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BOUNDS FOR HARDY DIFFERENCES

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Abstract

Bounds for Hardy differences, that is, improvements and reverses of the well-known Hardy inequality, are obtained.

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1. Introduction

Hardy [3] announced and then proved [4] a highly important classical integral inequality,

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty [f(x)]^p\,dx, \tag{1.1}$$

the so-called Hardy inequality, where $p > 1$ and $f \in L^p(0, \infty)$ is a nonnegative function. There are many extensions of (1.1) [1, 5, 6]. Boas [2] (see also Pečarić et al. [7, p. 225]) proved the following result.

THEOREM 1.1. *Let ϕ be a continuous convex function. Let f be a measurable and nonnegative function, λ an increasing and bounded function, and let $L = \lambda(\infty) - \lambda(0)$. Then*

$$\int_0^\infty x^{-1}\phi\left(L^{-1}\int_0^\infty f(ux)\,d\lambda(u)\right)dx \le \int_0^\infty x^{-1}\phi(f(x))\,dx. \tag{1.2}$$

If ϕ is a continuous concave function, then the inequality (1.2) holds in reverse order.

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As a special case, Boas obtained the following extension of (1.1) (as well as some inequalities similar to (1.1)) [7, p. 230]:

$$\int_0^\infty x^{-k}\Big(\int_0^x g(t)\,dt\Big)^p\,dx \le \Big(\frac{p}{k-1}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx,$$

where $k > 1$, $p > 1$ or $k < 1$, $p < 0$;

$$\int_0^\infty x^{-k}\Big(\int_x^\infty g(t)\,dt\Big)^p\,dx \le \Big(\frac{p}{1-k}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx,$$

where $k < 1$, $p > 1$ or $k > 1$, $p < 0$;

$$\int_0^\infty x^{-k}\Big(\int_0^x g(t)\,dt\Big)^p\,dx \ge \Big(\frac{p}{1-k}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx,$$

where $0 < p < 1$ and $k < 1$;

$$\int_0^\infty x^{-k}\Big(\int_x^\infty g(t)\,dt\Big)^p\,dx \ge \Big(\frac{p}{k-1}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx,$$

where $0 < p < 1$ and $k > 1$. Here $g \in L^p(0, \infty)$ is a nonnegative function.

In this paper we derive some improvements and reverses of these results.

2. Log-convexity of Boas differences

LEMMA 2.1. *Define the function*

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)} & \text{for } s \ne 0, 1 \\ -\log x & \text{for } s = 0 \\ x\log x & \text{for } s = 1. \end{cases}$$

Then $\varphi_s(x)$ is convex for $x > 0$.

The following lemma is equivalent to the definition of convex function [7, p. 2].

LEMMA 2.2. *If ϕ is continuous and convex for all s_1, s_2, s_3 in an open interval I such that $s_1 < s_2 < s_3$, then*

$$\phi(s_1)(s_3 - s_2) + \phi(s_2)(s_1 - s_3) + \phi(s_3)(s_2 - s_1) \ge 0.$$

We quote another useful lemma from log-convexity theory.

LEMMA 2.3 (Simic [8]). *A positive function f is log-convex in the Jensen sense on an open interval I, that is, for each $s, t \in I$,*

$$f(s)f(t) \ge f^2\Big(\frac{s+t}{2}\Big),$$

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if and only if the relation

$$u^2 f(s) + 2uwf\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0$$

holds for each real u, w *and* $s, t \in I$.

We now prove log-convexity of the Boas difference, that is, the difference between the expressions on either side of (1.2).

Theorem 2.4. *Let the conditions of Theorem 1.1 be satisfied and let* $F : \mathbb{R} \to \mathbb{R}_+$ *be a function defined by*

$$F(s) = \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\left(L^{-1}\int_0^\infty f(ux)\,d\lambda(u)\right)dx. \tag{2.1}$$

Then $F(s)$ *is log-convex, that is,*

$$[F(p)]^{r-s} \leq [F(r)]^{p-s}[F(s)]^{r-p} \tag{2.2}$$

for $-\infty < s < p < r < \infty$.

Proof. Let us consider the function ϕ defined by

$$\phi(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_p(x) \quad \text{where } r = \frac{s+p}{2} \text{ and } u, w \in \mathbb{R}.$$

Then

$$\phi''(x) = u^2x^{s-2} + 2uwx^{r-2} + w^2x^{p-2} = (ux^{s/2-1} + wx^{p/2-1})^2 \geq 0 \quad \text{for } x > 0.$$

The function ϕ is convex for $x > 0$; therefore (1.2) is equivalent to

$$u^2F(s) + 2uwF(r) + w^2F(p) \geq 0,$$

that is, by Lemma 2.3,

$$[F(r)]^2 \leq F(s)F(p).$$

So F is log-convex in the Jensen sense. Since

$$\lim_{s\to 0} F(s) = F(0) \quad \text{and} \quad \lim_{s\to 1} F(s) = F(1),$$

F is continuous for $s \in \mathbb{R}$ and therefore $\log F$ is convex.

