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THE ITERATED PROJECTION SOLUTION FOR THE FREDHOLM INTEGRAL EQUATION OF SECOND KIND

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Abstract

We are concerned with the solution of the second kind Fredholm equation (and eigenvalue problem) by a projection method, where the projection is either an orthogonal projection on a set of piecewise polynomials or an interpolatory projection at the Gauss points of subintervals.

We study these cases of superconvergence of the Sloan iterated solution: global superconvergence for a smooth kernel, and superconvergence at the partition points for a kernel of "Green's function" type. The mathematical analysis applies for the solution of the inhomogeneous equation as well as for an eigenvector.

1. Introduction

We consider some projection methods for the solution of second kind integral equations of the form

$$(Tx)(s) - zx(s) = f(s), \qquad 0 \leq s \leq 1, \tag{1}$$

where T is the operator defined by

$$x(s) \mapsto \int_0^1 k(s, t)x(t)\,dt, \qquad 0 \leq s \leq 1.$$

Along with (1), we consider the eigenvalue problem

$$(T\phi)(s) = \lambda\phi(s), \qquad 0 \leq s \leq 1, \phi \neq 0. \tag{2}$$

(1) and (2) are regarded as equations in an appropriate subspace X of the complex Banach space $L^\infty(0, 1)$ with the norm $\|\cdot\|_\infty$. T is supposed to be

compact and $z \in \rho(T)$, the resolvent set of T, so that $(T - z)^{-1}$ is bounded with domain X. Let X_n be a finite dimensional subspace of X and let Π_n be a projection onto X_n. Then the projection method consists in approximating (1) and (2) respectively by

$$(\Pi_n T - z)x_n = \Pi_n f, \qquad x_n \in X_n, \tag{3}$$

$$\Pi_n T\phi_n = \lambda_n \phi_n, \qquad 0 \neq \phi_n \in X_n, \tag{4}$$

where x_n (resp. ϕ_n) is the projection solution (resp. eigenvector), corresponding to the approximation $T_n^P = \Pi_n T$ of T (P for projection).

Given a projection $\Delta = \{t_i\}_0^n$ of $[0, 1]$, $t_0 = 0$, $t_n = 1$, let X_n be a space S_Δ of piecewise polynomials of degree $\leq r$ on each subinterval $\Delta_i = [t_{i-1}, t_i]$, $i = 1, \ldots, n$. We set $h = \max_{1 \leq i \leq n}(t_i - t_{i-1})$. We shall consider two types of projection methods:

(a) Π_n is the orthogonal projection (in $L^2(0, 1)$) on S_Δ,

(b) Π_n is an interpolatory projection defined so that $\Pi_n x$ is the piecewise polynomial of degree $\leq r$ which interpolates x at $r + 1$ points $\{\tau_j^i\}_{j=1}^{r+1}$, on each Δ_i, $i = 1, \ldots, n$.

Case (a) corresponds to a *Galerkin method*, and case (b) to a *collocation method* at the collocation points $\{\tau_j^i\}$.

If $z \neq 0$ (resp. $\lambda_n \neq 0$) we consider the iterated projection solution $\tilde{x}_n$ (resp. eigenvector $\tilde{\phi}_n$) introduced by Sloan [13], [14] and given by the formulae:

$$\tilde{x}_n = \frac{1}{z}(Tx_n - f), \qquad \tilde{\phi}_n = \frac{1}{\lambda_n} T\phi_n,$$

where $\tilde{x}_n$ and $\tilde{\phi}_n$ are solutions of the equations

$$(T\Pi_n - z)\tilde{x}_n = f, \tag{5}$$

and

$$T\Pi_n\tilde{\phi}_n = \lambda_n\tilde{\phi}_n, \tag{6}$$

corresponding to the approximation $T_n^S = T\Pi_n$ of T (S for Sloan). Now $\Pi_n\tilde{x}_n = x_n$ and $\Pi_n\tilde{\phi}_n = \phi_n$, so that in case (b), the iterated solutions and the solutions themselves agree at the collocation points.

If k and f are smooth enough, it is known that $\|x_n - x\|_\infty = O(h^{r+1})$, while $\|\tilde{x}_n - x\|_\infty = O(h^{2r+2})$ for case (b) for example, provided that the $\{\tau_j^i\}$ are the $r + 1$ Gauss points on Δ_i, $i = 1, \ldots, n$. The optimal rate of convergence, relative to S_Δ, which is $\inf_{y \in S_\Delta}\|x - y\|_\infty = O(h^{r+1})$, is then overshot by $\tilde{x}_n \notin S_\Delta$, when k and f are smooth. Such fast convergence is often called *superconvergence*.

When k is the Green's function of an ordinary differential equation (o.d.e.) of order p with smooth coefficients, $\tilde{x}_n$ is still superconvergent at the partition points $\{t_i\}_0^n$, but *not* globally: the global rate of convergence is now $O(h^{r+1+p})$.

Similar results hold for $\tilde{\phi}_n$. This problem is studied for the equation (1) and the Galerkin method in Chandler's thesis [6]. The collocation method for a non linear o.d.e. has been looked at by de Boor-Swartz (see [1] for the solution of (1), and [2], [3] for the linear eigenvalue problem (2)), where T is the associated differential operator. In de Boor-Swartz [4] the "essential" least squares method (or local moment method) for an o.d.e. is also studied.

