field of $A$ and put $B := K[[a_1, a_2, \ldots, a_d]]$. Then $B$ is a subring of $A$ and is a complete regular local ring with the unique maximal ideal $\mathfrak{n} = (a_1, a_2, \ldots, a_d)B$. Moreover, $A$ is a module-finite extension of $B$ and $[Q(A) : Q(B)] = e(J)$. Thus [WY, Theorem 2.7] yields that

$$e_{\mathrm{HK}}(J^n) = e_{\mathrm{HK}}(\mathfrak{n}^n, B)e(J) = l(B/\mathfrak{n}^n)e(J) = \binom{n+d-1}{d} e(J).$$

COROLLARY 1.4. *For any $\mathfrak{m}$-primary ideal $I$ of $A$, we have*

$$e_{\mathrm{HK}}(I^n) = \frac{e(I)}{d!} n^d + o(n^d).$$

From now on, we will give an upper bound for $e_{\mathrm{HK}}(I^n)$ in terms of $e_{\mathrm{HK}}(I)$ for any $\mathfrak{m}$-primary ideal $I$ in any Cohen-Macaulay local ring. Before doing that, we recall the notion of tight closure, weakly F-regular etc., which were introduced by Hochster and Huneke; see also [HH], [Hu2].

Definition 1.5. ([HH]) Let $I$ be an ideal of $A$. An element $x \in A$ is said to be in the *tight closure* of $I$ if there exists an element $c \in A \setminus \bigcup_{P \in \mathrm{Min}(A)} P$ such that for all large $q = p^e$, $cx^q \in I^{[q]}$. The tight closure of $I$ is denoted by $I^*$. Notice that $I \subseteq I^* \subseteq \overline{I}$, where $\overline{I}$ denotes the integral closure of $I$.

A local ring $A$ in which every ideal (resp. every parameter ideal) is tightly closed (i.e. $I^* = I$) is called *weakly F-regular* (resp. *F-rational*).

Any F-rational local ring is normal and if, in addition, it is a homomorphic image of a Cohen-Macaulay local ring then it is itself Cohen-Macaulay; see e.g. [Hu2]. Moreover, $A$ is F-rational if and only if $\mathfrak{q}^* = \mathfrak{q}$ for some parameter ideal $\mathfrak{q}$ of $A$. See [FW], [HH] and [Hu2] for more details.

As described in the previous section, the notion of tight closure is important in the study of Hilbert-Kunz multiplicity. Actually, we can illustrate it with the next lemma.

Lemma 1.6. ([HH, Theorem 8.17], [Hu2]) *Let $A$ be a local ring and let $I$, $J$ be $\mathfrak{m}$-primary ideals with $I \subseteq J$. Then*

(1) *If $J \subseteq I^*$, then $e_{\mathrm{HK}}(I) = e_{\mathrm{HK}}(J)$.*

(2) *The converse of* (1) *is also true, provided that $A$ is analytically unramified* (*i.e. $\widehat{A}$ is reduced*) *and quasi-unmixed* (*i.e. $\widehat{A}$ is equi-dimensional*).

Before stating our main theorem in this section, we also recall the notion of stable ideals. Various properties of stable ideals are well-known; see e.g. [Oo], [Hu1]. Let us summarize some of which we need later.

Proposition 1.7. (cf. [Oo, Theorem 4.3], [Hu1, Theorem 2.1]) *For any $\mathfrak{m}$-primary ideal $I$ of a Cohen-Macaulay local ring $A$, we have*

$$l_A(A/I^n) \le e(I)\binom{n+d-2}{d} + l_A(A/I)\binom{n+d-2}{d-1}. \tag{1.3}$$

*Furthermore, the following conditions are equivalent.*

(1) *$I^2 = JI$ for every minimal reduction $J$ of $I$.*

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(2) $I^2 = JI$ *for some minimal reduction* $J$ *of* $I$.

(3) *For all* $n \geq 1$, *the following equality holds*:

$$l_A(A/I^n) = e(I)\binom{n+d-2}{d} + l_A(A/I)\binom{n+d-2}{d-1}.$$

(4) $e_1(I) = e(I) - l_A(A/I)$, *where* $e_1(I)$ *is the first Hilbert coefficient of* $I$.

*An ideal* $I$ *is called stable if it satisfies one of the above equivalent conditions. In this case, the associated graded ring* $\mathrm{gr}_I(A) := \oplus_{n\geq 0} I^n/I^{n+1}$ *is Cohen-Macaulay.*

The following, which is an analogy of Proposition 1.7 is the main theorem of this section and will be frequently used later.

THEOREM 1.8. *Let* $A$ *be a Cohen-Macaulay local ring and* $I$ *an* $\mathfrak{m}$-*primary ideal in* $A$. *Put* $d := \dim A \geq 2$. *Then*

(1) *For any positive integer* $n$, *the following inequality holds*:

$$e_{\mathrm{HK}}(I^n) \leq e(I)\binom{n+d-2}{d} + e_{\mathrm{HK}}(I)\binom{n+d-2}{d-1}. \tag{1.4}$$

(2) *If* $I$ *is stable, then equality holds in Eq.*(1.4).

(3) *Suppose that the* $\mathfrak{m}$-*adic completion* $\widehat{A}$ *of* $A$ *is reduced. Then the following conditions are equivalent*:

   (a) *There exists a parameter ideal* $J$ *such that* $J \subseteq I \subseteq J^*$.

   (b) $e_{\mathrm{HK}}(I) = e(I)$.

   (c) $e_{\mathrm{HK}}(I^n) = e(I)\binom{n+d-1}{d}$.

(4) *Suppose that* $A$ *is weakly F-regular and* $\widehat{A}$ *is reduced. Then* $I$ *is stable if and only if equality holds in Eq.*(1.4) *for some integer* $n \geq 2$.