Lemma 2.2 for $-\infty < s < p < r < \infty$ yields

$$(r - s)\log F(p) \leq (r - p)\log F(s) + (p - s)\log F(r),$$

which is equivalent to (2.2). □

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3. Improvement and reverse of the Hardy inequality

We obtain an improvement and reverse of the Hardy inequality and of its dual inequality.

Theorem 3.1. *Let φ_s and F be given by Lemma 2.1 and Theorem 2.4, respectively. Let $g \in L^1(0, \infty)$ be a nonnegative function. Then, for $p \in \mathbb{R}\backslash\{0, 1\}$,*

$$\frac{1}{p(p-1)}\bigg\{\Big(\frac{p}{k-1}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx - \int_0^\infty x^{-k}\Big(\int_0^x g(t)\,dt\Big)^p dx\bigg\} \leq \Big(\frac{p}{k-1}\Big)^p [H(s)]^{(r-p)/(r-s)}\,[H(r)]^{(p-s)/(r-s)} \tag{3.1}$$

for $-\infty < s < p < r < \infty$, and

$$\frac{1}{p(p-1)}\bigg\{\Big(\frac{p}{k-1}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx - \int_0^\infty x^{-k}\Big(\int_0^x g(t)\,dt\Big)^p dx\bigg\} \geq \Big(\frac{p}{k-1}\Big)^p [H(s)]^{(r-p)/(r-s)}\,[H(r)]^{(p-s)/(r-s)} \tag{3.2}$$

for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$, where

$$H(r) = \int_0^\infty x^{-1}\varphi_r(x^{(p-k+1)/p}g(x))\,dx - \int_0^\infty x^{-1}\varphi_r\Big(\frac{k-1}{p}x^{(1-k)/p}\int_0^x g(t)\,dt\Big)\,dx.$$

Proof. For $\alpha > 0$, let $\lambda(u)$ be defined by

$$\lambda(u) = \begin{cases} \alpha^{-1}u^\alpha & \text{for } 0 \leq u \leq 1 \\ \alpha^{-1} & \text{for } u > 1. \end{cases}$$

In this case (2.1) becomes

$$\begin{aligned} F_\alpha(s) &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\alpha\int_0^1 f(ux)\,d\Big(\frac{u^\alpha}{\alpha}\Big)\Big)\,dx \\ &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\alpha\int_0^1 f(ux)u^{\alpha-1}\,du\Big)\,dx. \end{aligned}$$

Put $t = ux$, so that $dt = x\,du$. For $0 \leq u \leq 1$ we have $0 \leq t \leq x$, that is,

$$\begin{aligned} F_\alpha(s) &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\alpha\int_0^x f(t)\Big(\frac{t}{x}\Big)^{\alpha-1}\frac{dt}{x}\Big)\,dx \\ &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\alpha x^{-\alpha}\int_0^x t^{\alpha-1}f(t)\,dt\Big)\,dx. \end{aligned}$$

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For this function, inequality (2.2) becomes

$$[F_\alpha(p)]^{r-s} \le [F_\alpha(r)]^{p-s}[F_\alpha(s)]^{r-p}, \tag{3.3}$$

that is, $F_\alpha(p)$ is log-convex.

Now put $\alpha = (k-1)/p$ (for $p \neq 0$) and $f(t) = t^{1-\alpha} g(t)$ in (3.3) to obtain

$$\int_0^\infty x^{-1}\varphi_p(x^{(p-k+1)/p} g(x))\,dx - \int_0^\infty x^{-1}\varphi_p\Big(\frac{k-1}{p} x^{(1-k)/p}\int_0^x g(t)\,dt\Big)\,dx \\ \le [H(s)]^{(r-p)/(r-s)}[H(r)]^{(p-s)/(r-s)} \tag{3.4}$$

for $-\infty < s < p < r < \infty$. From (3.4) for $p \in \mathbb{R}\backslash\{0, 1\}$ we obtain (3.1).

If in (3.3) $s \to r$, $p \to s$, $r \to p$ and $s \to p$, $p \to r$, $r \to s$, then

$$[F_\alpha(p)]^{s-r} \ge [F_\alpha(r)]^{s-p}[F_\alpha(s)]^{p-r}.$$

Put $\alpha = (k-1)/p$ and $f(t) = t^{\alpha-1} g(t)$ to obtain

$$\int_0^\infty x^{-1}\varphi_p(x^{(p-k+1)/p} g(x))\,dx - \int_0^\infty x^{-1}\varphi_p\Big(\frac{k-1}{p} x^{(1-k)/p}\int_0^x g(t)\,dt\Big)\,dx \\ \ge [H(s)]^{(r-p)/(r-s)}[H(r)]^{(p-s)/(r-s)}, \tag{3.5}$$

from which for $p \in \mathbb{R}\backslash\{0, 1\}$ we obtain (3.2). □

REMARK 3.2. We have in fact proved a more general result, namely that (3.4) is valid for $-\infty < s < p < r < \infty$, and (3.5) for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$.

The following result is the dual to Theorem 3.1.