We present in this paper an analysis of the convergence rates which is a blend of the techniques of Chandler and of de Boor-Swartz. It applies for the iterated solution $\tilde{x}_n$ as well as for the iterated eigenvector $\tilde{\phi}_n$ (the result seems to be new for the *eigenvector* in the most general case). It is . ased on a study of the error at the point t of [0, 1] in terms of the scalar product $\langle l_t, (1 - \Pi_n)\tilde{x}_n\rangle$ (resp. $\langle l'_t, (1 - \Pi_n)\tilde{\phi}_n\rangle$) where l_t (resp. l'_t) is a function having the same smoothness properties as $k_t(\cdot) := k(t, \cdot)$, and where $\langle f, g\rangle = \int_0^1 f\bar{g}$.

In case (a), we use the orthogonality of Π_n:

$$\langle l_t, (1 - \Pi_n)\tilde{x}_n\rangle = \langle(1 - \Pi_n)l_t, (1 - \Pi_n)\tilde{x}_n\rangle.$$

In case (b) we use firstly that the function $(1 - \Pi_n)\tilde{x}_n$ vanishes at the collocation points τ_j^i, and secondly that the $\{\tau_j^i\}$ being the $r + 1$ Gauss points in Δ_i, then $\int_{\Delta_i} p(s)\prod_{j=1}^{r+1}(s - \tau_j^i)\, ds = 0$ for all polynomials p of degree $\leqslant r$.

The superconvergence in case (a) is proved under the assumption that Δ is quasi-uniform. In case (b), Δ is arbitrary but more smoothness properties are required for k and f.

2. The setting of the problem

2.1. Piecewise continuous functions

Let be given $\Delta = \{t_i\}_0^n$, a strict partition of [0, 1], $0 = t_0 < t_1 < \cdots < t_n = 1$. It is quasi-uniform if there exists $\sigma \geqslant 0$: $\max(t_i - t_{i-1})/\min(t_i - t_{i-1}) \leqslant \sigma$ for $n = 1, 2, \ldots$. Then $nh \leqslant \sigma$. $\Delta_i := [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. We define $C_\Delta := \prod_{i=1}^n C_{(\Delta_i)}$: $f \in C_\Delta$ consists of n components $f_i \in C_{(\Delta_i)}$, f is a piecewise continuous function having (possibly) different left and right values at the partition points t_i. With the norm $\|\cdot\|_\Delta$ defined by $\|f\|_\Delta = \max_{i=1,\ldots,n}\|f_i\|_\infty$, C_Δ is a Banach space. $C_\Delta \subset L^\infty(0, 1)$ by $\|f\|_\Delta \leqslant \|f\|_\infty$ and if f is continuous on [0, 1], then $\|f\|_\infty = \|f\|_\Delta$. We define, more generally, C_Δ^l for positive integer l by $C_\Delta^l = \prod_{i=1}^n C_{(\Delta_i)}^l$ where $f_i \in C_{(\Delta_i)}^l$ iff its lth derivative $f_i^{(l)}$ is continuous on Δ_i. Clearly $S_\Delta \subset C_\Delta$ and the projection Π_n is defined $C_\Delta \to S_\Delta$ with $f = (f_1, \ldots, f_n) \mapsto \Pi_n f = (\Pi f_1, \ldots, \Pi f_n)$, where Πf_i is the projection of $f_i \in C_{(\Delta_i)}$ on the polynomials of degree $\leqslant r$ on Δ_i.

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2.2. Spectral definitions

T is supposed to be compact in the complex Banach space $X = C_\Delta$. $\mathcal{L}(X)$ is the algebra of bounded operators on X. The *resolvent set* of T is $\rho(T) = \{z \in \mathbf{C};\ (T - z)^{-1} \in \mathcal{L}(X)\}$ where z stands for $z1$. For z in $\rho(T)$, $R(z) = (T - z)^{-1}$ is the *resolvent* of T and $TR(z) = R(z)T$. The unique solution of (1) is then $x = R(z)f$.

Let $\lambda \neq 0$ be an *isolated eigenvalue* of T with algebraic (resp. geometric) multiplicity m (resp. g), and ascent μ, $1 \leq \mu \leq m$, $1 \leq g \leq m$. The associated eigenspace is $E = \ker(T - \lambda)$, the null space of $T - \lambda$ so $\dim E = g$; the invariant subspace is

$$M = \ker(T - \lambda)^m, \quad \dim M = m, \quad \text{and} \quad \ker(T - \lambda)^\mu \equiv \ker(T - \lambda)^m.$$

Let Γ be a Jordan curve in $\rho(T)$, around λ, which contains neither 0 nor any other eigenvalue of T. $P := -1/2i\pi \int_\Gamma R(z)\,dz$ is the *spectral projection* associated with λ, $M = PX$. Let T_n be a sequence of operators in $\mathcal{L}(X)$ such that T_n converge to T pointwise. T_n will be either $T_n^P = \Pi_n T$ or $T_n^S = T\Pi_n$. If $\Gamma \subset \rho(T_n)$, we may define for T_n the resolvent $R_n(z)$ for $z \in \Gamma$ and the spectral projection $P_n := -1/2i\pi \int_\Gamma R_n(z)\,dz$. If T_n is strongly stable inside Γ (Chatelin [8], [9]), there are, for n large enough, exactly m eigenvalues $\{\lambda_{in}\}_{i=1}^m$ of T_n inside Γ (counting their algebraic multiplicities), $\hat{\lambda}_n$ is their arithmetic mean, and λ_n is any one of them.

