A DETERMINANT EXPRESSION OF TCHEBYCHEV POLYNOMIALS

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1. A certain $n \times n$ determinant, namely that in (1), provides a rather simple expression for the Tchebychev polynomial $T_n(x) = \cos n(\cos^{-1}x)$. This determinant also leads to an interesting combinatorial interpretation of the coefficients in that polynomial.

2. We wish to prove the following identity:

$$(1) \qquad \begin{vmatrix} 1 & -2x & 1 & \dots & . & . & 0 & 0 \\ 0 & 1 & -2x & \dots & . & . & 0 & 0 \\ 0 & 0 & 1 & \dots & . & . & 0 & 0 \\ . & . & . & & & & & . \\ . & . & . & & & & 1 & . \\ 0 & 0 & 0 & \dots & . & . & -2x & 1 \\ 1 & 0 & 0 & \dots & . & . & 1 & -2x \\ -2x & 1 & 0 & \dots & . & . & 0 & 1 \end{vmatrix} = 2(1 - T_n(x)) .$$

Let $A_n(x)$ be the $n \times n$ matrix with the entries of the above determinant. Consider the $n \times n$ matrix

$$\Omega_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

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which is a particular example of a circulant matrix [1]. It has the following properties.

(a) For each integer r, Ω_n^r is obtained from Ω_n by displacing the generalized diagonal of 1's by r - 1 positions to the right; that is, each entry 1 is displaced cyclically by r - 1 positions to the right. Thus Ω_n^n is equal to the $n \times n$ identity I_n.

(b) The determinant $\left|\sum_{\lambda=1}^{n} a_\lambda \Omega_n^\lambda\right|$ is the product of the n sums (c.f. [1])

$$\sum_{\lambda=1}^{n} a_\lambda e^{2\pi i k\lambda/n}, \quad k = 1, 2, \ldots, n.$$

From (a), we form the expansion of $A_n(x)$ in terms of powers of Ω_n,

$$A_n(x) = I_n - 2x\,\Omega_n + \Omega_n^2 .$$

Hence

$$|A_n(x)| = (-1)^{n-1}\left|\Omega_n^{-1} A_n(x)\right| = -(-2)^n\left|\tfrac{1}{2}(\Omega_n + \Omega_n^{n-1}) - xI_n\right| .$$

It follows from (b) that the zeros of $|A_n(x)|$ are the sums

$$\tfrac{1}{2}\{e^{2\pi ik/n} + e^{2\pi ik(n-1)/n}\} = \cos 2\pi k/n, \quad k = 1, 2, \ldots, n,$$

which in turn are the n zeros of $1 - T_n(x)$. Finally, by expanding $|A_n(0)|$, we find that $|A_n(x)|$ and $2(1 - T_n(x))$ are polynomials of degree n with constant term

$$2(1 - \cos \tfrac{1}{2}n\pi) = \begin{cases} 2, & n \text{ odd} \\ 2 - 2(-1)^{\frac{1}{2}n}, & n \text{ even} \end{cases}$$

and so they are identical.

3. In expanding the determinant $|A_n(x)|$, it can be checked that its non-zero terms are obtained in exactly the following ways.

(i) We form the product of all the entries in one of the three non-zero generalized diagonals.

(ii) We partition off r distinct 2×2 submatrices of

A_n, each containing non-zero entries from two cyclically adjacent rows and columns of A_n. We then form the product of the entries 1 in these submatrices with the remaining entries (-2x) of the second generalized diagonal. Thus

$$(2) \qquad A_n(x) = 2 - \sum_{r=0}^{[n/2]} (-1)^r S_{n,r} (2x)^{n-2r}$$

where $S_{n,r}$ equals the number of possible selections of r integers from the first n such that the difference of any two is not congruent to 1 modulo n, and $0 \leqslant r \leqslant [n/2]$. Clearly $S_{n,o} = 1$.

4. From (2) and (1), we now know the form of $T_n(x)$.

$$T_n(x) = \sum_{r=o}^{[n/2]} T_{n,r}\, x^{n-2r} \quad ,$$

where $T_{n,r} = (-1)^r 2 \qquad S_{n,r}$; r =

We next determine $S_{n,r}$ for $1 \leqslant r \leqslant [n/2]$ as a function of n and r. In the selection of the r integers, we consider two cases .

(i) The integer 1 is not chosen. Let $\{a_\lambda \mid \lambda = 1, 2, \ldots, r\}$ be a selection. Thus we may assume

$$(3) \qquad 1 < a_\lambda < a_{\lambda+1} - 1 < n; \quad \lambda = 1, 2, \ldots, r - 1 .$$

Put $b_\lambda = a_\lambda - \lambda$; $\lambda = 1, 2, \ldots, r$. Then from (3) we obtain

$$(4) \qquad 1 \leq b_\lambda < b_{\lambda+1} \leq n - r; \quad \lambda = 1, 2, \ldots, r - 1.$$

(ii) The integer 1 is chosen. If $r \geqslant 2$, let $\{c_\lambda \mid \lambda = 2, 3, \ldots, r\}$ be a selection of the remaining r - 1 integers. Thus similarly

$$(5) \qquad 2 < c_\lambda < c_{\lambda+1} - 1 < n - 1; \quad \lambda = 2, 3, \ldots, r - 1.$$

Put $d_\lambda = c_{\lambda+1} - \lambda - 1$; $\lambda = 1, 2, \ldots, r - 1$. Then from (5) we have

$$(6) \qquad 1 \leqslant d_\lambda < d_{\lambda+1} \leqslant n - r - 1; \quad \lambda = 1, 2, \ldots, r - 2.$$

It is readily seen from (4) (from (6)) that the sets $\{b_\lambda\}$ (the sets $\{d_\lambda\}$) are all the possible selections of r integers from the first n - r (of r - 1 integers from the first n - r - 1) with no restrictions.

Hence for $1 \leq r \leq [n/2]$,

$$S_{n,r} = \binom{n-r}{r} + \binom{n-r-1}{r-1}.$$

REFERENCE

1. A. C. Aitken, Determinants and Matrices, (New York, 1944), § 51.

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