

BIPARTITE DIVISOR GRAPH FOR THE PRODUCT OF SUBSETS OF INTEGERS

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Abstract

The bipartite divisor graph $B(X)$, for a set X of positive integers, and some of its properties have recently been studied. We construct the bipartite divisor graph for the product of subsets of positive integers and investigate some of its properties. We also give some applications in group theory.

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1. Introduction

Graphs associated with various algebraic structures have been actively investigated and many interesting results have recently been obtained, for example in [14–16, 19]. An extensive bibliography concerning applications of graphs of this kind is given in [12]; see also [8].

Let G be a finite group. For $x \in G$, let $x^G = \{x^g : g \in G\}$ be the conjugacy class of x , where $x^g = g^{-1}xg$. By $cs(G)$, we mean the set of conjugacy class sizes of G . There are several graphs associated with conjugacy class sizes, or the character degrees of a finite group, which have been studied in [1–3, 6, 7, 11]. The main graph-theoretic parameters considered in these papers are closely related. A survey regarding the influence of the size of conjugacy classes on the structure of finite groups can be found in [5].

Recently, Lewis [13] elucidated most of these connections first by analysing analogues of these graphs for positive integer subsets. Inspired by this paper, in [9] Praeger and the second author of the present paper considered the bipartite divisor graph $B(X)$ for a finite set X of positive integers and studied some properties of this graph such as the diameter, girth, number of connected components and clique number.

For a nonempty subset X of positive integers, let $\pi(x)$ be the set of primes dividing $x \in X$ and let $X^* = X \setminus \{1\}$. Then we define:

- (i) the prime vertex graph
- $\Delta(X)$
- as a graph with vertex set

$$V(\Delta(X)) = \rho(X) = \bigcup_{x \in X} \pi(x)$$

and edge set

$$E(\Delta(X)) = \{\{p, q\} : pq \text{ divides } x, \text{ for some } x \in X\};$$

- (ii) the common divisor graph
- $\Gamma(X)$
- as a graph with vertex set

$$V(\Gamma(X)) = X^*$$

and edge set

$$E(\Gamma(X)) = \{\{x, y\} : \gcd(x, y) \neq 1\};$$

- (iii) the bipartite divisor graph
- $B(X)$
- as a graph with vertex set

$$V(B(X)) = \rho(X) \sqcup X^*$$

and edge set

$$E(B(X)) = \{\{p, x\} : p \in \rho(X), x \in X^* \text{ and } p \text{ divides } x\}.$$

Thus $B(X)$ is bipartite, and $\{\rho(X) \mid X^*\}$ forms a bipartition of the vertex set, which is unique if $B(X)$ is connected.

Let H, K be two finite groups. Since $\text{cs}(H \times K) = \{xy : x \in \text{cs}(H), y \in \text{cs}(K)\}$, it is important to consider a bipartite graph related to $\text{cs}(H \times K)$, which is a set of positive integers.

Suppose that X is a subset of positive integers such that X has a decomposition $X = YZ$, where Y and Z are some positive integer subsets, each of size at least two. The aims of this paper are to construct a bipartite divisor graph $B(YZ)$ for the product of two integer subsets Y and Z and to find a relation between the graph $B(YZ)$ and the graphs $B(Y)$ and $B(Z)$.

In Section 2, we first construct a special bipartite graph and use this construction to define the bipartite graph for the product of two integer subsets as follows.

DEFINITION 1.1. Let Y and Z be two nonempty sets of positive integers, each of size at least two. By $B(YZ)$ we mean a bipartite graph with vertex set

$$(\rho(Y) \cup \rho(Z)) \sqcup (\{yz : y \in Y, z \in Z\})$$

and edge set $E(B(YZ))$, where $\{p, yz\} \in E(B(YZ))$ if and only if $\{p, y\} \in E(B(Y))$ or $\{p, z\} \in E(B(Z))$.

Then we study certain important properties of the diameter of $B(X)$. In Section 3 we consider the girth of $B(X)$ in special cases. Finally, in Section 4, we apply the results to finite group theory.

