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ON THE INDEPENDENCE OF SOJOURN TIMES IN TANDEM QUEUES

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Abstract

Reich (1957) proved that the sojourn times in two tandem queues are independent when the first queue is $M/M/1$ and the second has exponential service times. When service times in the first queue are not exponential, it has been generally expected that the sojourn times are not independent. A proof for the case of deterministic service times in the first queue is offered here.

1. Problem formulation

Let us consider two tandem FCFS queues, the first queue being $M/D/1$ and the second queue having exponential service times. Let λ denote the rate of Poisson arrivals at the first queue, T the deterministic service time in the first queue, and μ^{-1} the mean service time in the second queue. The state of the system is defined as the vector of queue lengths at the embedded times $\{t_1, t_2, \cdots\}$ when customers move from the first queue to the second queue. Let $n_1(t_k)$ be the number of customers left behind in the first queue by customer k, and $n_2(t_k-)$ be the number of customers found by customer k upon arrival at the second queue. It will be shown that the steady-state probabilities $\pi_{ij} = P\{(n_1, n_2) = (i, j)\}$ do not have a product-form solution, and hence n_1 and n_2 are not independent. This will imply that the sojourn times in the two queues, T_1 and T_2, are not independent.

2. Analysis

The transition probabilities $P_{ijkl} = P\{(n_1(t_{n+1}), n_2(t_{n+1}-)) = (k, l) \mid (n_1(t_n), n_2(t_n-)) = (i, j)\}$ are

$$P_{ijkl} = \begin{cases} a_{k+1}b_{j+1-l}, & i>0, \quad j \geqq 0, \quad j+1 \geqq l > 0, \quad k \geqq i-1 \\ a_{k+1-i} \sum_{m=j+1}^{\infty} b_m, & i>0, \quad j \geqq 0, \quad l=0, \quad k \geqq i-1 \\ a_k c_{j+1-l}, & i=0, \quad j \geqq 0, \quad j+1 \geqq l > 0, \quad k \geqq 0 \\ a_k \sum_{m=j+1}^{\infty} c_m, & i=0, \quad j \geqq 0, \quad l=0, \quad k \geqq 0 \\ 0, & \text{otherwise}, \end{cases}$$

where

$$a_k = \Pr\{k \text{ arrivals in } T \text{ time units}\} = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

$$b_k = \Pr\{k \text{ departures in } T \text{ time units}\} = \frac{(\mu T)^k}{k!} e^{-\mu T}$$

$$c_k = \Pr\{k \text{ departures in } T+t \text{ time units}\} = \sum_{m=0}^{k} \frac{\lambda}{\lambda+\mu} \left(\frac{\mu}{\lambda+\mu}\right)^{k-m} b_m.$$

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The steady-state probabilities, assuming that they exist, must satisfy

$$\pi_{kl} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \pi_{ij} P_{ijkl}.$$

We assume a product form $\pi_{kl} = \pi_k^1 \pi_l^2$ and substitute the known steady-state probabilities for the $M/D/1$ queue (see for example Gross and Harris (1974), pp. 241–243). From the expression for $\pi_0^1 \pi_l^2$ and the fact $\pi_l^2 = \sum_{k=0}^{\infty} \pi_k^1 \pi_l^2$, we obtain the set of equations

$$\sum_{j=l-1}^{\infty} \pi_j^2 (c_{j+1-l} - b_{j+1-l}) = 0, \quad l > 0.$$

By determining the left-side inverse of the matrix

$$\mathbf{A}_{ij} \begin{cases} c_{j-i} - b_{j-i}, & j \geqq i, \quad i \geqq 1 \\ 0, & 1 \leqq j < i, \quad i \geqq 1 \end{cases}$$

column by column, we can show that $\{\pi_j^2\} \equiv 0$ is the only solution of $c_0 \neq b_0$, which is valid under the conditions $\lambda > 0$, $\mu > 0$. Hence the steady-state probabilities $\{\pi_{ij}\}$ cannot have product form.

Lemma 1. For two tandem queues consisting of an $M/G/1$ queue followed by a $G/M/1$ queue, T_1 and T_2 are dependent if n_1 and n_2 are dependent.

Proof. It is shown that T_1 and n_2 are dependent when n_1 and n_2 are dependent. As noted by Reich (1957), n_1 and T_1 are related by

$$E\{z^{n_1} \mid n_2\} = \sum_{n_1=0}^{\infty} z^{n_1} \int_0^{\infty} p(n_1 \mid T_1, n_2) p(T_1 \mid n_2) \, dT_1 = \int_0^{\infty} \exp(\lambda T_1(z-1)) p(T_1 \mid n_2) \, dT_1.$$

The left-hand side is dependent on n_2, and so $p(T_1 \mid n_2)$ depends on n_2.

Now it is shown that T_1 and T_2 are dependent when T_1 and n_2 are dependent. Note that T_2 is the sum of $n_2 + 1$ i.i.d. exponential service times. We find

$$\begin{aligned} E[\exp(-sT_2) \mid T_1\} &= \int_0^{\infty} \exp(-sT_2) \sum_{n_2=0}^{\infty} p(T_2 \mid n_2, T_1) p(n_2 \mid T_1) \, dT_2 \\ &= \sum_{n_2=0}^{\infty} (1 + \mu^{-1} s)^{-n_2-1} p(n_2 \mid T_1). \end{aligned}$$

The right-hand side depends on T_1, so T_2 and T_1 are dependent.

References

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