For the projections Π_n under consideration, both T_n^P and T_n^S are strongly stable around any non-zero eigenvalue of T (Chatelin [7], [9]). The solution x_n of (3) is such that $x_n = R_n^P(z)\Pi_n f$, and $\tilde{x}_n = R_n^S(z)f$. Similarly ϕ_n is an eigenvector of T_n^P and $\tilde{\phi}_n$ of T_n^S, associated with the same eigenvalue λ_n.

2.3. The errors $x_n - x$, $\phi_n - P\phi_n$, $\tilde{x}_n - x$, $\tilde{\phi}_n - P\tilde{\phi}_n$ and $\lambda - \hat{\lambda}_n$

C is a generic constant, which may depend on r and σ, but is otherwise independent of Δ.

2.3.1. The projection method

We recall the following equality:

$$x_n - x = zR_n^P(z)(1 - \Pi_n)x, \quad \text{then } \|x - x_n\|_\infty \leq C\|(1 - \Pi_n)x\|_\infty.$$

As for the resolvents,

$$\left(R_n^P(z) - R(z)\right)\phi_n = R(z)\left(T - T_n^P\right)R_n^P(z)\phi_n = \frac{R(z)}{\lambda_n - z}(1 - \Pi_n)T\phi_n,$$

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because $R_n^P(z)\phi_n = \phi_n/\lambda_n - z$. To integrate on Γ, we distinguish whether $\lambda_n = \lambda$ or not. If $\lambda_n = \lambda$, then $-1/2i\pi\int_\Gamma(R(z)/\lambda - z)dz = S = \lim_{z\to\lambda} R(z)(1-P)$; S is the *reduced resolvent* with respect to λ. If $\lambda_n \neq \lambda$, $R(z) - R(\lambda_n) = (z-\lambda_n)R(\lambda_n)R(z)$, and

$$\frac{-1}{2i\pi}\int_\Gamma \frac{R(z)}{\lambda_n - z}\,dz = R(\lambda_n)\left[\frac{-1}{2i\pi}\int_\Gamma\frac{dz}{\lambda_n - z} + \frac{1}{2i\pi}\int_\Gamma R(z)\,dz\right] = R(\lambda_n)(1-P).$$

λ is the only pole of $R(z)$ inside Γ, $R(\lambda_n)(1-P)$ is well defined and when $n\to\infty$, $\lambda_n\to\lambda$, $R(\lambda_n)(1-P)\to S$. $R(\lambda_n)(1-P)$ is then uniformly bounded in n, for n large enough. To have a unique formula for the cases $\lambda_n = \lambda$ and $\lambda_n \neq \lambda$, we set $R(\lambda)(1-P) = S$.

By integration in z on Γ, we get $\phi_n - P\phi_n = R(\lambda_n)(1-P)(1-\Pi_n)T\phi_n$, and

$$\operatorname{dist}(\phi_n, M) = \inf_{\phi\in M}\|\phi_n - \phi\|_\infty \leqslant \|\phi_n - P\phi_n\|_\infty \leqslant C\|(1-\Pi_n)T\phi_n\|_\infty.$$

2.3.2. The Sloan method

1) $\tilde{x}_n - x = (R_n^S(z) - R(z))f = R(z)(T - T_n^S)R_n^S f = R(z)T(1-\Pi_n)\tilde{x}_n = TR(z)(1-\Pi_n)\tilde{x}_n$.

Then for any fixed t in $[0, 1]$, and any fixed z in $\rho(T)$,

$$\begin{aligned}(\tilde{x}_n - x)(t) &= \int_0^1 k(t,s)\left[R(z)(1-\Pi_n)\tilde{x}_n\right](s)\,ds\\ &= \langle k_t, R(z)(1-\Pi_n)\tilde{x}_n\rangle\\ &= \langle (R(z))^* k_t, (1-\Pi_n)\tilde{x}_n\rangle = \langle l_t, (1-\Pi_n)\tilde{x}_n\rangle.\end{aligned}$$

Because $R^*(z) := (R(z))^* = (T^* - \bar{z})^{-1}$, $l_t := R^*(z)k_t$ is the solution of $(T^* - \bar{z})l_t = k_t$; the solution l_t (which depends on z) is unique since $z\in\rho(T)\Leftrightarrow \bar{z}\in\rho(T^*)$.

2) Similarly

$$\left(R_n^S(z) - R(z)\right)\tilde{\phi}_n = R(z)\left(T - T_n^S\right)R_n^S(z)\tilde{\phi}_n = \frac{R(z)}{\lambda_n - z}T(1-\Pi_n)\tilde{\phi}_n.$$

By integration on Γ, we get for any fixed t on $[0, 1]$

$$\begin{aligned}\tilde{\phi}_n(t) - \left(P\tilde{\phi}_n\right)(t) &= \left[T\left(\frac{-1}{2i\pi}\int_\Gamma\frac{R(z)}{\lambda_n - z}\,dz\right)(1-\Pi_n)\tilde{\phi}_n\right](t)\\ &= \left[TR(\lambda_n)(1-P)(1-\Pi_n)\tilde{\phi}_n\right](t).\end{aligned}$$