For positive integers m and n , we denote the greatest common divisor of m and n by $\gcd(m, n)$; the diameter of a graph G by $\text{diam}(G)$ (where by the diameter we mean the maximum distance between vertices in the same connected component of the graph); and the girth of the graph G (the length of the shortest cycle) by $\mathbf{g}(G)$ (which is ∞ if there is no cycle in the graph). Also, by P_n , C_n and K_n we mean a path of length n , a cycle of length n and a complete graph with n vertices, respectively. Other notation throughout the paper is standard.

2. Bipartite graph for the product of integer subsets

Throughout this paper we suppose that X and Y are two finite and nonempty sets of positive integers.

Let Σ_1 and Σ_2 be two bipartite graphs with vertex sets $V\Sigma_i = \Sigma_i^{(1)} \cup \Sigma_i^{(2)}$, such that Σ_1 and Σ_2 have no isolated vertices. For $i = 1, 2$, let $\Sigma'_i \subset \Sigma_i^{(1)}$ and suppose that there is a bijection $\varphi : \Sigma'_1 \rightarrow \Sigma'_2$. Suppose also that Π is a partition of $\Sigma_1^{(2)} \times \Sigma_2^{(2)}$ such that whenever $(x_1, y_1), (x_2, y_2)$ lie in the same part of Π , the following conditions hold:

- (i) for $p \in \Sigma_1^{(1)} \setminus \Sigma'_1$, $\{p, x_1\} \in E\Sigma_1$ if and only if $\{p, x_2\} \in E\Sigma_1$;
- (ii) for $p \in \Sigma_2^{(1)} \setminus \Sigma'_2$, $\{p, y_1\} \in E\Sigma_2$ if and only if $\{p, y_2\} \in E\Sigma_2$;
- (iii) for $p \in \Sigma'_1$ both $\{p, x_1\} \notin E\Sigma_1$ and $\{\varphi(p), y_1\} \notin E\Sigma_2$ hold if and only if both $\{p, x_2\} \notin E\Sigma_1$ and $\{\varphi(p), y_2\} \notin E\Sigma_2$ hold.

According to this notation, we may construct a bipartite graph $\Sigma = \Sigma(\Sigma_1, \Sigma_2, \varphi, \Pi)$ with vertex set $V\Sigma = \Phi \cup \Pi$, where $\Phi = \Sigma_1^{(1)} \cup (\Sigma_2^{(1)} \setminus \Sigma'_2)$, and edges of the form $\{p, \pi\}$ where $p \in \Phi, \pi \in \Pi$ and there exists $(x_1, y_1) \in \pi$ such that for each $(x_2, y_2) \in \pi$ one of the above conditions holds.

THEOREM 2.1. *The graph $\Sigma = \Sigma(\Sigma_1, \Sigma_2, \varphi, \Pi)$ is a connected graph and $\text{diam } \Sigma \leq 6$.*

PROOF. Let $\sigma_1, \sigma_2 \in V\Sigma$. We prove that there is a path in Σ from σ_1 to σ_2 . Let $p \in \Sigma_1^{(1)} \setminus \Sigma'_1$. Since p is not isolated in Σ_1 , there exists $x \in \Sigma_1^{(2)}$ such that $\{p, x\} \in E\Sigma_1$. By definition of $E\Sigma$, $\{p, \pi\} \in E\Sigma$ for all π containing a pair of the form (x, z) for some $z \in \Sigma_2^{(2)}$.

Consider $q \in \Sigma_2^{(1)} \setminus \Sigma'_2$ and $q' \in \Sigma'_2$. Since Σ_2 has no isolated vertices, there exist $y, y' \in \Sigma_2^{(2)}$ such that $\{q, y\}$ and $\{q', y'\} \in E\Sigma_2$. Let π, π' be the points of Π containing (x, y) and (x, y') , respectively. Then, by definition of $E\Sigma$, (p, π, q) and $(p, \pi', \varphi^{-1}(q'))$ are both paths in Σ . Thus p has distance 2 from each element of $\Sigma'_1 \cup (\Sigma_2^{(1)} \setminus \Sigma'_2)$. A similar argument shows that each $q \in \Sigma_2^{(1)} \setminus \Sigma'_2$ is at distance 2 from each element of $\Sigma_1^{(1)}$. In particular, for any $r, r' \in \Sigma_1^{(1)} \cup (\Sigma_2^{(1)} \setminus \Sigma'_2)$, we have $\mathbf{d}_\Sigma(r, r') \leq 4$.