*Proof.* First we recall the proof of Eq.(1.3) for convenience of the reader. Let $J$ be a minimal reduction of $I$. Then since $J$ is generated by a maximal regular sequence, we have $l_A(J^{n-1}/J^{n-1}I) = l_A(A/I) \cdot \mu_A(J^{n-1})$. Thus we get

$$\begin{aligned} l_A(A/J^{n-1}I) &= l_A(A/J^{n-1}) + l_A(J^{n-1}/J^{n-1}I) \\ &= e(J)\binom{n+d-2}{d} + l_A(A/I) \cdot \mu_A(J^{n-1}) \\ &= e(I)\binom{n+d-2}{d} + l_A(A/I)\binom{n+d-2}{d-1}. \end{aligned} \tag{1.5}$$

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Since $J^{[q]}$ is a minimal reduction of $I^{[q]}$ for all $q = p^e$, utilizing Eq.(1.5), we get

$$\begin{aligned} l_A\left(A/(J^{n-1}I)^{[q]}\right) &= l_A\left(A/(J^{[q]})^{n-1}I^{[q]}\right) \\ &= e\left(I^{[q]}\right)\binom{n+d-2}{d} + l_A(A/I^{[q]})\binom{n+d-2}{d-1} \\ &= q^d \cdot e(I)\binom{n+d-2}{d} + l_A(A/I^{[q]})\binom{n+d-2}{d-1}. \end{aligned}$$

Dividing this by $q^d$ and letting $e$ tend to $\infty$, we obtain

$$(1.6) \qquad e_{\mathrm{HK}}(J^{n-1}I) = e(I)\binom{n+d-2}{d} + e_{\mathrm{HK}}(I)\binom{n+d-2}{d-1}.$$

Now we prove (1) and (2). Since $I^n \supseteq J^{n-1}I$, we have $e_{\mathrm{HK}}(I^n) \leq e_{\mathrm{HK}}(J^{n-1}I)$. Moreover, if $I$ is stable then $I^2 = JI$; hence $I^n = J^{n-1}I$ for all $n \geq 1$. Thus (1) and (2) follow from Eq.(1.6).

In order to see (3) and (4), in the rest of the proof, we assume that $\widehat{A}$ is reduced. First, (3). $(a) \Longleftrightarrow (b)$ follows from Lemma 1.3 and Lemma 1.6. Moreover, since $I \subseteq J^*$ implies $I^n \subseteq ((J^*)^n)^* = (J^n)^*$, $(a) \Longrightarrow (c)$ also follows from Lemma 1.3 and Lemma 1.6. Thus it suffices to show $(c) \Longrightarrow (b)$. Suppose that $e_{\mathrm{HK}}(I^n) = e(I)\binom{n+d-1}{d}$ for some $n \geq 1$ and $e_{\mathrm{HK}}(I) < e(I)$. Then by virtue of Eq.(1.4), we have

$$\begin{aligned} e_{\mathrm{HK}}(I^n) &\leq e(I)\binom{n+d-2}{d} + e_{\mathrm{HK}}(I)\binom{n+d-2}{d-1} \\ &= e(I)\binom{n+d-1}{d} - [e(I) - e_{\mathrm{HK}}(I)]\binom{n+d-2}{d-1} \\ &< e(I)\binom{n+d-1}{d}. \end{aligned}$$

This gives a contradiction. Hence we obtain $(c) \Longrightarrow (b)$.

In order to see (4), we further assume that $A$ is weakly F-regular. Suppose that equality in Eq.(1.4) holds for some $m \geq 2$. Then we must show that $I$ is stable. We have $e_{\mathrm{HK}}(I^m) = e_{\mathrm{HK}}(J^{m-1}I)$ by Eq.(1.6). Since $J^{m-1}I \subseteq I^m$ and $A$ is weakly F-regular, we have $I^m \subseteq (J^{m-1}I)^* = J^{m-1}I$. Then $I^n = J^{n-1}I$ for all $n \geq m$. For such an integer $n$, we have by Eq.(1.5),

$$\begin{aligned} (1.7) \qquad l_A(A/I^n) &= l_A(A/J^{n-1}I) \\ &= e(I)\binom{n+d-2}{d} + l_A(A/I)\cdot\binom{n+d-2}{d-1}. \end{aligned}$$

On the other hand, since $l_A(A/I^n) = P_I(n)$ for large enough $n$, where $P_I(n)$ denotes the Hilbert-Polynomial of $I$, we may assume that $l_A(A/I^n) = P_I(n)$ for all $n \geq m$. In particular, for such an integer $n$, we have

$$(1.8)\quad l_A(A/I^n) = P_I(n) \;:=\; e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I),$$

where $e_i(I) \in \mathbb{Z}$ is called the $i$th *Hilbert coefficient* of $I$ for each $i = 0, \ldots, d$. Comparing Eq.(1.7) with Eq.(1.8), we get $e_1(I) = e(I) - l_A(A/I)$. Thus $I$ is stable by Proposition 1.7. □

The following corollary indicates the importance of "$e_{\mathrm{HK}}(I) - l_A(A/I)$".

COROLLARY 1.9. *If $I$ is a stable ideal of a Cohen-Macaulay local ring $A$, then for all $n \geq 1$, we have*

$$e_{\mathrm{HK}}(I^n) - l_A(A/I^n) = (e_{\mathrm{HK}}(I) - l_A(A/I))\binom{n+d-2}{d-1}.$$

*Proof.* It follows from Proposition 1.7 and Theorem 1.8(2). □

If $A$ is a Cohen-Macaulay local ring, then $e(A) \geq \mu_A(\mathfrak{m}) - d + 1$. The ring $A$ is called a Cohen-Macaulay local ring with *minimal multiplicity* if equality holds.

Using Theorem 1.8, we show that Conjecture 2 is true in case of Cohen-Macaulay local rings with minimal multiplicity.

COROLLARY 1.10. *Suppose that $A$ is a Cohen-Macaulay local ring with minimal multiplicity. Then for all $n \geq 1$, we have*

$$e_{\mathrm{HK}}(\mathfrak{m}^n) = e(A)\binom{n+d-2}{d} + e_{\mathrm{HK}}(A)\binom{n+d-2}{d-1}.$$

*In particular, $e_{\mathrm{HK}}(\mathfrak{m}^n) \geq l_A(A/\mathfrak{m}^n)$ for all $n \geq 1$. Furthermore, the following conditions are equivalent.*

(1) *$A$ is regular.*

(2) *$e_{\mathrm{HK}}(\mathfrak{m}^n) = l_A(A/\mathfrak{m}^n)$ for all $n \geq 1$.*

(3) *$e_{\mathrm{HK}}(\mathfrak{m}^n) = l_A(A/\mathfrak{m}^n)$ for some $n \geq 1$.*

*Proof.* Since $A$ is Cohen-Macaulay with minimal multiplicity, $\mathfrak{m}$ is stable; see [Sa2]. Thus the first statement follows from Theorem 1.8(2). The second statement follows from the first one and Proposition 1.7, because $e_{\mathrm{HK}}(A) \geq 1$.