We define $l_t' := R^*(\lambda_n)(1-P^*)k_t$, that is l_t' is the unique solution of $(T^* - \bar{\lambda}_n)l_t' = (1-P^*)k_t$. We define accordingly $R^*(\lambda)(1-P^*) := S^*$. Then $\tilde{\phi}_n(t) - (P\tilde{\phi}_n)(t) = \langle l_t', (1-\Pi_n)\tilde{\phi}_n\rangle$. We have just proved that the error $(x - \tilde{x}_n)(t)$ (resp. $(\tilde{\phi}_n - P\tilde{\phi}_n)(t)$) at $t\in[0,1]$ can be expressed in terms of the scalar product $\langle l_t, (1-\Pi_n)\tilde{x}_n\rangle$ (resp. $\langle l_t', (1-\Pi_n)\tilde{\phi}_n\rangle$).

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REMARK. Another way to bound

$$(\tilde{\phi}_n - P\tilde{\phi}_n)(t) = \left[T(-1/2i\pi \int_\Gamma (R(z)/\lambda_n - z)\, dz)(1-\Pi_n)\tilde{\phi}_n\right](t)$$

is the following (Lebbar [10]). Let Γ' be the circle centered at λ_n, with radius r, containing λ and contained in Γ (for n large enough, there exists such a circle). We set $z = \lambda_n + re^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$, for $z \in \Gamma'$.

$$\begin{aligned}\frac{-1}{2i\pi}\int_\Gamma \frac{R(z)}{\lambda_n - z}\, dz &= \frac{-1}{2i\pi}\int_{\Gamma'} \frac{R(z)}{\lambda_n - z}\, dz\\ &= \frac{-1}{2i\pi}\int_0^{2\pi} \frac{R(\lambda_n + re^{i\theta})}{-re^{i\theta}} rie^{i\theta}\, d\theta.\end{aligned}$$

Then

$$\begin{aligned}|(\tilde{\phi}_n - P\tilde{\phi}_n)(t)| &= \frac{1}{2\pi}\left|\int_0^{2\pi}\left[TR(\lambda_n + re^{i\theta})(1-\Pi_n)\tilde{\phi}_n\right](t)\, d\theta\right|\\ &\leqslant \sup_{0 \leqslant \theta \leqslant 2\pi} \left|\left[TR(\lambda_n + re^{i\theta})(1-\Pi_n)\tilde{\phi}_n\right](t)\right|\\ &= \sup_{z\in\Gamma'} \left|\left[TR(z)(1-\Pi_n)\tilde{\phi}_n\right](t)\right|.\end{aligned}$$

For $z \in \Gamma'$, we define $l_t(z) := R^*(z)k_t$. Then

$$|(\tilde{\phi}_n - P\tilde{\phi}_n)(t)| \leqslant \sup_{z\in\Gamma'} |\langle l_t(z), (1-\Pi_n)\tilde{\phi}_n\rangle|.$$

As for the global bounds on [0, 1], they are easy to get:

$$r\tilde{x}_n - x = R(z)T(1-\Pi_n)\tilde{x}_n$$

implies $\|\tilde{x}_n - x\|_\infty \leqslant C\|T(1-\Pi_n)\tilde{x}_n\|_\infty$, and

$$\|T(1-\Pi_n)\tilde{x}_n\|_\infty = \sup_{t\in[0,1]} |\langle k_t, (1-\Pi_n)\tilde{x}_n\rangle|.$$

$\tilde{\phi}_n - P\tilde{\phi}_n = R(\lambda_n)(1-P)T(1-\Pi_n)\tilde{\phi}_n$ implies

$$\operatorname{dist}(\tilde{\phi}_n, M) := \inf_{\phi\in M} \|\tilde{\phi}_n - \phi\|_\infty \leqslant \|\tilde{\phi}_n - P\tilde{\phi}_n\|_\infty \leqslant C\|T(1-\Pi_n)\tilde{\phi}_n\|_\infty,$$

and

$$\|T(1-\Pi_n)\tilde{\phi}_n\|_\infty = \sup_{t\in[0,1]} |\langle k_t, (1-\Pi_n)\tilde{\phi}_n\rangle|.$$

3) Now we set $M_n := P_n X$. For n large enough, $P_{\restriction M_n}$ has a bounded inverse and $m(\lambda - \hat{\lambda}_n) = \sum_{i=1}^m \langle x_i^*, (1-\Pi_n)T(P_{\restriction M_n})^{-1}x_i\rangle$ where $\{x_i\}_1^m$ (resp. $\{x_i^*\}_1^m$) is a basis of M (resp. the adjoint basis of M^*) (see de Boor-Swartz [2]). The error $\lambda - \hat{\lambda}_n$ is then of the same type as the errors $(\tilde{x}_n - x)(t)$ and $(\tilde{\phi}_n - P\tilde{\phi}_n)(t)$.