Suppose that $\pi \in \Pi$. Let $(x, y) \in \pi$, so $x \in V\Sigma_1^{(2)}$ and $y \in V\Sigma_2^{(2)}$. Since Σ_1 and Σ_2 have no isolated vertices, there exist $p \in \Sigma_1^{(1)}$ and $q \in \Sigma_2^{(1)}$ such that $\{p, x\} \in E\Sigma_1$ and $\{q, y\} \in E\Sigma_2$. By the definition of $E\Sigma$, the following properties hold:

- (i) if $p \in \Sigma_1^{(1)} \setminus \Sigma'_1$, then $\{p, \pi\} \in E\Sigma$;

- (ii) if $q \in \Sigma_2^{(1)} \setminus \Sigma'_2$, then $\{q, \pi\} \in E\Sigma$;
- (iii) if $p \in \Sigma'_1$ or $\varphi^{(-1)}(q) \in \Sigma'_1$, then $\{p, \pi\} \in E\Sigma$ or $\{\varphi^{(-1)}(q), \pi\} \in E\Sigma$, respectively.

Since at least one of (i) or (iii) holds for p and at least one of (ii) or (iii) holds for q , we deduce that at least one of $\{p, \pi\}$, $\{q, \pi\}$ or $\{\varphi^{(-1)}(q), \pi\} \in E\Sigma$.

Let $\pi, \pi' \in \Pi$. Then we have shown that $\{r, \pi\}, \{r', \pi'\} \in E\Sigma$ for some $r, r' \in \Sigma^{(1)} = \Sigma_1^{(1)} \cup (\Sigma_2^{(1)} \setminus \Sigma'_2)$. We have proved that $\mathbf{d}_\Sigma(r, r') \leq 4$ and hence $\mathbf{d}_\Sigma(\pi, \pi') \leq 6$. Finally, for $\pi \in \Pi, r' \in \Sigma_1^{(1)}$, since $\mathbf{d}_\Sigma(\pi, r) = 1$ for some $r \in \Sigma_1^{(1)}$, it follows that $\mathbf{d}_\Sigma(\pi, r') \leq 5$. \square

Let X and Y be two nonempty sets of positive integers. Let $B(X)$ and $B(Y)$ be bipartite divisor graphs related to X and Y , respectively. Note that, by Definition 1.1, $B(XY)$ is a bipartite graph with vertex set $(\rho(X) \cup \rho(Y)) \cup (\{xy : x \in X, y \in Y\})$ and edge set $E(B(XY))$, where $\{p, xy\} \in E(B(XY))$ if and only if $\{p, x\} \in E(B(X))$ or $\{p, y\} \in E(B(Y))$.

LEMMA 2.2. *Let $\Sigma_1 = B(X_1)$ and $\Sigma_2 = B(X_2)$. For $(x, y), (x', y') \in X_1 \times X_2$ define $(x, y), (x', y') \in \pi \in \Pi$ if and only if $xy = x'y'$. Let φ be the identity function on $\Sigma'_1 = \Sigma'_2 = \rho(X_1) \cap \rho(X_2)$. Then $\Sigma = \Sigma(\Sigma_1, \Sigma_2, \varphi, \Pi) \cong B(X_1X_2)$.*

PROOF. By our assumption, we conclude that

$$\Phi = (\rho(X_1) \cup \rho(X_2)) \setminus (\rho(X_1) \cap \rho(X_2)) = \rho(X_1X_2).$$