In order to complete the proof, it is enough to show (3) $\Longrightarrow$ (1). Suppose that (3) holds for some $n \geq 1$. By virtue of Corollary 1.9, we may assume that $n = 1$. Then $e_{\mathrm{HK}}(A) = 1$. Hence $A$ is regular by [WY, Theorem 1.5]. $\square$

It seems to be useful to find such an asymptotic formula as in Theorem 1.8(2), because sometimes it is easy to compute $e_{\mathrm{HK}}(I)$ but not $e_{\mathrm{HK}}(I^n)$ directly. Moreover, in case of $d \geq 3$, $I^n$ is not necessarily stable in general.

Also, we notice that we need the assumption that $A$ is weakly F-regular in Theorem 1.8(4). For example, let $A = k[[s^3, s^4, t]]$ and put $\mathfrak{m} = (s^3, s^4, t)A$, the unique maximal ideal of $A$. Then $\mathfrak{m}^3 = (s^3, t)\mathfrak{m}^2$ and $\mathfrak{m}^2 \neq (s^3, t)\mathfrak{m}$; thus $\mathfrak{m}$ is not stable. But equality holds in Eq.(1.4) because $\mathfrak{m} = (s^3, t)^*$.

## §2. Hilbert-Kunz multiplicity and colength

In this section, let $A$ be a two-dimensional Cohen-Macaulay local ring. The main purpose of this section is to prove that Conjecture 2 for two-dimensional Cohen-Macaulay local rings.

The following lemma gives a lower bound for $e_{\mathrm{HK}}(I)$ in terms of the multiplicity $e(I)$. Although the proof of the lemma is very similar to that of [WY, Lemma 5.5], we give a proof for convenience of the reader.

LEMMA 2.1. *Let $A$ be a two-dimensional Cohen-Macaulay local ring and $I$ an $\mathfrak{m}$-primary ideal. If $r \geq \mu_A(I) - 2$, then*

$$e_{\mathrm{HK}}(I) \geq \frac{r+2}{2(r+1)} \cdot e(I). \tag{2.1}$$

*In particular, we have*

$$e_{\mathrm{HK}}(I) \geq \frac{e(I)+1}{2}. \tag{2.2}$$

*Proof.* In the following, $I^{xq}$ for $x \in \mathbb{Q}$ means $I^n$, where $n$ is the greatest integer which does not exceed $xq$. Since we are interested in the highest power of $q$, difference of a fixed integer independent of $q$ does not matter in our argument.

Let $J$ be a minimal reduction of $I$. Since $l_A(A/J^{[q]}) = e(I)q^2$, we want to find the lower bound for $e_{\mathrm{HK}}(I)$ by finding an upper bound for $l_A(I^{[q]}/J^{[q]})$. Let $u_1, \ldots, u_r \in I$ be generators of $I/J$, and let $s$ be a rational number such that $1 \le s < 2$. Then we have

$$(2.3) \quad l_A(I^{[q]}/J^{[q]}) \le l_A\left(\frac{I^{[q]} + I^{sq}}{J^{[q]} + I^{sq}}\right) + l_A\left(\frac{J^{[q]} + I^{sq}}{J^{[q]}}\right) =: (A_1) + (A_2).$$

Since $I^{[q]} = J^{[q]} + u_1^q A + \cdots + u_r^q A$ and $u_i^q \cdot I^{(s-1)q} \subseteq I^{sq}$, we have

$$(A_1) \le \sum_{i=1}^{r} l_A\left(\frac{u_i^q A + J^{[q]} + I^{sq}}{J^{[q]} + I^{sq}}\right) \le \sum_{i=1}^{r} l_A(A/I^{sq} : u_i^q) \le r l_A\left(A/I^{(s-1)q}\right).$$

Moreover, since we have

$$l_A(A/I^{(s-1)q}) = \frac{(s-1)^2}{2} q^2 e(I) + o(q^2)$$

and

$$(A_2) = \frac{(2-s)^2}{2} q^2 e(I) + o(q^2),$$

if we put $s = \frac{r+2}{r+1}$, then we get

$$l_A(I^{[q]}/J^{[q]}) \le \frac{e(I)q^2}{2}\left\{r(s-1)^2 + (2-s)^2\right\} + o(q^2) = \frac{r}{2r+2} q^2 e(I) + o(q^2).$$

This shows the inequality (2.1).

Now we show the second statement. Since

$$1 + l_A(\mathfrak{m}/I) + \mu_A(I) + l_A(\mathfrak{m}I/\mathfrak{m}J) = l_A(A/\mathfrak{m}J) = e(I) + 2,$$

we have $\mu_A(I) = e(I) + 1 - l_A(\mathfrak{m}/I) - l_A(\mathfrak{m}I/\mathfrak{m}J) \le e(I) + 1$. If we put $r = e(I) - 1$, then we obtain the desired inequality. ∎

Since the right-hand side of Eq.(2.1) is a decreasing function of $r$, we obtain the following.

Corollary 2.2. *Under the same notation as in Lemma 2.1, if $r \ge \mu_A(I) - 2$ and equality $e_{\mathrm{HK}}(I) = \frac{r+2}{2(r+1)} \cdot e(I)$ holds, then $r = \mu_A(I) - 2$.*

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2.3. *Mixed multiplicity.* We now recall the notion of *mixed multiplicity*. Let $I, J$ be $\mathfrak{m}$-primary ideals of $A$. Then we put

$$e(I|J) := \frac{1}{2}\{e(IJ) - e(I) - e(J)\}$$

and call it the (first) *mixed multiplicity* of $I$ and $J$.

Let us summarize some properties of the mixed multiplicities; see e.g. [Ve1], [Ve2].

(1) There exists some integers $f$, $g$ and $h$ such that for all large enough $r$ and $s$,

$$l_A(A/I^rJ^s) = e(I)\binom{r}{2} + e(I|J)rs + e(J)\binom{s}{2} + fr + gs + h.$$

(2) There exists a parameter ideal $(a, b)$ such that $a \in I$, $b \in J$ and $aJ + bI$ is a reduction of $IJ$. Such a parameter ideal $(a, b)$ is called a *joint reduction* of $(I, J)$.

(3) If $(a, b)$ is a joint reduction of $(I, J)$, then $e(I|J) = e(a, b)$.

(4) If $I'$ (resp. $J'$) is a reduction of $I$ (resp. $J$), then $e(I'|J') = e(I|J)$.

It seems that the following lemma is well-known. Actually, (1) follows from [Sa1, Theorem 1.2]. However, as for (2), we cannot find the suitable literature. So we give a proof for the sake of completeness.