4) Let Q be the eigenprojection on $E = \ker(T - \lambda)$, along a supplementary subspace F.

$$\tilde{\phi}_n - Q\tilde{\phi}_n = [(T-\lambda)(1-Q)]^{-1}(T-\lambda)(1-Q)\tilde{\phi}_n,$$

and

$$(T-\lambda)(1-Q)\tilde{\phi}_n = (T-\lambda)\tilde{\phi}_n = T(1-\Pi_n)\tilde{\phi}_n + (\lambda_n - \lambda)\tilde{\phi}_n.$$

Therefore

$$\begin{aligned}\operatorname{dist}(\tilde{\phi}_n, E) &= \inf_{\phi \in E} \|\tilde{\phi}_n - \phi\|_\infty \leqslant \|\tilde{\phi}_n - Q\tilde{\phi}_n\|_\infty \\ &\leqslant C\big(\|T(1-\Pi_n)\tilde{\phi}_n\|_\infty + |\lambda_n - \lambda|\big).\end{aligned}$$

Note that this method does not provide a pointwise estimate for $\tilde{\phi}_n - Q\tilde{\phi}_n$. This is due to the fact that, unlike the spectral projection P, the eigenprojection Q has no expression in terms of the resolvent.

3. Two basic results

We shall be concerned with two types of continuous kernels k that we define now.

i) k is *smooth* (of order $l \geqslant 0$) if $k \in C^l_\Delta([0,1] \times [0,1])$, that is $k_{ij} \in C^l_{(\Delta_i \times \Delta_j)}$, for $1 \leqslant i, j \leqslant n$, and k is continuous on $[0,1] \times [0,1]$.

ii) k is a *Green's* kernel (of order $l \geqslant 1$, and continuity δ, $0 \leqslant \delta < l$) if

$$k(t,s) = \begin{cases} k_1(t,s) & \text{for } t \geqslant s, \\ k_2(t,s) & \text{for } t \leqslant s, \end{cases}$$

is such that

$$\begin{aligned} &k_1 \in C^l(\{0 \leqslant s \leqslant t \leqslant 1\}), \\ &k_2 \in C^l(\{0 \leqslant t \leqslant s \leqslant 1\}), \\ &k \in C^\delta([0,1] \times [0,1]). \end{aligned}$$

An obvious example of case ii) is the Green's function of an o.d.e. of order $\delta + 2$.

For any z in $\rho(T)$, we consider the solution $x = R(z)f$ of (1), along with $\tilde{x}_n$ and $\tilde{\phi}_n$, solutions of (5) and (6).

LEMMA 1. *Let T be an integral operator with a kernel k of order l, of type* i) *or* ii). *If $f \in C^l_\Delta$ then, in both cases, x, $\tilde{x}_n$ and $\tilde{\phi}_n$ are in C^l_Δ.*

Now with $k_t(\cdot) := k(t, \cdot)$ for t fixed in $[0,1]$, we consider the equation $(T^* - \bar{z})l_t = k_t$, for $z \in \rho(T)$.

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LEMMA 2. *When k is a smooth kernel of order l, then $l_t \in C^l_\Delta(0, 1)$ for any t in $[0, 1]$. When k is a Green's kernel of order l and continuity δ, then $l_{t_i} \in C^l_\Delta(0, 1)$ for $t_i \in \Delta$, $i = 0, \ldots, n$, and $l_t \in C^\delta(0, 1)$ for $t \notin \Delta$.*

It is left to the reader to check the two lemmas (see Lebbar [10]). Note that when k is a Green's kernel, l_t is defined by the functions $l_{1t} \in C^l(0, t)$, $l_{2t} \in C^l(t, 1)$.

Lemma 2 shows that l_t has the same smoothness properties as k_t. The same is true for l_t'.

We define $\alpha := \min(l, r + 1)$ and $\alpha^* := \min(l, r + 1, \delta + 2)$.

3.1. Π_n is an orthogonal projection

THEOREM 3. *Let Δ be quasi-uniform. With the above definitions, then for $f \in C^l_\Delta$, and z in $\rho(T)$:*

i) if k is a smooth kernel of order l, then for $t \in [0, 1]$: $|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{2\alpha}\|l_t^{(\alpha)}\|_\Delta$, and globally $\|T(1 - \Pi_n)f\|_\infty \leqslant Ch^{2\alpha}$.

ii) if k is a Green's kernel of order l and continuity δ, $0 \leqslant \delta < l$, then for $t_i \in \Delta$,

$$|\langle l_{t_i}, (1 - \Pi_n)f\rangle| \leqslant Ch^{2\alpha}\|l_{t_i}^{(\alpha)}\|_\Delta, \qquad i = 0, \ldots, n,$$

for $t \notin \Delta$,

$$|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{\alpha+\alpha^*} \max\left(\|l_{1t}^{(\delta+1)}\|_\infty, \|l_{2t}^{(\delta+1)}\|_\infty\right),$$

and globally, $\|T(1 - \Pi_n)f\|_\infty \leqslant Ch^{\alpha+\alpha^*}$.

PROOF. It is adapted from Chandler [6]. Since Π_n is an orthogonal projection: $\langle l_t, (1 - \Pi_n)f\rangle = \langle (1 - \Pi_n)l_t, (1 - \Pi_n)f\rangle$. And

$$\int_0^1 (1 - \Pi_n)l_t(s)(1 - \Pi_n)f(s)\, ds = \sum_{i=1}^n \int_{\Delta_i} (1 - \Pi)l_{it}(s)(1 - \Pi)f_i(s)\, ds.$$

Given $f_i \in C_{(\Delta_i)}$, Πf_i is the orthogonal projection of f_i on the set of polynomials of degree $\leqslant r$ on Δ_i. When $f, l_t \in C^l_\Delta(0, 1)$, $f_i, l_{it} \in C^l_{(\Delta_i)}$, and

$$\|(1 - \Pi)f_i\|_1 \leqslant Ch^{\alpha+1}\|f_i^{(\alpha)}\|_\infty, \|(1 - \Pi)l_{it}\|_\infty \leqslant Ch^\alpha\|l_{it}^{(\alpha)}\|_\infty.$$

When $l_t \in C^\delta(0, 1)$, with $t_{i-1} < t < t_i$, then on Δ_i, if $\delta \leqslant r$,

$$\|(1 - \Pi)l_{it}\|_\infty \leqslant Ch^{\delta+1} \max\left(\|l_{it}^{(\delta+1)}\|_\infty, \|l_{2t}^{(\delta+1)}\|_\infty\right).$$

The result follows by summing over i, and using $nh \leqslant \sigma$.