Define $\psi : \Phi \cup \Pi \rightarrow \rho(X_1X_2) \cup X_1X_2$ such that $\psi(p) = p$, for all $p \in \Phi$ and for each $\pi \in \Pi, \psi(\pi) = xy$, such that $(x, y) \in \pi$. It is enough to show that ψ preserves the adjacency. Let $p \in \Phi$ and $\pi \in \Pi$ such that $\{p, \pi\} \in E\Sigma$. By the definition of ψ and Φ , we have $\psi(p) = p, \psi(\pi) = xy$, where $(x, y) \in \pi$. Since $\Phi = \rho(X_1X_2)$, by the definition of Σ and $B(X_1X_2)$ we conclude that $\{p, xy\} \in E(B(X_1X_2))$. Conversely, suppose that $\{p, xy\} \in E(B(X_1X_2))$, such that $(x, y) \in \pi \in \Pi$. By the definition of Φ , there are the following three cases for p :

- (i) $p \in \rho(X_1) \setminus \rho(X_2)$;
- (ii) $p \in \rho(X_2) \setminus \rho(X_1)$;
- (iii) $p \in \rho(X_1) \cap \rho(X_2)$.

Since $\psi^{-1}(p) = p$ and $\psi^{-1}(xy) = \pi$, in each case we conclude that $\{p, \pi\} \in E\Sigma$, so ψ preserves the adjacency. This completes the proof. \square

THEOREM 2.3. *Suppose that X and Y are two nonempty sets of positive integers. Then $B(XY)$ is a connected graph and $\text{diam}(B(XY)) \leq 6$.*

PROOF. Lemma 2.2 and Theorem 2.1 yield the desired result. \square

Let X, Y be two nonempty sets of positive integers. By $\Gamma(XY)$ and $\Delta(XY)$, we mean the common divisor and prime graphs related to the product of two integer sets X, Y . Then the diameters of $\Gamma(XY)$ and $\Delta(XY)$ are less than or equal to 3. This is an immediate consequence of Theorem 2.3 and [9, Lemma 1]. Note that all cases may hold, as we may see in Table 1.

TABLE 1. Illustration of all cases in the note after Theorem 2.3.

X	Y	XY	$\text{diam}(\Gamma(XY))$	$\text{diam}(\Delta(XY))$
{2}	{3, 4}	{6, 8}	1	1
{2, 5}	{3, 4}	{6, 8, 15, 20}	2	1
{5, 7}	{3, 4}	{15, 21, 20, 28}	2	2
{1, 2, 3}	{1, 5}	{1, 2, 3, 5, 10, 15}	3	2

LEMMA 2.4. *Let X, Y be two nonempty sets of positive integers. Let $B(X)$ and $B(Y)$ be bipartite divisor graphs for X and Y , respectively. Suppose that $\Delta(X)$, $\Delta(Y)$ and $\Gamma(XY)$ are complete graphs. Then $\text{diam}(B(XY)) \leq 3$.*

PROOF. Since $\Delta(X)$, $\Delta(Y)$ and $\Gamma(XY)$ are complete, it follows that $\Delta(XY)$ is complete. Now by [9, Lemma 1], we conclude that

$$\text{diam}(B(XY)) = 2\text{diam } \Delta(XY) + 1 = 3.$$

This completes the proof. □

3. Girth of the $B(XY)$

In this section, we prove that, under some conditions, $\mathbf{g}(B(XY)) \leq 8$. Note that we have the same results if we swap X and Y .

THEOREM 3.1. *Let X, Y be two nonempty sets of positive integers. Let $B(X)$ and $B(Y)$ be bipartite divisor graphs for X and Y , respectively. Then $\mathbf{g}(B(XY)) = 4$ if one of the following conditions holds.*

- (i) $B(X)$ has a cycle and $|Y^*| \geq 1$.
- (ii) $B(X)$ is connected, $|X^*| \geq 2$ and there exists $q \in \rho(Y) \setminus \rho(X)$.
- (iii) Both $B(X)$ and $B(Y)$ are acyclic and disconnected and there is a component of $B(X)$ which contains P_2 .

Otherwise, if both $B(X)$ and $B(Y)$ are acyclic and disconnected, such that all components of both $B(X)$ and $B(Y)$ are paths of length one and $X \neq Y$, then $\mathbf{g}(B(XY)) \leq 8$.