Lemma 2.4. *Let $A$ be a two-dimensional Cohen-Macaulay local ring with multiplicity $e := e(A)$. Then*

(1) *For all $n \geq 1$, $\mu_A(\mathfrak{m}^n) \leq ne + 1$.*

(2) *The following conditions are equivalent.*

   (a) *$A$ has minimal multiplicity, that is, $\mu_A(\mathfrak{m}) = e + 1$.*
   (b) *$\mu_A(\mathfrak{m}^n) = ne + 1$ for all $n \geq 1$.*
   (c) *$\mu_A(\mathfrak{m}^n) = ne + 1$ for some $n \geq 1$.*

*Proof.* First we prove (1). Let $(a, b)$ be a minimal reduction of $\mathfrak{m}$. Notice that $(a^n, b)$ gives a joint reduction of $(\mathfrak{m}^n, \mathfrak{m})$. In fact, let $r$ be a positive

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integer such that $\mathfrak{m}^{r+1} = (a,b)\mathfrak{m}^r$. Then $(\mathfrak{m}^{n+1})^r = (a^n\mathfrak{m} + b\mathfrak{m}^n)(\mathfrak{m}^{n+1})^{r-1}$. By [Ve1, Lemma 3.1], we have

$$(2.4)\quad ne = e(\mathfrak{m}\,|\,\mathfrak{m}^n) \geq l_A(A/\mathfrak{m}\mathfrak{m}^n) - l_A(A/\mathfrak{m}^n) - l_A(A/\mathfrak{m}) = \mu_A(\mathfrak{m}^n) - 1$$

for any positive integer $n$. Moreover, equality holds in Eq.(2.4) if and only if $\mathfrak{m}^{n+1} = a^n\mathfrak{m} + b\mathfrak{m}^n$; see [Ve1, Theorem 3.2].

(2) To see (a)$\Longrightarrow$(b), it is enough to show that $\mathfrak{m}^{k+1} = a^k\mathfrak{m} + b\mathfrak{m}^k$ for all $k \geq 1$. But it follows easily from $\mathfrak{m}^2 = (a,b)\mathfrak{m}$ by induction on $k \geq 1$. Thus in order to complete the proof of this lemma, it suffices to show only (c)$\Longrightarrow$(a).

Suppose $\mu_A(\mathfrak{m}^n) = ne + 1$ for some integer $n \geq 1$. Then $\mathfrak{m}^{n+1} = a^n\mathfrak{m} + b\mathfrak{m}^n$. When $n = 1$, clearly $\mathfrak{m}$ is stable; thus $A$ has minimal multiplicity by [Sa2]. So we suppose $n > 1$ and $\mathfrak{m}^2 \neq (a,b)\mathfrak{m}$. Then for $z \in \mathfrak{m}^2 \setminus (a,b)\mathfrak{m}$, since $a^{n-1}z \in \mathfrak{m}^{n+1}$, we can write $a^{n-1}z = a^n x + by$ for some $x \in \mathfrak{m}$ and $y \in \mathfrak{m}^n$; hence $a^{n-1}(z - ax) = by$. Since $a^{n-1}, b$ is an $A$-sequence, we can write $z - ax = bs$, $y = a^{n-1}s$ for some $s \in A$. As elements $a$, $b$ are analytically independent, we obtain $s \in \mathfrak{m}$. However, this implies that $z = ax + bs \in (a,b)\mathfrak{m}$; this is a contradiction. $\square$

The following theorem is the main result in this section.

Theorem 2.5. *Let $A$ be a two-dimensional Cohen-Macaulay local ring with multiplicity $e := e(A)$. Then*

(1) *For all $n \geq 1$, we have*

$$e_{\mathrm{HK}}(\mathfrak{m}^n) \geq \frac{e}{2}n^2 + \frac{n}{2} \geq l_A(A/\mathfrak{m}^n). \tag{2.5}$$

(2) *The following conditions are equivalent:*

(a) $e_{\mathrm{HK}}(A) = \frac{e+1}{2}$.

(b) $e_{\mathrm{HK}}(\mathfrak{m}^n) = \frac{e}{2}n^2 + \frac{n}{2}$ *holds for all* $n \geq 1$.

(c) $e_{\mathrm{HK}}(\mathfrak{m}^n) = \frac{e}{2}n^2 + \frac{n}{2}$ *holds for some* $n \geq 1$.

(3) *The following conditions are equivalent:*

(a) *$A$ is regular.*

(b) $e_{\mathrm{HK}}(A) = 1$.

(c) $e_{\mathrm{HK}}(\mathfrak{m}^n) = l_A(A/\mathfrak{m}^n)$ *for all* $n \geq 1$.

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(d) $e_{\mathrm{HK}}(\mathfrak{m}^n) = l_A(A/\mathfrak{m}^n)$ *for some* $n \geq 1$.

(e) $l_A(A/\mathfrak{m}^n) = \frac{e}{2}n^2 + \frac{n}{2}$ *for all* $n \geq 1$.

(f) $l_A(A/\mathfrak{m}^n) = \frac{e}{2}n^2 + \frac{n}{2}$ *for some* $n \geq 1$.

*Remark* 1. According to Theorem 1.8, for all $n \geq 1$, we have

$$e_{\mathrm{HK}}(\mathfrak{m}^n) \leq \frac{e}{2}n^2 + \left(e_{\mathrm{HK}}(A) - \frac{e}{2}\right)n.$$

*Proof of Theorem* 2.5. First, we prove the inequality of the left-hand side of Eq.(2.5). By Lemma 2.4, we have $\mu_A(\mathfrak{m}^n) \leq ne + 1$ for all $n \geq 1$. Putting $r = ne - 1$ in Lemma 2.1, we obtain

$$e_{\mathrm{HK}}(\mathfrak{m}^n) \geq \frac{ne - 1 + 2}{2ne} \cdot e(\mathfrak{m}^n) = \left(\frac{1}{2} + \frac{1}{2ne}\right)n^2 e = \frac{e}{2}n^2 + \frac{n}{2}.$$

Next, we check the inequality of the right-hand side. By Proposition 1.7, we get

$$\begin{aligned} \frac{e}{2}n^2 + \frac{n}{2} &= e\binom{n}{2} - 1 \cdot n + \frac{e-1}{2}n \\ &\geq l_A(A/\mathfrak{m}^n) + \frac{e-1}{2}n \geq l_A(A/\mathfrak{m}^n). \end{aligned} \tag{2.6}$$

We now prove (2). Suppose that $e_{\mathrm{HK}}(\mathfrak{m}^n) = \frac{e}{2}n^2 + \frac{n}{2}$ for some $n \geq 1$. Then $\mu_A(\mathfrak{m}^n) = ne + 1$ by Corollary 2.2. This implies that $A$ has minimal multiplicity by Lemma 2.4. Thus in the proof of (2), we may assume that $A$ has minimal multiplicity. Then by virtue of Theorem 1.8, we have

$$e_{\mathrm{HK}}(\mathfrak{m}^n) = \frac{e}{2}n^2 + \left(e_{\mathrm{HK}}(A) - \frac{e}{2}\right)n$$

for all $n \geq 1$. The required assertion follows from the above equation.