3.2. Π_n is an interpolatory projection

Let f be a function of $C^{l+1}(a, b)$, such that $f(w_j) = 0, j = 1, \ldots, r + 1$, where the $\{w_i\}_1^{r+1}$ are $r + 1$ distinct points in (a, b). The $(r + 1)$th divided difference

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of f on the points $w_1, \ldots, w_{r+1}$ is denoted by $\delta[w_1, w_2, \ldots, w_{r+1}, \cdot]f$. Then

$$f(s) = (s - w_1) \cdots (s - w_{r+1})\delta[w_1, w_2, \ldots, w_{r+1}, s]f, \quad \text{for } s \notin \{w_j\}_1^{r+1}.$$

We set, for $s \in [a, b]$,

$$g^*(s) = \begin{cases} f(s)/v(s) & \text{if } s \notin \{w_j\}_1^{r+1}, \\ \lim\limits_{t \to w_j} \left(\dfrac{f(s)}{v(s)}\right) & \text{if } t = w_j, j = 1, \ldots, r+1, \end{cases}$$

where $v(s) = (s - w_1) \cdots (s - w_{r+1})$.

LEMMA 4. *If $f \in C^{l+1}(a, b)$, then $g^* \in C^l(a, b)$.*

There is only a need to prove that g^* is C^l in the neighborhood of any w_j, $j = 1, \ldots, r + 1$ (see Lebbar [10]). If $f \in C^{l+1}(a, b)$, the divided difference $\delta[w_1, \ldots, w_{r+1}, \cdot]f$ may therefore be prolongated by continuity on $[a, b]$, up to the order l.

We shall apply this lemma on each Δ_i, with the $\{w_j\}_1^{r+1}$ being the Gauss points $\{\tau_j^i\}_{j=1}^{r+1}$. For $f \in C_\Delta^{l+1}$, Πf_i is the polynomial of degree $\leqslant r$ on Δ_i which interpolates f_i at the Gauss points $\{\tau_j^i\}_{j=1}^{r+1}$. Hence $(1 - \Pi)f_i(\tau_j^i) = 0$ for $j = 1, \ldots, r + 1$. We consider the divided difference $\delta[\tau_1^i, \ldots, \tau_{r+1}^i, \cdot](1 - \Pi)f_i$, and set $q_{it} := l_{it}\delta[\tau_1^i, \ldots, \tau_{r+1}^i, \cdot](1 - \Pi)f_i$. $k_{it} \in C_{(\Delta_i)}^{l+1}$ (resp. $C_{(\Delta_i)}^{\delta}$) implies that $l_{it} \in C_{(\Delta_i)}^{l+1}$ (resp. $C_{(\Delta_i)}^{\delta}$) and $q_{it} \in C_{(\Delta_i)}^{l}$ (resp. $C_{(\Delta_i)}^{\delta}$, for $\delta \leqslant l$).

THEOREM 5. *With the above definitions, then for $f \in C_\Delta^{l+1}$ and z in $\rho(T)$*

i) *if k is a smooth kernel of order l, then for $t \in [0, 1]$, $|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha}\|q_t^{(\alpha)}\|_\Delta$ and globally $\|T(1 - \Pi_n)f\|_\infty \leqslant M_\Delta h^{r+1+\alpha}$,*

ii) *if k is a Green's kernel, of order l and continuity δ, $0 \leqslant \delta < l$, then for $t_i \in \Delta$,*

$$|\langle l_{t_i}, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha}\|q_{t_i}^{(\alpha)}\|_\Delta, \qquad i = 0, \ldots, n,$$

for $t \notin \Delta$,

$$|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha^*}\max(\|q_{1t}^{(\delta+1)}\|_\infty, \|q_{2t}^{(\delta+1)}\|_\infty),$$

and globally $\|T(1 - \Pi_n)f\|_\infty \leqslant M_\Delta h^{r+1+\alpha^*}$.
If $f \in C_\Delta^{l+r+1}$, then $M_\Delta \leqslant C$.