PROOF. (i) If one of the graphs $B(X)$ or $B(Y)$ contains a cycle of length four, then $\mathbf{g}(B(XY)) = 4$. So suppose that $\min\{\mathbf{g}(B(X)), \mathbf{g}(B(Y))\} \geq 6$. Since $\mathbf{g}(B(X)) \geq 6$, so $B(X)$ has at least six vertices. Suppose that $\rho(X) = \rho(Y)$ and $B(X)$ has a cycle of the form

$$p_1 - x_1 - p_2 - x_2 - \cdots - p_n - x_n - p_1. \tag{*}$$

Since $\rho(X) = \rho(Y)$, so there exists $y \in Y$ such that p_1 divides y . Now $p_1 - x_1y - p_2 - x_2y - p_1$ is a cycle of length four in $B(XY)$, so $\mathbf{g}(B(XY)) = 4$. Suppose that $\rho(X) \neq \rho(Y)$

and $B(X)$ has a cycle similar to $(*)$. If there exists $q \in \rho(Y) \setminus \rho(X)$, then we may choose $y \in Y$ such that q divides y . In this case $q - x_1y - p_2 - x_2y - q$ is a cycle in $B(XY)$, so $\mathbf{g}(B(XY)) = 4$. On the other hand, if $\rho(Y)$ is a proper subset of $\rho(X)$ then, for $p_1 \in \rho(Y)$, we have a cycle $p_1 - x_1y - p_2 - x_2y - p_1$ in $B(XY)$; otherwise $q - x_1y - p_2 - x_2y - q$ is a cycle in $B(XY)$ where $q \in \rho(Y)$. So in this case $\mathbf{g}(B(XY)) = 4$.

(ii) Let x, x' be two distinct elements of X^* . Since $B(X)$ is connected, there is a path between x, x' , say $x - p_1 - x_2 - p_2 - \dots - x'$. Let $y \in Y$ with $\{q, y\} \in E(B(Y))$. Then $p_1 - xy - q - x_2y - p_1$ is a cycle of length four in $B(XY)$.

(iii) First suppose that $x_1 - p - x_2$ is a path of length two in a component of $B(X)$. Since $B(X)$ is disconnected, there must be $x_3 \in X \setminus \{x_1, x_2\}$ and $q \in \rho(X)$ such that q divides x_3 . If $\rho(X) = \rho(Y)$, then there exists $y \in Y$ such that q divides y . Hence $x_1y - p - x_2y - q - x_1y$ is a cycle of length four in $B(XY)$.

So suppose that $\rho(X) \neq \rho(Y)$. Let $r \in \rho(Y) \setminus \rho(X)$. Then $B(XY)$ has a cycle of length four.

Finally, let $\rho(Y)$ be a proper subset of $\rho(X)$. Since $B(Y)$ is disconnected, there must be $t \in \rho(Y)$ such that $t \neq p$ and there exists $y \in Y$ such that t divides y . Now $p - x_2y - t - x_1y - p$ is a cycle of length four. For the case where there is a path of the form $p_1 - x - p_2$ in a component of $B(X)$, the proof is similar.

Suppose that both $B(X)$ and $B(Y)$ are acyclic and disconnected such that all components of both $B(X)$ and $B(Y)$ are paths of length one. First suppose that there exists $p \in \rho(X) \cap \rho(Y)$. Let x_1, y_1 be two elements of X, Y such that p divides x_1 and p divides y_1 . Since both $B(X)$ and $B(Y)$ are disconnected, both X^* and Y^* have size greater than one, which implies that there is $x_2 \in X \setminus \{x_1\}$ and $y_2 \in Y \setminus \{y_1\}$. Hence

$$p - x_1y_2 - q - x_2y_2 - p_2 - x_2y_1 - p$$

is a cycle of length six in $B(XY)$ where p_2 divides x_2 and q divides y_2 . Therefore $\mathbf{g}(B(XY)) \leq 6$. On the other hand, if $\rho(X) \cap \rho(Y) = \emptyset$, since both $B(X)$ and $B(Y)$ are disconnected and all components of both $B(X), B(Y)$ are paths of length one, so there exist $x_1, x_2 \in X, y_1, y_2 \in Y, p_1, p_2 \in \rho(X)$ and $q_1, q_2 \in \rho(Y)$ such that p_i divides x_i and q_i divides y_i , for $i = 1, 2$. Now it is easy to see that there is a path of length eight in $B(XY)$. □