Finally, we prove (3). (a) $\Longleftrightarrow$ (b) follows [WY, Theorem 1.5]. Moreover, (a) $\Longrightarrow$ (c) is well-known. (c) $\Longrightarrow$ (e) follows from (1). Hence it suffices to check (f) $\Longrightarrow$ (a). Suppose (f). Then equality holds in Eq.(2.6). This implies that $e = 1$; hence $A$ is regular. $\square$

*Remark* 2. In *Eq.*(2.5), we cannot replace $l_A(A/\mathfrak{m}^n)$ with $P_{\mathfrak{m}}(n)$, the Hilbert-Polynomial of $\mathfrak{m}$; see Example 4.3.

## §3. Local rings with "minimal" Hilbert-Kunz multiplicity

Let $(A, \mathfrak{m})$ be a two-dimensional Cohen-Macaulay local ring. In the previous section, we showed that

$$e_{\mathrm{HK}}(A) \geq \frac{e(A)+1}{2}. \tag{3.1}$$

In this section, we characterize the cases where we have equality in Eq.(3.1). Surprisingly, for each multiplicity $e$, there is essentially only one ring with this property; that is, Veronese subring of degree $e$ of $k[X, Y]$.

THEOREM 3.1. *Let $A$ be a Cohen-Macaulay local ring of characteristic $p > 0$ with $\dim A = 2$. Assume that $k = A/\mathfrak{m}$ is algebraically closed. Then $e_{\mathrm{HK}}(A) = (e(A)+1)/2$, if and only if $G := \mathrm{gr}_{\mathfrak{m}}(A) \cong (k[X, Y])^{(e(A))}$, where $G$ is the associated graded ring with respect to the maximal ideal and $(k[X, Y])^{(e(A))}$ is the subring of $k[X, Y]$ generated by all forms of degree $e(A)$.*

To prove the "only if" part, we use the following proposition concerning "curves of minimal multiplicity".

PROPOSITION 3.2. *Let $R$ be a graded ring of dimension 2 over a field $k = R_0$, generated by elements of degree 1. Assume $R$ is Cohen-Macaulay with $a(R) < 0$ and multiplicity $e$. Then we have;*

(1) *Let $I$ be a graded ideal of $R$ of pure height 0. Then the following statements are equivalent.*

   (a) *$R/I$ is Cohen-Macaulay.*

   (b) *The Poincaré series of $I$ is of the form $P(I,t) = \frac{bt}{(1-t)^2}$ for some $b \in \mathbb{Z}$.*

   (c) *If we put $X' = \mathrm{Proj}(R/I)$, then $H^0(X', \mathcal{O}_{X'}(0)) = k(= \overline{k})$ and $H^0(X', \mathcal{O}_{X'}(n)) = 0$ for all $n < 0$.*

   *If these equivalent conditions are satisfied, then $I$ is generated by elements of degree 1 and $P(R/I,t) = \frac{1+at}{(1-t)^2}$ for some $a \in \mathbb{Z}$.*

(2) *If $R$ is not reduced, then the nilradical $N$ of $R$ is generated by elements of degree 1 and $N^e = 0$. Also, $R_{\mathrm{red}}$ is Cohen-Macaulay with $a(R_{\mathrm{red}}) < 0$.*

(3) *Assume that $R$ is reduced. Let $E$ be a subset of the set* $\mathrm{Min}(R)$ *of the minimal prime ideals of $R$. Let $I$ be the intersection of the prime ideals in $E$ and $J$ the intersection of those in* $\mathrm{Min}(R)\setminus E$*. Also, assume that* $\mathrm{Proj}(R/I)$ *and* $\mathrm{Proj}(R/J)$ *are both connected. Then there exists a minimal system of generators $\{x, y_1, \ldots, y_s, z_1, \ldots, z_t\}$ of $R_1$ so that $I = (y_1, \ldots, y_s)$ and $J = (z_1, \ldots, z_t)$.*

*In this case, $R/(I + J) = k[x]$,* $\deg(R/I) = t$ *and* $\deg(R/J) = s$*. Also, for an integer $n > 0$, we have*

$$\dim_k(R/(\mathfrak{m}^{2n} + I^n)) = n\Big\{(2n-1)e + 2\Big\} - \frac{sn(n+1)}{2}.$$

*Note that $a(R)$ is defined by* $a(R) = \max\{n \in \mathbb{Z} : [H^2_{\mathfrak{m}}(R)]_n \neq 0\}$ (cf. [GW]).

We will first prove Theorem 3.1 assuming the results of Proposition 3.2. Let $J$ be a minimal reduction of $\mathfrak{m}$. If $\mathfrak{m}/J$ is generated by $r$ elements, then

$$e_{\mathrm{HK}}(A) \geq \frac{r+2}{2(r+1)} e(A). \tag{3.2}$$

Recall that $\mathfrak{m}/J$ is generated by at most $e(A) - 1$ elements. Hence we have Eq.(3.1) substituting $r = e(A) - 1$ in Eq.(3.2).

If we have equality in Eq.(3.1) then $\mathfrak{m}/J$ is minimally generated by $e(A) - 1$ elements by Corollary 2.2. Hence $\mathfrak{m}^2 = J\mathfrak{m}$ and we know that $\mathrm{gr}_{\mathfrak{m}}(A) = \oplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is Cohen-Macaulay with $a(G) < 0$ ([Sa2]); see also Proposition 1.7.

Put $e = e(A)$. We will show that if $G$ is not a domain, then

$$l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{(e+1)q/e}}{J^{[q]} + \mathfrak{m}^{(e+1)q/e}}\right) < \frac{e-1}{2e} q^2 + o(q^2) \tag{3.3}$$

and then the Hilbert-Kunz multiplicity becomes strictly bigger.

Now, assume that $G$ is *not* reduced. By Proposition 3.2 (2) there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $\mathrm{ini}(x)^e = 0$. Since $x^e \in \mathfrak{m}^{e+1}$, we have $x^q \in \mathfrak{m}^{(e+1)q/e}$ and thus

the left-hand side of Eq.(3.3)

$$\leq \left\{(e-2) \cdot \frac{e}{2}\left(\frac{e+1}{e} - 1\right)^2\right\} q^2 = \frac{e-2}{2e} q^2 + o(q^2).$$

Hence we have $e_{\mathrm{HK}}(A) \geq \frac{e+1}{2} + \frac{1}{2e}$.