THEOREM 3.3. *Let φ_s and F be given by Lemma 2.1 and Theorem 2.4, respectively. Let $g \in L^1(0, \infty)$ be a nonnegative function. Then, for $p \in \mathbb{R}\backslash\{0, 1\}$,*

$$\frac{1}{p(p-1)}\Big\{\Big(\frac{p}{1-k}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx - \int_0^\infty x^{-k}\Big(\int_x^\infty g(t)\,dt\Big)^p\,dx\Big\} \\ \le \Big(\frac{p}{1-k}\Big)^p [\tilde{H}(s)]^{(r-p)/(r-s)}[\tilde{H}(r)]^{(p-s)/(r-s)} \tag{3.6}$$

for $-\infty < s < p < r < \infty$, and

$$\frac{1}{p(p-1)}\Big\{\Big(\frac{p}{1-k}\Big)^p \int_0^\infty x^{p-k}[g(x)]^p\,dx - \int_0^\infty x^{-k}\Big(\int_x^\infty g(t)\,dt\Big)^p\,dx\Big\} \\ \ge \Big(\frac{p}{1-k}\Big)^p [\tilde{H}(s)]^{(r-p)/(r-s)}[\tilde{H}(r)]^{(p-s)/(r-s)} \tag{3.7}$$

for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$, where

$$\tilde{H}(r) = \int_0^\infty x^{-1}\varphi_r(x^{(p-k+1)/p} g(x))\,dx \\ - \int_0^\infty x^{-1}\varphi_r\Big(\frac{1-k}{p} x^{(1-k)/p}\int_x^\infty g(t)\,dt\Big)\,dx.$$

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PROOF. For $\beta > 0$, let $\lambda(u)$ be defined by

$$\lambda(u) = \begin{cases} 0 & \text{for } 0 \le u \le 1 \\ \beta^{-1}(1 - u^{-\beta}) & \text{for } u > 1. \end{cases}$$

In this case (2.1) becomes

$$\begin{aligned} F_\beta(s) &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\int_1^\infty f(ux)\,d(1-u^{-\beta})\Big)\,dx \\ &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\beta\int_1^\infty f(ux)u^{-\beta-1}\,du\Big)\,dx. \end{aligned}$$

Put $t = ux$, so that $dt = x\,du$. For $u > 1$ we have $t > x$, that is,

$$\begin{aligned} F_\beta(s) &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\beta\int_x^\infty f(t)\Big(\frac{t}{x}\Big)^{1+\beta}\frac{dt}{x}\Big)\,dx \\ &= \int_0^\infty x^{-1}\varphi_s(f(x))\,dx - \int_0^\infty x^{-1}\varphi_s\Big(\beta x^\beta\int_x^\infty t^{-\beta-1}f(t)\,dt\Big)\,dx. \end{aligned}$$

For this function, inequality (2.2) becomes

$$[F_\beta(p)]^{r-s} \le [F_\beta(r)]^{p-s}[F_\beta(s)]^{r-p}, \tag{3.8}$$

that is, $F_\beta(p)$ is log-convex.

Now put $\beta = (1-k)/p$ (for $p \ne 0$) and $f(t) = t^{1+\beta}g(t)$ in (3.8) to obtain

$$\begin{aligned} &\int_0^\infty x^{-1}\varphi_p(x^{(p-k+1)/p}g(x))\,dx - \int_0^\infty x^{-1}\varphi_p\Big(\frac{1-k}{p}x^{(1-k)/p}\int_x^\infty g(t)\,dt\Big)\,dx \\ &\quad \le [\tilde{H}(s)]^{(r-p)/(r-s)}[\tilde{H}(r)]^{(p-s)/(r-s)} \end{aligned} \tag{3.9}$$

for $-\infty < s < p < r < \infty$. From (3.9) for $p \in \mathbb{R}\backslash\{0, 1\}$ we obtain (3.6).

If in (3.8) $s \to r$, $p \to s$, $r \to p$ and $s \to p$, $p \to r$, $r \to s$, then

$$[F_\beta(p)]^{s-r} \ge [F_\beta(r)]^{s-p}[F_\beta(s)]^{p-r}.$$

Put $\beta = (1-k)/p$ and $f(t) = t^{1+\beta}g(t)$ to obtain

$$\begin{aligned} &\int_0^\infty x^{-1}\varphi_p(x^{(p-k+1)/p}g(x))\,dx - \int_0^\infty x^{-1}\varphi_p\Big(\frac{1-k}{p}x^{(1-k)/p}\int_x^\infty g(t)\,dt\Big)\,dx \\ &\quad \ge [\tilde{H}(s)]^{(r-p)/(r-s)}[\tilde{H}(r)]^{(p-s)/(r-s)}, \end{aligned} \tag{3.10}$$

from which for $p \in \mathbb{R}\backslash\{0, 1\}$ we obtain (3.7). □

REMARK 3.4. We have in fact proved a more general result, namely that (3.9) is valid for $-\infty < s < p < r < \infty$, and (3.10) for $-\infty < r < s < p < \infty$ and $-\infty < p < r < s < \infty$.

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