REMARK 3.2. The condition $|X^*| \geq 2$ is necessary, since otherwise there may exist sets of positive integers, say X, Y , such that $B(XY)$ is acyclic and both $B(X)$ and $B(Y)$ are connected. Also, there may exist X, Y such that both $B(X)$ and $B(Y)$ are acyclic, but $\mathbf{g}(B(XY)) = 4$. In case (ii) of Theorem 3.1 it is necessary that at least one of $B(X)$ or $B(Y)$ be connected, since there are sets of positive integers such that both $B(X), B(Y)$ are disconnected and acyclic and $\mathbf{g}(B(XY)) = 6$. Table 2 contains some examples of these cases.

REMARK 3.3. The case $X \neq Y$ in the last part of the previous theorem is essential. Since for $X = Y = \{1, 4, 9\}$, we can see that both $B(X)$ and $B(Y)$ are disconnected and acyclic and all components are paths of length one. But $B(XY)$ is acyclic and therefore $\mathbf{g}(B(XY)) = \infty$.

QUESTION 3.4. Determine all sets of positive integers X, Y such that $\mathbf{g}(B(XY)) = 4$.

TABLE 2. Illustration of all cases of Remark 3.2.

X	Y	$n(B(X))$	$n(B(Y))$	$g(B(X))$	$g(B(Y))$	$g(B(XY))$
{1, 3, 6, 8}	{1, 3, 6, 8}	1	1	∞	∞	4
{1, 2, 3}	{1, 2, 3, 5}	2	3	∞	∞	6
{1, 2}	{1, 3}	1	1	∞	∞	∞

4. Some examples and applications in group theory

Let H, K be groups. It is a well-known fact that $cs(H \times K) = \{xy : x \in cs(H), y \in cs(K)\}$, so by the definition we conclude that $B(H \times K) = B(XY)$, where $X = cs(H), Y = cs(K)$. Let $G = H \times K$ such that both groups are finite and H is abelian but K is not. Since $cs(H) = \{1\}$, $cs(G) = cs(K)$. Therefore $B(G) = B(K)$. For example, let $G = D_{12} = \mathbb{Z}_2 \times D_6$. Let $X = cs(\mathbb{Z}_2)$ and $Y = cs(D_6)$, so $X = \{1\}$ and $Y = \{1, 2, 3\}$. It is easy to see that $B(XY) \cong B(G)$. So $B(D_{12}) = B(D_6)$. Also, $B(SL(2, \mathbb{Z}_6))$ is isomorphic to $B(XY)$ where X, Y are $cs(S_3), cs(A_4)$, respectively.

Throughout this section, let P be a p -group, Q a q -group and R an r -group for distinct primes p, q, r . We consider the following three types of groups.

- (i) Type(A): $G = P \rtimes (Q \times R)$, with P and Q abelian, $r = 2, Z(G) = O_2(G)$ and $G/Z(G)$ is a Frobenius group and $R/Z(G) \cong Q_8$.
- (ii) Type(B): $G = (P \times R) \rtimes Q$, with P and Q abelian, $G/Z(G)$ a Frobenius group and $|cs^*(R)| = 1$.
- (iii) Type(C): $G = R \rtimes PQ$, with $R = C_G(R)$ minimal normal in G and $PQ \subseteq \Gamma L(1, R)$ a Frobenius group.