Next, if $G$ is reduced and *not* an integral domain, then take a minimal generator system $\{x, y_1, \ldots, y_s, z_1, \ldots, z_t\}$ of $\mathfrak{m}$ corresponding to those basis of $G_1$ as in Proposition 3.2(3). Further, we may assume that $(x, y_1 + z_1)$ is a minimal reduction of $\mathfrak{m}$. Then if we put $I_1 = (y_1, \ldots, y_s)$ and $I_2 = (z_1, \ldots, z_t)$, we have $I_1.I_2 \subset \mathfrak{m}^3$ and $y_i^q I_2^{q/2e} \subset \mathfrak{m}^{(e+1)q/e}$ and by Proposition 3.2(3), we have strict inequality in Eq.(3.1). In fact, if we put $n = q/2e$, then

$$
\begin{aligned}
&\text{the left-hand side of Eq.(3.3)}\\
&\leq \sum_{i=2}^{s} l_A\left(\mathfrak{m}^{(e+1)q/e} : y_i^q\right) + \sum_{i=1}^{t} l_A\left(\mathfrak{m}^{(e+1)q/e} : z_i^q\right)\\
&\leq l_A(A/\mathfrak{m}^{2n} + I_2^n) \times (s-1) + l_A(A/\mathfrak{m}^{2n} + I_1^n) \times t\\
&= n\Big\{(2n-1)e + 2\Big\}(s-1+t) - \Big\{t(s-1) + st\Big\}\frac{n(n+1)}{2}\\
&= \frac{e-1}{2e}q^2 - \frac{(2s-1)t}{8e^2}q^2 + o(q^2).
\end{aligned}
$$

Thus we have shown that if $e_{\mathrm{HK}}(A) = (e(A)+1)/2$, then $\mathrm{gr}_{\mathfrak{m}}(A)$ is an integral domain. Then it is well known that any 2-dimensional graded domain with minimal multiplicity is isomorphic to a Veronese subring (cf. [EG],[EH],[Xa]).

Conversely, assume that $\mathrm{gr}_{\mathfrak{m}}(A)$ is an integral domain. In [WY, Theorem 2.15] we showed that $e_{\mathrm{HK}}(\mathrm{gr}_{\mathfrak{m}}(A)) \geq e_{\mathrm{HK}}(A)$ in general. Then $(e+1)/2 = e_{\mathrm{HK}}(\mathrm{gr}_{\mathfrak{m}}(A)) \geq e_{\mathrm{HK}}(A) \geq (e+1)/2$ and we have equality.

Now, we will prove Proposition 3.2. In the following proof, we write $f(t) \geq 0$ if $f(t) \in \mathbb{Z}[t, t^{-1}]$ whose coefficients are non-negative integers.

(1) First, recall that if $R$ is an unmixed graded ring of dimension 2, then there is a finite overring $\overline{R}$ with $\mathrm{depth}_R \overline{R} \geq 2$ and $\overline{R}/R \cong H_{\mathfrak{m}}^1(R)$ has finite length. Note that if we put $X = \mathrm{Proj}(R)$, then $\overline{R} = \oplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n))$. We always use $\overline{R}$ in this sense. Moreover, in this case, the condition (c) means that $[\overline{R/I}]_0 = k$ and $[\overline{R/I}]_n = 0$ for all $n \leq -1$.

Now let $I$ be a graded ideal of pure height 0. From the exact sequence

$$0 \to I \to R \to R/I \to 0 \tag{3.4}$$

we easily see that $I$ is a Cohen-Macaulay $R$-module. Also, since $H_{\mathfrak{m}}^2(R)$ surjects to $H_{\mathfrak{m}}^2(R/I) = H_{\mathfrak{m}}^2(\overline{R/I})$, we also have $a(\overline{R/I}) < 0$.

Now, let us compute the Poincaré series of each term. We will write

$$P(M,t) = \sum_{n\in\mathbb{Z}} \dim_k M_n t^n.$$

Also, since $R, \overline{R/I}$ are Cohen-Macaulay with $a(R), a(\overline{R/I}) < 0$,

$$P(R,t) = \frac{1+(e-1)t}{(1-t)^2} \quad \text{and} \quad P(\overline{R/I},t) = \frac{f(t)}{(1-t)^2}$$

with $f(t) \geq 0, \deg f \leq 1$. Also, since $I$ is also Cohen-Macaulay, $P(I,t) = \frac{g(t)}{(1-t)^2}$, where $g(t) = P(I/(x,y)I,t) \geq 0$.

Now, from Eq.(3.4), we have

$$P(R,t) = P(I,t) + P(R/I,t) = P(I,t) + P(\overline{R/I},t) - P((\overline{R/I})/(R/I),t).$$

Hence

$$\frac{1+(e-1)t}{(1-t)^2} = \frac{g(t)}{(1-t)^2} + \frac{f(t)}{(1-t)^2} - h(t), \tag{3.5}$$

where $h(t) \geq 0$.

If we assume the condition (a), then $h(t) = 0$ and we have $1+(e-1)t = g(t)+f(t)$. Hence $g(t) = bt$ and $f(t) = 1+at$ for some non-negative integers $a,b$ with $a+b = e-1$ and we have conditions (b), (c).

Now, assume the condition (b). Then Eq.(3.5) looks as

$$1+(e-1)t = bt + f(t) - (1-t)^2 h(t)$$

with $\deg(f) \leq 1$. Now it is easy to see that the above equality is possible only when $h(t) = 0$ and $f(t) = 1+(e-1-b)t$. Thus we have conditions (a), (c). Since the argument is the same if we start from the condition (c), the proof of (1) is complete.

If $R$ is not reduced, let $N$ be the nilradical of $R$. Since $R$ is Cohen-Macaulay, $H^0(X,\mathcal{O}_X) = R_0 = k$ and hence $X$ is connected. Since $X_{\text{red}}$ is a reduced connected curve, condition (c) holds for $X_{\text{red}}$ and $N$ is generated by elements of degree 1 by (1).

If $\mathcal{N}$ is the nilradical of $\mathcal{O}_X$, then $\{\mathcal{N}^i\}$ $(i = 1,2,\ldots)$ is a strictly decreasing sequence of ideals of $\mathcal{O}_X$. Since the degree of such ideals are strictly decreasing, we have $\mathcal{N}^e = 0$ and hence $N^e \subset H^0(X,\mathcal{N}^e) = 0$.