PROOF. It is adapted from de Boor-Swartz [1].

$$\int_0^1 l_t(s)(1 - \Pi_n)f(s)\,ds = \sum_{i=1}^n \int_{\Delta_i} l_{it}(s)(1 - \Pi)f_i(s)\,ds$$

$$= \sum_{i=1}^n \int_{\Delta_i} \underbrace{l_{it}(s)\delta[\tau_1^i, \ldots, \tau_{r+1}^i, s](1 - \Pi)f_i(s)}_{q_{it}(s)}\ \underbrace{(s - \tau_1^i) \cdots (s - \tau_{r+1}^i)}_{v(s)}\,ds.$$

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When $q_{it} \in C^{l}_{(\Delta_i)}$, $q_{it}(s) = q_{it}(t_{i-1}) + \cdots + ((s - t_{i-1})^l / l!) q_{it}^{(l)}(\theta_s)$, $t_{i-1} < \theta_s < s$. Making use of $\int_{\Delta_i} v(s)p(s)\, ds = 0$ for all polynomial p of degree $< r$ on Δ_i, we get $|\int_{\Delta_i} q_{it}(s)v(s)\, ds| \leqslant Ch^{r+2+\alpha} \|q_{it}^{(\alpha)}\|_\infty$, which gives, for $l_t \in C^l_\Delta$,

$$|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha} \|q_t^{(\alpha)}\|_\Delta.$$

When $l_{it} \in C^{\delta}_{(\Delta_i)}$,

$$\left| \int_{\Delta_i} l_{it}(s)(1 - \Pi) f_i(s)\, ds \right| \leqslant Ch^{r+2+\min(r+1,\delta+2)}$$

and $|\langle l_t, (1 - \Pi_n)f\rangle| \leqslant Ch^{r+1+\alpha^*}$, by summing over i.

Theorems 3 and 5 play a central role to derive the convergence rates, as we shall see in the next section.

4. Convergence rates

We recall that $\alpha = \min(l, r + 1)$ and $\alpha^* = \min(l, r + 1, \delta + 2)$. In practice $\delta + 2 \leqslant r + 1 \leqslant l$, so that $\alpha = r + 1$ and $\alpha^* = \delta + 2$. We assume throughout this section that the kernel k is of order l for the Galerkin method ($f \in C^l_\Delta \Rightarrow \tilde{x}_n \in C^l_\Delta$, $\tilde{\phi}_n \in C^l_\Delta$) and of order $l + r + 1$ for the collocation method ($f \in C^{l+r+1}_\Delta \Rightarrow \tilde{x}_n \in C^{l+r+1}_\Delta$ and $\tilde{\phi}_n \in C^{l+r+1}_\Delta$).

4.1. Convergence rate for the eigenvalues

The definitions are those of Section 2.2.

THEOREM 6. *For both types of kernel k*

$$\lambda - \hat{\lambda}_n = O(\varepsilon_n), \quad \max_i |\lambda - \lambda_{in}| = O\left(\varepsilon_n^{1/\mu}\right), \quad \min_i |\lambda - \lambda_{in}| = O\left(\varepsilon_n^{g/m}\right)$$

where (a) $\varepsilon_n = h^{2\alpha}$ *for the Galerkin method, and* (b) $\varepsilon_n = h^{r+1+\alpha}$ *for the collocation method.*

PROOF. It is adapted from de Boor-Swartz [2] where it is noticed that λ (resp. λ_{in}) are the eigenvalues of two $m \times m$ matrices such that the (i, j)th coefficient of the difference is $\langle x_i^*, (1 - \Pi_n)T(P_{\restriction M_n})^{-1}x_j\rangle$. Theorem 3 applies where l_t is replaced by $x_i^* \in C^l_\Delta$ and $T(P_{\restriction M_n})^{-1}x_j \in C^l_\Delta$, if the kernel is of order l. Similarly, Theorem 5 applies if k is of order $l + r + 1$. And the results follow from classical theorems in *matrix* theory (see Wilkinson [18], pp. 80–81).

4.2. Convergence rate for the solutions and the eigenvectors

(a) **The Galerkin method.** We suppose that $f \in C_\Delta^l$ and k is of order l. Then $\tilde{x}_n$, $\tilde{\phi}_n \in C_\Delta^l$. Δ is quasi-uniform.

THEOREM 7. *With a smooth kernel,* $\|x - \tilde{x}_n\|_\infty$ *and* $\operatorname{dist}(\tilde{\phi}_n, M)$ *are of the order* $h^{2\alpha}$, $\operatorname{dist}(\tilde{\phi}_n, E) = O(h^{2\alpha/\mu})$. *With a Green's kernel, then: at* $t_i \in \Delta$, $|x(t_i) - \tilde{x}_n(t_i)|$ *and* $|\tilde{\phi}_n(t_i) - (P\tilde{\phi}_n)(t_i)|$ *are of the order* $h^{2\alpha}$, $i = 0, \ldots, n$, *whereas globally* $\|x - \tilde{x}_n\|_\infty$ *and* $\operatorname{dist}(\tilde{\phi}_n, M)$ *are of the order* $h^{\alpha+\alpha^*}$, $\operatorname{dist}(\tilde{\phi}_n, E) = O(h^{2\alpha/\mu})$, *for* $\mu > 1$.