THEOREM 4.1. *Let G be a finite group. If $B(G)$ is a path, then G is solvable. Furthermore, G is one of the following groups.*

- (i) G is one of the groups of type (A), (B) or (C).
- (ii) G is a p -group for some prime p .
- (iii) $G = KL$, with $K \trianglelefteq G, \gcd(|K|, |L|) = 1$ and one of the following cases occurs:
 - (a) both K and L are abelian, $Z(G) < L$ and G is a quasi-Frobenius group;
 - (b) K is abelian, L is a nonabelian p -group for some prime $p, O_p(G)$ is an abelian subgroup of index p in L and $G/O_p(G)$ is a Frobenius group;
 - (c) K is a p -group of conjugate rank one for some prime p, L is abelian, $Z(K) = Z(G) \cap K$ and G is quasi-Frobenius.
- (iv) G is a group such that $cs(G) = \{1, p^a, q^b, p^c q^d\}$.
- (v) G is a direct product of a p -group and an abelian p' -group.

PROOF. Suppose that $B(G)$ is a path. By [4, Proposition 22 and Theorem 5], we know that $B(G) \cong P_n$, such that $n \leq 5$, and for the case $n = 5, G$ is solvable and is one of the

groups of type(A), (B), or (C). Define

$$\sigma^*(G) := \max\{|\pi(x^G)| : x \in G\}.$$

If $B(G)$ is isomorphic to P_4 or P_3 , then $\sigma^*(G) = 2$. Suppose that G is not solvable; then by [18] we conclude that $B(G)$ is a cycle of length six which is in contradiction to the hypothesis, so G is solvable. Finally, suppose that $B(G)$ is isomorphic to P_2 . In this case, $cs^*(G)$ has at most two elements. If $cs^*(G)$ has only one element then G is nilpotent, so is solvable, so suppose that $cs^*(G)$ has two nontrivial elements, say m, n . Hence G is a group of conjugate rank two. Ito [10] proved that a finite group of conjugate rank two is solvable.

If $B(G) = P_3$, or $B(G) = P_4$ and G has conjugate rank two, then $cs(G) = \{1, p^a q^b, q^c r^d\}$, so by [5, Theorem 19], we conclude that up to abelian direct factor, $G = KL$, with $K \trianglelefteq G$, $\gcd(|K|, |L|) = 1$ and one of the following cases occurs:

- (i) both K and L are abelian, $Z(G) < L$ and G is a quasi-Frobenius group;
- (ii) K is abelian, L is a nonabelian p -group for some prime p , $O_p(G)$ is an abelian subgroup of index p in L and $G/O_p(G)$ is a Frobenius group; or
- (iii) K is a p -group of conjugate rank one for some prime p , L is abelian, $Z(K) = Z(G) \cap K$ and G is quasi-Frobenius.

If $B(G) = P_4$ and G has conjugate rank three, then there exist some positive integers, say a, b, c, d , such that $cs(G) = \{1, p^a, q^b, p^c q^d\}$. If $B(G) = P_2$ and G has conjugate rank two, then both of its conjugacy class sizes must be a power of a prime number, say p , so by [5, Theorem 19], G is a p -group. But if it has only one conjugacy class size, then $cs(G) = \{1, p^a q^b\}$. By [5, Theorem 13], if a group has only one nontrivial conjugacy class size, it must be a power of a prime number. So this case will not occur. Finally, if $B(G) = P_1$, then by [5, Theorem 13], G is a direct product of a p -group and an abelian p' -group (which is a group such that its order is not divisible by p). The p -group P has an abelian normal subgroup A , such that P/A has exponent p . □

COROLLARY 4.2. *Let G and H be finite groups such that $B(G)$ and $B(H)$ are both paths. Then the direct product of these groups is solvable.*

PROOF. This is an immediate consequence of Theorem 4.1. □

LEMMA 4.3. *Suppose that $G \cong H \times K$, and H, K are finite groups. If $\text{diam}(B(G)) = 4$, and $\Gamma(T)$ is a complete graph for every subgroup T of G , then we have the following properties:*

- (i) G is a solvable group;
- (ii) there exist two prime numbers $p, q \in \rho(G)$, such that G is p -nilpotent or q -nilpotent.