Now let $I$, $J$ be as in (3). Since they define reduced connected curves, we have condition (c) of (1). Hence $R/I$ and $R/J$ are Cohen-Macaulay.

From the exact sequence

$$0 \to R \to R/I \oplus R/J \to R/(I+J) \to 0, \tag{3.6}$$

we have the equality of Poincaré series (we can easily see that $R/(I+J)$ is Cohen-Macaulay of dimension 1 from Eq.(3.6)):

$$\frac{1+(e-1)t}{(1-t)^2} = \frac{1+at}{(1-t)^2} + \frac{1+bt}{(1-t)^2} - \frac{g(t)}{1-t},$$

where $P(R/(I+J),t) = \frac{g(t)}{1-t}$ and $a, b \geq 0$ are integers. Hence $g(t) = 1$ and $a+b = e-2$. Hence $I, J$ are generated respectively by $b+1, a+1$ elements of degree 1. Since $\mathfrak{m}$ is generated by $e+1$ elements, we have the desired conclusion. ∎

## §4. Several examples

Throughout this section, let $A$ be a two-dimensional Cohen-Macaulay local ring with infinite residue field. The purpose of this section is to present several examples of Hilbert-Kunz multiplicities of stable ideals.

### 4.1. Rational double points

Assume $e(A) = 2$. Then we have $e_{\mathrm{HK}}(A) < 2$ (resp. $e_{\mathrm{HK}}(A) = 2$) if $A$ is F-rational (resp. not F-rational). In both cases, since $\mathfrak{m}$ is stable, we can apply Theorem 1.8 to the maximal ideal $\mathfrak{m}$. Namely, we have

$$e_{\mathrm{HK}}(\mathfrak{m}^n) = n^2 + \Big(e_{\mathrm{HK}}(A) - 1\Big)n \qquad \text{for all } n \geq 1. \tag{4.1}$$

Hence if $A$ is not F-rational, $e_{\mathrm{HK}}(\mathfrak{m}^n) = n^2 + n$ for all $n \geq 1$. If $A$ is a complete F-rational double point and if $k = A/\mathfrak{m}$ is algebraically closed, then $A$ is isomorphic to one of following rings in Example 4.1 below. Using this fact, we can compute $e_{\mathrm{HK}}(\mathfrak{m}^n)$ for any Cohen-Macaulay local ring with $e(A) = 2$.

Example 4.1. (cf. [WY, Sect.5]) *Put $A = k[[x,y,z]]/(f(x,y,z))$, where $k$ is a field of characteristic $p$.*

| *type* | *equation* | *char A* | $e_{\mathrm{HK}}(A)$ | |
|---|---|---|---|---|
| $(A_n)$ | $f = xy + z^{n+1}$ | $p \geq 2$ | $2 - 1/(n+1)$ | $(n \geq 1)$ |
| $(D_n)$ | $f = x^2 + yz^2 + y^{n-1}$ | $p \geq 3$ | $2 - 1/4(n-2)$ | $(n \geq 4)$ |
| $(E_6)$ | $f = x^2 + y^3 + z^4$ | $p \geq 5$ | $2 - 1/24$ | |
| $(E_7)$ | $f = x^2 + y^3 + yz^3$ | $p \geq 5$ | $2 - 1/48$ | |
| $(E_8)$ | $f = x^2 + y^3 + z^5$ | $p \geq 7$ | $2 - 1/120$ | |

*Remark* 3. For the polynomial $f$ in the list of Example 4.1, we have

$$\begin{array}{llcl} \mathrm{gr}_{\mathfrak{m}}(A) & \cong k[x,y,z]/(xy+z^2) & \Longleftrightarrow & f \text{ is type } (A_1) \\ \mathrm{gr}_{\mathfrak{m}}(A) & \cong k[x,y,z]/(xy) & \Longleftrightarrow & f \text{ is type } (A_n) \text{ for some } n \geq 2 \\ \mathrm{gr}_{\mathfrak{m}}(A) & \cong k[x,y,z]/(x^2) & \Longleftrightarrow & f \text{ is either type } (D_n), (E_6), (E_7) \\ & & & \text{or } (E_8) \end{array}$$

By the similar argument as in the proof of Theorem 3.1, we can show that if $e \geq 3$ and $e_{\mathrm{HK}}(A) < \frac{e+1}{2} + \frac{1}{2(e-1)}$ then $\mathrm{gr}_{\mathfrak{m}}(A)$ is reduced. In general, the converse is not true. Namely, the reducedness of $\mathrm{gr}_{\mathfrak{m}}(A)$ does not yield the above inequality.

EXAMPLE 4.2. (cf. [WY, Sect.5]) *Let $A = k[[T,\, xT^a,\, x^{-1}T^b,\, \frac{1}{x+1}T^c]]$, where $k$ is a field of characteristic $p$ and $1 \leq a \leq b \leq c$ are given integers. Then $A$ is a two-dimensional Cohen-Macaulay local ring with rational triple point and is defined by the ideal $I = (UV - T^{a+b},\, UW - T^{a+c} + T^a W,\, VW - T^c V + T^b W)$ in $k[[T,U,V,W]]$. Then we have*

$$e_{\mathrm{HK}}(A) = 3 - \frac{a+b+c}{ab+bc+ca}.$$

*In this case, $\mathrm{gr}_{\mathfrak{m}}(A)$ is always reduced, but $e_{\mathrm{HK}}(A) < \frac{9}{4}$ if and only if $(a,b,c) = (1,1,1)$ or $(1,1,2)$.*

### 4.2. Hilbert-Polynomial

Let $P_{\mathfrak{m}}(n)$ be the Hilbert Polynomial of $\mathfrak{m}$. Namely, $P_{\mathfrak{m}}(n)$ is a rational polynomial in $n$ of degree $d (= \dim A)$ having leading coefficient $e(A)/d!$ such that $P_{\mathfrak{m}}(n) = l_A(A/\mathfrak{m}^n)$ for large enough $n$. In particular, in case of $d = 2$, it has the following expression:

$$P_{\mathfrak{m}}(n) = e_0 \binom{n+1}{2} - e_1 n + e_2 = \frac{e_0}{2} n^2 + \left(\frac{e_0}{2} - e_1\right) n + e_2,$$

where $e_0$, $e_1$, $e_2 \in \mathbb{Z}$.

If $A$ has minimal multiplicity, then we have $e_{\mathrm{HK}}(\mathfrak{m}^n) \geq P_{\mathfrak{m}}(n)$ for all $n \geq 1$, since $P_{\mathfrak{m}}(n) = l_A(A/\mathfrak{m}^n)$ for all $n \geq 1$ (cf. Theorem 2.5). But this is not true in general, as is shown in the following example.