PROOF. We apply Theorem 3 to $(\tilde{x}_n - x)(t) = \langle l_t, (1 - \Pi_n)\tilde{x}_n\rangle$, $(\tilde{\phi}_n - P\tilde{\phi}_n)(t) = \langle l_t', (1 - \Pi_n)\tilde{\phi}_n\rangle$, and Theorem 6 to

$$\operatorname{dist}(\tilde{\phi}_n, E) \leqslant C\left(\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty + |\lambda_n - \lambda|\right).$$

(b) **The collocation method.** We suppose that $f \in C_\Delta^{l+r+1}$ and k is of order $l + r + 1$. Then $\tilde{x}_n$, $\tilde{\phi}_n \in C_\Delta^{l+r+1}$. We get, as Theorem 8, the analog of Theorem 7, where $h^{2\alpha}$ (resp. $h^{\alpha+\alpha^*}$) is replaced by $h^{r+1+\alpha}$ (resp. $h^{r+1+\alpha^*}$). The convergence rates in Theorems 7 and 8 are the best we could hope from the known results. It should be noticed that the computation of $\tilde{x}_n$ (resp. $\tilde{\phi}_n$) from x_n (resp. λ_n, ϕ_n) does not require much extra work: let $\dim X_n = n$ (say), let $\{e_i^n\}_1^n$ be a basis of X_n: if $x_n = \sum_{i=1}^n \xi_i^n e_i^n$, then $Tx_n = \sum_{i=1}^n \xi_i^n Te_i^n$ where the $\{Te_i^n\}_1^n$ have already been computed to get the coefficients of the matrix associated with the projection method.

5. Numerical Example

We end this paper with a numerical example illustrating the behavior of the iterated collocation solution for the Fredholm equation

$$\int_0^1 k(t, s)x(s)\, ds - \frac{1}{4}x(t) = -\cosh(1), \qquad 0 \leqslant t \leqslant 1,$$

with

$$k(t, s) = \begin{cases} -t(1 - s) & \text{if } s \geqslant t, \\ -s(1 - t) & \text{if } s \leqslant t. \end{cases}$$

The exact solution is $x(t) = \cosh(2t - 1)$.

We choose the partition $\Delta = \{i/5\}_0^5$, $h = \frac{1}{5}$, and on each interval Δ_i, the $r + 1 = 4$ Gauss points. We display in Table 1 the values of $x - x_n$ and $x - \tilde{x}_n$

at the partition points t_i, $i = 1, 2, 3, 4$. The kernel k is of "Green's function" type with $\delta = 0$.

TABLE 1

Error values at the partition points

i	$(x - x_n)(t_i^-)$	$(x - x_n)(t_i^+)$	$(x - \tilde{x}_n)(t_i)$
1	8.10^{-1}	7.10^{-5}	-5.10^{-12}
2	6.10^{-5}	6.10^{-5}	-7.10^{-12}
3	6.10^{-5}	6.10^{-5}	-7.10^{-12}
4	7.10^{-5}	8.10^{-5}	-5.10^{-12}

References

[1] C. de Boor and B. Swartz, "Collocation at Gaussian points", *SIAM J. Num. Anal.* 10 (1973), 582–606.

[2] C. de Boor and B. Swartz, "Collocation approximation to eigenvalues of an ordinary differential equation: the principle of the thing", *Math. Comp.* 35 (1980), 679–694.

[3] C. de Boor and B. Swartz, "Collocation approximation to eigenvalues of an ordinary differential equation: numerical illustrations", *Math. Comp.* 36 (1981).

[4] C. de Boor and B. Swartz, "Local piecewise polynomial projection methods for an ode which give high-order convergence at knots", *Math. Comp.* 36 (1981).

[5] G. A. Chandler, "Global superconvergence of iterated Galerkin solutions for second kind integral equations", Australian National University Technical Report (1978).

[6] G. A. Chandler, *Superconvergence of numerical solutions of second kind integral equations* (Ph. D. Thesis, Australian National University, 1979).

[7] F. Chatelin, "Numerical computation of the eigenelements of linear integral operators by iterations", *SIAM J. Num. Anal.* 15 (1978), 1112–1124.

[8] F. Chatelin, "Sur les bornes d'erreur a posteriori pour les éléments propres d'opérateurs linéaires", *Numer. Math.* 32 (1979), 233–246.

[9] F. Chatelin, *Linear spectral approximation in Banach spaces* (Academic Press, New-York, to appear).

[10] R. Lebbar, "Méthode de Galerkin iterée pour l'équation intégrale de Fredholm de 2e espèce" (Rapport D. E. A., Université de Grenoble, 1980).

[11] Lin Qun and Liu Jiaquan, "Extrapolation methods for Fredholm integral equations with non-smooth kernels" Numer. Math. 35 (1980), 459–464.

[12] G. R. Richter, "Superconvergence of piecewise polynomial Galerkin approximations for Fredholm integral equations of the 2nd kind", *Numer. Math.* 31 (1978), 63–70.

[13] I. H. Sloan, "Error analysis for a class of degenerate kernel methods", *Numer. Math.* 25 (1976), 231–238.

[14] I. H. Sloan, "Iterated Galerkin method for eigenvalue problems", *SIAM J. Num. Anal.* 13 (1976), 753–760.

[15] I. H. Sloan, "A review of numerical methods for Fredholm equations of the 2nd kind", in the *Application and numerical solution of integral equations* (eds. Anderssen, de Hoog, Lukas, Sijthoff & Noordhoff, Leyden, 1980).

[16] I. H. Sloan and B. J. Burn, "Collocation with polynomials for integral equations of the 2nd kind: a new approach to the theory", *Journ. Integral Equ.* 1 (1979), 77–94.
[17] I. H. Sloan, E. Noussair and B. J. Burn, "Projection method for equations of the 2nd kind" *J. Math. Anal. Appl.* 69 (1979), 84–103.
[18] J. H. Wilkinson, *The algebraic eigenvalue problem* (Clarendon Press, Oxford, 1965).

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