PROOF. Since $\text{diam}(B(G)) = 4$, by [9, Lemma 1] we have the following cases:

- (i) $\text{diam}(\Gamma(G)) = 1$ and $\text{diam}(\Delta(G)) = 2$;
- (ii) $\text{diam}(\Gamma(G)) = 2$ and $\text{diam}(\Delta(G)) = 1$;
- (iii) $\text{diam}(\Delta(G)) = \text{diam}(\Gamma(G)) = 2$.

Also, by Theorem 2.1, $B(G)$ is a connected graph, so both $\Gamma(G)$ and $\Delta(G)$ are also connected. By the hypothesis, $\Gamma(G)$ is a complete graph, so we conclude that only the first case will occur. Since $\Gamma(T)$ is a complete graph for every subgroup T of G , by [17] we conclude that G is solvable. As $\text{diam}(\Delta(G)) = 2$, so there exist at least two prime numbers $p, q \in \text{cs}^*(G)$ such that p is not adjacent to q . Since G is solvable, by [10] G is either p -nilpotent or q -nilpotent. \square

QUESTION 4.4. If the second case in the proof of the last lemma occurs, then what can we say about the group G ?

LEMMA 4.5. Suppose that H and K are finite groups such that their bipartite divisor graphs are both cycles. Let $G = H \times K$. Then either $\Gamma(G)$ is a complete graph or G has a normal t -complement, abelian Sylow t -subgroup for some prime number t and $\Gamma(G)$ can be obtained by omitting just three edges of the complete graph K_{15} or four edges of the complete graph K_9 .

PROOF. Both $B(H)$ and $B(K)$ are cycles, so by [18, Theorem 1] $B(H) \simeq B(K) \simeq C_6$, and also H and K are isomorphic to $A \times \text{SL}_2(q)$, where A is an abelian group and $q \in \{4, 8\}$. Since $\text{cs}(\text{SL}_2(4)) = \{1, 12, 15, 20\}$ and $\text{cs}(\text{SL}_2(8)) = \{1, 72, 63, 50\}$, we have three cases as follows:

- (i) $\text{cs}^*(G) = \{12, 15, 20, 144, 180, 240, 225, 300, 400\}$;
- (ii) $\text{cs}^*(G) = \{12, 15, 20, 72, 63, 50, 864, 756, 600, 1080, 945, 750, 1440, 1260, 1000\}$;
- (iii) $\text{cs}^*(G) = \{72, 63, 50, 5184, 4536, 3600, 3969, 3150, 2500\}$.

In the first case, we can see that $\Gamma(G)$ is a complete graph. But in the second case, $\text{gcd}(63, 1000) = 1$, $\text{gcd}(63, 20) = 1$ and $\text{gcd}(63, 50) = 1$. Also, in the last case, $\text{gcd}(63, 50) = 1$, $\text{gcd}(63, 2500) = 1$, $\text{gcd}(50, 3969) = 1$, $\text{gcd}(3969, 2500) = 1$ and other vertices are joined, so $\Gamma(G)$ can be obtained by omitting just three edges of the complete graph K_{15} or four edges of K_9 , respectively. Also, $\{5, 7\} \subseteq V(\Delta(G))$ and they are not adjacent. So by [4, Lemma 13], G has a t -complement, abelian Sylow t -subgroup for some $t \in \{5, 7\}$. \square

REMARK 4.6. Suppose that G is a finite group. It is proved in [4, Theorem 3] that under certain conditions both $\Delta(G)$ and $\Gamma(G)$ are acyclic. So for all groups G except those mentioned in [4, Theorem 3] and for all groups H such that $\text{cs}(G) \neq \text{cs}(H)$ and $|\text{cs}^*(H)| \geq 2$, $\mathbf{g}(B(G \times H)) = 4$.

EXAMPLE 4.7. Let $G = H \times K$, where H, K are finite nonabelian groups. By Theorem 2.3, $B(G)$ is connected. Now by [4, Theorem 1] there are no abelian groups A, B of coprime orders such that $G = AB$ and $G/Z(G)$ is a Frobenius group of order ab , where $a = |A : (Z(G) \cap A)|$, $b = |B : (Z(G) \cap B)|$.

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