EXAMPLE 4.3. (cf. [Sa4]) *Let $k$ be any field of characteristic $p > 0$, and let $e \geq 4$ be an integer. Then $A = k[[s, t^e, t^{e+1}, t^{2e+3}, t^{2e+4}, \ldots, t^{3e-1}]]$ is a two-dimensional Cohen-Macaulay local ring with multiplicity $e(A) = e$ and embedding dimension $\mathrm{emb}(A) = e$. Furthermore, we have*

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(1) $e_{\mathrm{HK}}(\mathfrak{m}^n) = \binom{n+1}{2}e$ *for all* $n \geq 1$.

(2) $P_{\mathfrak{m}}(n) = \binom{n+1}{2}e - (2e-3)n + \binom{e-1}{2}$ *for all* $n \geq 1$.

(3) $n(\mathfrak{m}) := \max\{n \in \mathbb{Z} : H_{\mathfrak{m}}(n) \neq P_{\mathfrak{m}}(n)\} = e-3$.

*In particular,* $e_{\mathrm{HK}}(\mathfrak{m}^n) \geq P_{\mathfrak{m}}(n)$ *if and only if* $n \geq \frac{(e-1)(e-2)}{2(2e-3)}$.

*Proof.* Since $B := k[[s,t]]$ is the integral closure of $A$ in its fraction field, we get

$$e_{\mathrm{HK}}(\mathfrak{m}^n) = e_{\mathrm{HK}}(\mathfrak{m}^n B) = l_B(B/\mathfrak{m}^n B) = l_B(B/(s,t^e)^n B) = \binom{n+1}{2}e.$$

On the other hand, we have $P_{\mathfrak{m}/sA}(n) = (n-1)e - (e-3)$ for all $n \geq 1$ by [Sa4, Example 4.4]. Moreover, as $s$ is a superficial element in $\mathfrak{m}$, we get

$$P_{\mathfrak{m}}(n) = \binom{n+1}{2}e - (2e-3)n + e_2.$$

Hence $e_1 = 2e-3 = 2(e-2)+1$. Thus we obtain that $e_2 = \binom{e-1}{2}$ by [ERV, Proposition 3.3]. This shows the assertion (2).

Moreover, by [Sa4, Corollary 5.8], we have $\operatorname{depth} \mathrm{gr}_{\mathfrak{m}}(A) = 1$. Hence by [Sa3, Proposition 3] or [Ma, Theorem 2], we have $n(\mathfrak{m}) = r(\mathfrak{m}) - 2 = e-3$, where $r(\mathfrak{m}) = e-1$ follows from [Sa4, Example 4.4]. $\square$

### 4.3. Pseudo-rational local rings

In the rest of this section, we will show that we can apply our theory to complete ideals of pseudo-rational local rings.

We first recall the notion of pseudo-rational local rings: Let $A$ be a two-dimensional normal, analytically unramified local ring. Then for an $\mathfrak{m}$-primary ideal $I$, there exists an integer $n_0$ such that, for all $n \geq n_0$,

$$l_A(A/\overline{I^n}) = e(I)\binom{n}{2} + f(I)n + g(I).$$

Definition 4.4. ([Li], [Re2]) Let $A$ be a two-dimensional normal, analytically unramified local ring. It is *pseudo-rational* if the normal genus $g(I) = 0$ for every $\mathfrak{m}$-primary ideal $I$.

*Remark* 4. Let $A$ be as above. For an $\mathfrak{m}$-primary ideal $I$ of $A$, we put $X = \operatorname{Proj}\left(\oplus_{n\geq 0}\overline{I^n}\right)$. Let $\mathcal{O}_X$ be the structure sheaf of $X$, then $g(I) = l_A(H^1(X, \mathcal{O}_X))$.

Any excellent F-rational local ring is pseudo-rational. But the converse is not true. For example, if $k$ is a field of characteristic $p = 2$, then $A = k[[x, y, z]]/(x^2 + y^3 + z^3)$ has a rational singularity (hence pseudo-rational), but not F-rational.

Now, let us summarize properties of pseudo-rational local rings; see [Li], [Re2], [Ve1] and [Ve2].

Fact 4.5. *Let $A$ be a two-dimensional pseudo-rational local ring, and let $I$, $J$ be $\mathfrak{m}$-primary complete (i.e. integrally closed) ideals of $A$. Then*

(1) *$IJ$ is also complete; see* [Li, Theorem 7.1].

(2) *For any joint reduction $(a, b)$ of $(I, J)$, $IJ = aJ + bI$. In particular, $I$ is stable. See* [Ve2, Proposition 3.5].

For pseudo-rational local rings (of dimension 2), we have a stronger result than Theorem 1.8 as follows:

Proposition 4.6. *Let $A$ be a two-dimensional pseudo-rational local ring, and let $I, J$ be complete $\mathfrak{m}$-primary ideals of $A$. Then*

$$e_{\mathrm{HK}}(I^r J^s) = e(I)\binom{r}{2} + e(I|J)rs + e(J)\binom{s}{2} + e_{\mathrm{HK}}(I)r + e_{\mathrm{HK}}(J)s$$

*for all non-negative integers $r, s$ with $r + s \geq 1$.*

*Proof.* By Fact 4.5(2), there exists a joint reduction $(a, b)$ of $(I, J)$ such that $IJ = aJ + bI$. Then we have $I^{[q]}J^{[q]} = a^q J^{[q]} + b^q I^{[q]}$ for all $q = p^e, e \geq 1$. Moreover, we have that both $I^{[q]}$ and $J^{[q]}$ are stable. Thus by [Ve2, Proposition 2.9], we get

$$\begin{aligned}(4.2)\quad & l_A(A/(I^r J^s)^{[q]}) \\ = \quad & e(I^{[q]})\binom{r}{2} + e(I^{[q]} \mid J^{[q]})rs + e(J^{[q]})\binom{s}{2} + l_A(A/I^{[q]})r + l_A(A/J^{[q]})s.\end{aligned}$$

Moreover, we have

$$\begin{aligned} e(I^{[q]} \mid J^{[q]}) &= \frac{1}{2}\left\{e((IJ)^{[q]}) - e(I^{[q]}) - e(J^{[q]})\right\} \\ &= \frac{q^2}{2}\{e(IJ) - e(I) - e(J)\} = q^2 e(I|J). \end{aligned}$$

Dividing the both sides in Eq.(4.2) by $q^2$ and letting $e$ tend to $\infty$, we obtain the required formula. ∎

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